6.5 – Logarithmic Properties

Learning Objectives

1. Properties of logarithms.
2. Laws of logarithms
3. Expand logarithmic expressions.
4. Combine logarithmic expressions.
5. Use the change-of-base formula for logarithms.

1 – Properties of Logarithms

When we apply the Inverse Function Property to $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, we get

$$\log_b(b^x) = x \quad \text{for} \quad (-\infty, \infty)$$

$$b^{\log_b x} = x \quad \text{for} \quad (0, \infty)$$

Properties of Logarithms

1. $\log_b 1 = 0$
2. $log_b b = 1$
3. $\log_b(b^x) = x$
4. $b^{\log_b x} = x$

Example 1 – Applying Properties of Logarithms

We illustrate the properties of logarithms when the base is 5.

$\log_5 1 = 0$	Property 1	$\log_5 5 = 1$	Property 2
$\log_5 5^8 = 8$	Property 3	$5^{\log_5 12} = 12$	Property 4

Since logarithms are exponents, the Laws of Exponents give rise to the Laws of Logarithms.

LAWS OF LOGARITHMS

Let b be a positive number, with $b \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_b(AB) = \log_b A + \log_b B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_b\left(\frac{A}{B}\right) = \log_b A - \log_b B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_b(A^C) = C \log_b A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

The Product Rule for Logarithms

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(AB) = \log_b A + \log_b B$$

HOW TO

Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms.

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Example 2 – Using the Product Rule for Logarithms

Expand $\log_3(30x(3x+4))$.

Solution:

The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{A}{B}\right) = \log_a A - \log_b B$$

HOW TO

Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms.

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Example 3 – Using the Quotient Rule for Logarithms

Expand $\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

Solution:

The Power Rule for Logarithms

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(A^C) = C \log_b A$$

HOW TO

Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm.

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Example 4 – Expanding a Logarithm with Powers

Expand $\log_2 x^5$.

Solution:

The Laws of Logarithms allow us to write the logarithm of a product or a quotient as the sum or difference of logarithms. This process, called expanding a logarithmic expression, is illustrated in the next example.

Example 5 – Expanding Logarithmic Expressions

Use the Laws of Logarithms to expand each expression.

(a) $\log_2(6x)$ (b) $\log_5(x^3y^6)$ (c) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right)$

Solution:

The Laws of Logarithms also allow us to reverse the process of expanding that was done in Example 5. That is, we can write sums and differences of logarithms as a single logarithm. This process, called combining logarithmic expressions, is illustrated in the next example.

HOW TO

Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Use the Laws of Logarithms to combine each expression into a single logarithm.

(a) $3 \log x + \frac{1}{2} \log(x + 1)$

(b) $3 \ln s + \frac{1}{2} \ln t - 4 \ln(t^2 + 1)$

Solution:

5 – Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

Change of Base Formula

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$.

$$\log_b M = \frac{log_n M}{\log_n b}$$

The change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

HOW TO

Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$.

1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$, has base e.
2. Rewrite the log as a quotient using the change-of-base formula
   a. The numerator of the quotient will be a logarithm with base n and argument M.
   b. The denominator of the quotient will be a logarithm with base n and argument b.

Example 7 – Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.
Solution: