6.3 and 6.4 – Logarithmic Functions and Their Graphs

Learning Objectives

1. Convert between logarithmic and exponential form
2. Evaluate logarithms expressions.
3. Use common and natural logarithms.
4. Find the domain and range of a logarithmic function.
5. Graph logarithmic functions.
6. Graphing transformations of logarithmic functions.

Definition of the Logarithmic Function

The exponential function $y = f(x) = b^x$, where $b > 0$ and $b \neq 1$, is a one-to-one function and has an inverse function that is defined implicitly by the equation

$$x = b^y \quad b > 0 \quad b \neq 1$$

This inverse function is called the **logarithmic** function.

Definition – Logarithmic Function with Base b

The logarithmic function with base b, where $b > 0$ and $b \neq 1$, is denoted by $y = \log_b x$ (read as " y is the logarithm with base b of x") and is defined by

$$y \log_b x \quad \text{if and only if} \quad x = b^y$$

The domain of the logarithmic function is $(0, \infty)$. The range is $(-\infty, \infty)$.

Logarithmic form	Exponential form
Exponent	Exponent
$\log_b x = y$	$b^y = x$
Base	Base

HOW TO

Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form.

1. Examine the equation $y = \log_b(x)$ and identify $b, y, \text{and} x$.
2. Rewrite $\log_b(x) = y$ as $b^y = x$.

Example 1 – Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

(a) $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$ (b) $\log_3(9) = 2$

Solution:

Example 2 – Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

(a) $2^3 = 8$ (b) $10^{-4} = \dfrac{1}{10,000}$

Solution:

2 – Evaluating Logarithmic Expressions

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally.

Example: Consider $\log_2 8$.
Question: "To what exponent must 2 be raised in order to get 8?".

Answer: Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

HOW TO

Given a logarithm of the form $y = \log_b(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of b : $b^y = x$.
2. Use previous knowledge of powers of b identify y by asking, "To what exponent should b be raised in order to get x? "

Example 3 – Solving Logarithms Mentally

Solve $\log_4 64 = y$ for y.

Solution:

Example 4 – Evaluating the Logarithm of a Reciprocal

Evaluate $\log_3\left(\frac{1}{27}\right)$.

Solution:

COMMON LOGARITHM

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

NATURAL LOGARITHM

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

Example 5 – Evaluating Logarithms with base 10 and e

Evaluate each logarithmic expression using Desmos.

(a) $\log(1000)$ (b) $\ln(500)$

Solution:

4 – Finding the Domain and Range of a Logarithmic Function

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

1. The domain of $y = b^x$ is $(-\infty, \infty)$.
2. The range of $y = b^x$ is $(0, \infty)$.

The logarithmic function $y = \log_b x$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

1. The domain of $y = log_b x$ is the range of $y = b^x$: $(0, \infty)$.
2. The range of $y = \log_b x$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Domain of a Logarithmic Function

Applying transformations to the parent function $y = \log_b x$ can change the domain. When finding the domain of a logarithmic function, therefore, it is important to remember that **the domain consists only of positive real numbers**.

The argument of the logarithmic function must be greater than zero.

HOW TO

Given a logarithmic function, identify the domain.

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

Example 6 – Identifying the Domain of a Logarithmic Shift

Find the domain of $f(x) = \log_2(x+3)$. Write your answer using interval notation.

Solution:

Example 7 – Identifying the Domain of a Logarithmic Shift and Reflection

Find the domain of $f(x) = \log(5 - 2x)$. Write your answer using interval notation.

Solution:

Example 8 – Finding the Domain of a Logarithmic Function

Find the domain of $f(x) = \ln(4 - x^2)$. Write your answer using interval notation.

Solution:

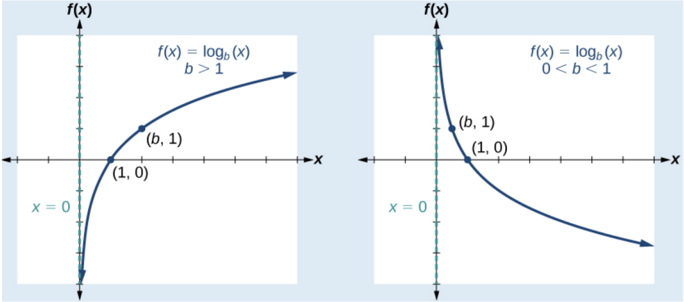
Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions.

The family of logarithmic functions includes the parent function $y = \log_b x$ along with all its transformations.

Characteristics of the Graph of the Parent Function, $f(x) = \log_b x$

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b x$:

1. one-to-one function
2. vertical asymptote: $x = 0$
3. domain: $(0, \infty)$
4. range: $(-\infty, \infty)$
5. $x-$intercept: $(1, 0)$ and key point $(b, 1)$
6. $y-$intercept: none
7. increasing if $b > 1$
8. decreasing if $0 < b < 1$



Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.
3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote, $x = 0$.

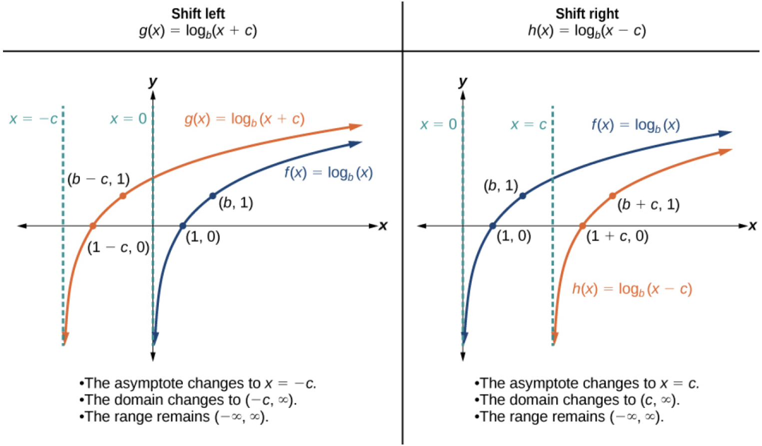
Example 9 – Graphing a Logarithmic Function with the Form $f(x) = \log_b x$

Graph $f(x) = \log_5 x$. State the domain, range, and asymptote.
Solution:

6 – Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can transform the parent function $f(x) = \log_b x$ without loss of shape.

Graphing a Horizontal Shift of $f(x) = \log_b x$



Graphing a Horizontal Shift of $f(x) = \log_b x$

For any constant c, the function $f(x) = \log_b(x - c)$

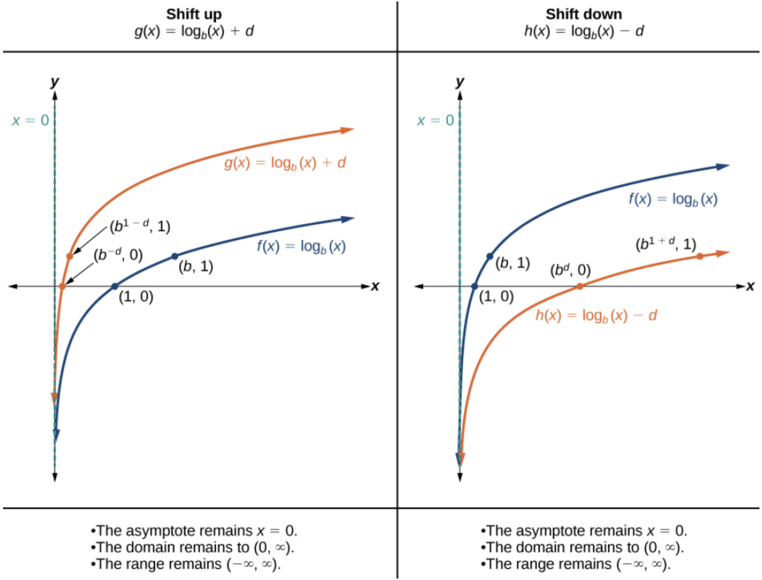
1. shifts the parent function $y = \log_b x$ left c units if $c < 0$.
2. shifts the parent function $y = \log_b x$ right c units if $c > 0$.
3. has vertical asymptote $x = c$.
4. has domain (c, ∞).
5. has range $(-\infty, \infty)$.

Example 10 – Graphing a Horizontal Shift of the Parent Function

Sketch the horizontal shift $f(x) = \log_3(x-2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution:

Graphing a Vertical Shift of $f(x) = \log_b x$



Graphing a Vertical Shift of $f(x) = \log_b x$

For any constant d, the function $f(x) = \log_b + d$

1. shifts the parent function $y = \log_b x$ up d units if $d > 0$.
2. shifts the parent function $y = \log_b x$ down d units if $d < 0$.
3. has vertical asymptote $x = 0$.
4. has domain $(0, \infty)$.
5. has range $(-\infty, \infty)$.

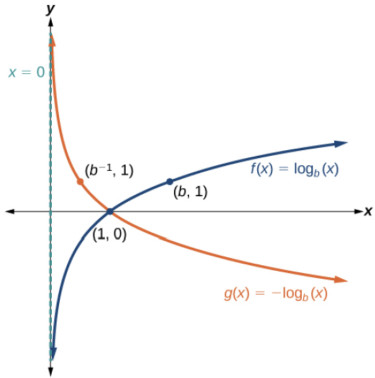
Example 11 – Graphing a Vertical Shift of the Parent Function

Sketch the horizontal shift $f(x) = \log_3(x) + 2$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

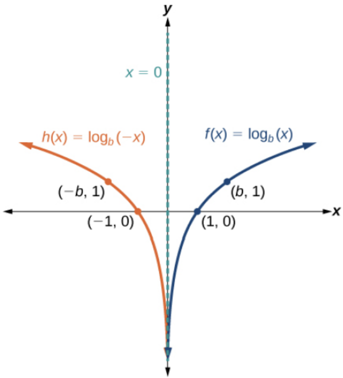
Solution:

Graphing Reflections of $f(x) = \log_b x$

Reflection about the *x*-axis $g(x) = -\log_b(x)$, $b > 1$	Reflection about the *y*-axis $h(x) = \log_b(-x)$, $b > 1$



- The reflected function is decreasing as x moves from zero to infinity.
- The asymptote remains $x = 0$.
- The x-intercept remains $(1, 0)$.
- The key point changes to $(b^{-1}, 1)$
- The domain remains $(0, \infty)$.
- The range remains $(-\infty, \infty)$.



- The reflected function is decreasing as x moves from negative infinity to zero.
- The asymptote remains $x = 0$.
- The x-intercept changes to $(-1, 0)$.
- The key point changes to $(-b, 1)$
- The domain changes to $(-\infty, 0)$.
- The range remains $(-\infty, \infty)$.

Graphing Reflections of $f(x) = \log_b x$

Reflections of the Parent Function $y = \log_b(x)$

The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the x-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

- reflects the parent function $y = \log_b(x)$ about the y-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

Example 12 – Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution: