6.1 and 6.2 – Exponential Functions and Their Graphs

Learning Objectives

1. Evaluate exponential functions.
2. Find the equation of an exponential function.
3. Define the number e.
4. Graph exponential functions.
5. Graph exponential functions using transformations.

Exponential Function

For any real number x, an exponential function is a function with the form

$$f(x) = ab^x$$

where

1. a is a non-zero real number called the initial value
2. b is any positive real number such that $b \neq 1$
3. The domain of $f(x)$ is $(-\infty, \infty)$
4. The range of $f(x)$ is $(0, \infty)$
5. The $y-$intercept is $(0, a)$
6. The horizontal asymptote is $y = 0$

1 – Evaluating Exponential Functions

The base of an exponential function must be a positive real number other than 1.

Question: Why do we limit the base to positive values?

Answer:To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

Let $b = -9$ and $x = \frac{1}{2}$. The $f(x) = f\left(\frac{1}{2}\right) = (-9)^{1/2} = \sqrt{-9}$, which is not defined.

Question: Why do we limit the base to positive values other than 1?

Answer:Because base 1 results in the constant function. Observe what happens if the base is 1: Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x.

To evaluate an exponential function with the form $f(x) = ab^x$, we simply simply substitute x with the given value, and calculate the resulting power.

Example 1 – Evaluating Exponential Functions

Let $f(x) = 5(3)^{x+1}$. Find $f(2)$.
Solution:

Example 2 – Evaluating Exponential Functions

Let $f(x) = 3^x$, and evaluate the following:

(a) $f(5)$ **(b)** $f\left(-\frac{2}{3}\right)$

(c) $f(\pi)$ **(d)** $f(\sqrt{2})$

Solution:

2 – Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b, and evaluate the function.

HOW TO

Given two data points, write an exponential function.

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Example 3 – Writing an Exponential Function When the Initial Value Is Known

Find an exponential Function that passes through the points $(0, 1)$ and $(2, 25)$.

Solution:

Example 4 – Writing an Exponential Function When the Initial Value Is Not Known

Find an exponential Function that passes through the points $(-2, 6)$ and $(2, 1)$.

Solution:

Define the Number e

Many problems that occur in nature require the use of an exponential function whose base is a certain irrational number, symbolized by the letter e.

Definition: – Number e

The **number e** is defined as the number that the expression

$$\left(1 + \frac{1}{n}\right)^n$$

approaches as $n \to \infty$.

The Number e

n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

$$e \approx 2.718281828459045235 36$$

THE NATURAL EXPONENTIAL FUNCTION

The **natural exponential function** is the exponential function

$$f(x) = e^x \text{ with base } e.$$

Example 5 – Evaluating the Natural Exponential Function

Evaluate each expression rounded to five decimal places.

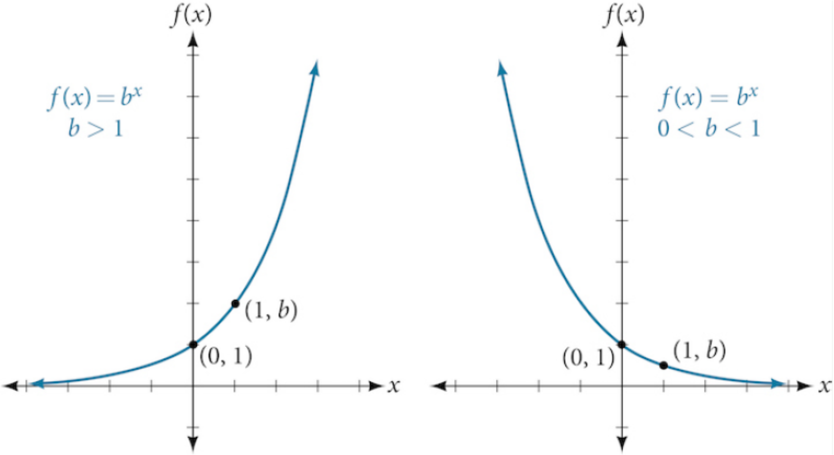
(a) e^3 **(b)** $2e^{-0.53}$ **(c)** $e^{4.8}$

Solution:

Characteristics of the Graph of the Parent Function $f(x) = b^x$

An exponential function with the form $f(x) = b^x$, $b > 0$, $b \neq 0$, has these characteristics:

1. one-to-one function
2. horizontal asymptote: $y = 0$
3. domain: $(-\infty, \infty)$
4. range: $(0, \infty)$
5. $x-$intercept: none
6. $y-$intercept: $(0, 1)$
7. increasing if $b > 1$
8. decreasing if $b < 1$


f(x)
f(x) = b^x
b > 1
(1, b)
(0, 1)
x
f(x)
f(x) = b^x
0 < b < 1
(0, 1)
(1, b)
x

How to Graph $f(x) = b^x$

HOW TO

Given an exponential function of the form $f(x) = b^x$, graph the function.

1. Create a table of points.
2. Plot at least 3 point from the table, including the y-intercept $(0, 1)$.
3. Draw a smooth curve through the points.
4. State the domain, $\left(-\infty, \infty\right)$, the range, $\left(0, \infty\right)$, and the horizontal asymptote, $y = 0$.

Example 6 – Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch the graph of each function below. State the domain, range, and asymptote.

(a) $f(x) = 3^x$ **(b)** $g(x) = \left(\frac{1}{3}\right)^x$

Solution:

5 – Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the transformations to the parent function without loss of shape.

Shifts of the Parent Function $f(x) = b^x$

For any constants d and d, the function $f(x) = b^{x-c} + d$ shifts the parent function $f(x) = b^x$

1. vertically d units, in the same direction of the sign of d
2. horizontally c units, in the opposite direction of the sign of c
3. The $y-$intercept becomes $(0, b^c + d)$
4. The horizontal asymptote becomes $y = d$
5. The range becomes (d, ∞)
6. The domain $(-\infty, \infty)$, remains unchanged

HOW TO

Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation.

1. Draw the horizontal asymptote $y = d$.
2. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left c units if c is positive, and right c units if c is negative.
3. Shift the graph of $f(x) = b^x$ up d units if d is positive, and down d units if d is negative.
4. State the domain, $\left(-\infty, \infty\right)$, the range, $\left(d, \infty\right)$, and the horizontal asymptote $y = d$.

Example 7 – Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.
Solution:

Reflections of the Parent Function $f(x) = b^x$

The function $f(x) = -b^x$

1. reflects the parent function $f(x) = b^x$ over the $x-$axis
2. has a $y-$intercept of $(0, -1)$
3. has a range of $(-\infty, 0)$
4. has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$, which are unchanged from the parent function

The function $f(x) = b^{-x}$

1. reflects the parent function $f(x) = b^x$ over the $y-$axis
2. has a $y-$intercept of $(0, 1)$
3. has a range of $(0, \infty)$
4. has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$, which are unchanged from the parent function

Example 8 – Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = \left(\frac{1}{4}\right)^x$ over the $x-$axis. State its domain, range, and asymptote.

Solution: