5.6 – Rational Functions

Learning Objectives

1. Find the domain of a rational function
2. Find intercepts of a rational function
3. Find the asymptotes of a rational function
4. Graph a rational function using transformations

1 – Find the Domain of a Rational Function

Definition – Rational Function

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

The **domain** of $R(x)$ is the set of all real numbers, except those for which the denominator $q(x) = 0$.

In other words: The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

How to Find the Domain of a Rational Function

HOW TO

Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the *x*-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Example 1 – Finding the Domain of a Rational Function

Find the domain of $f(x) = \dfrac{x+3}{x^2-9}$. Write your answer using interval notation.

Solution:

Example 2 – Finding the Domain of a Rational Function

(a) The domain of $R(x) = \dfrac{2x^2 - 4}{x + 5}$ is the set of all real numbers x except -5; that is, the domain is $\{x | x \neq -5\}$.

(b) The domain of $R(x) = \dfrac{1}{x^2 - 4} = \dfrac{1}{(x + 2)(x - 2)}$ is the set of all real numbers x except -2 and 2; that is, the domain is $\{x | x \neq -2, x \neq 2\}$.

(c) The domain of $R(x) = \dfrac{x^3}{x^2 + 1}$ is the set of all real numbers.

(d) The domain of $R(x) = \dfrac{x^2 - 1}{x - 1}$ is the set of all real numbers x except 1; that is, the domain is $\{x | x \neq 1\}$.

A closer look at 2(d)

Although $R(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x-1)(x+1)}{(x-1)} = x + 1,$

It is important to observe that the functions

$$R(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad f(x) = x + 1$$

are not equal, since the domain of $R(x)$ is $(-\infty, 1) \cup (1, \infty)$ and the domain of $f(x)$ is the set of all real numbers.

If $R(x) = \dfrac{p(x)}{q(x)}$ is a rational function, and if $p(x)$ and $q(x)$ have no common factors, then the rational function $R(x)$ is said to be in **lowest terms**.

Note: The domain of a rational function must be found before writing the function in lowest terms.

Intercepts of Rational Functions

A rational function $R(x)$ will have a $y-$intercept at $R(0)$, if the function is defined at zero. A rational function will not have a $y-$intercept if the function is not defined at zero.

Likewise, a rational function $R(x)$ will have $x-$intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, $x-$intercepts can only occur when the numerator of the rational function is equal to zero.

Example 3 – Finding the Intercepts of a Rational Function

Find the intercepts of $R(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution:

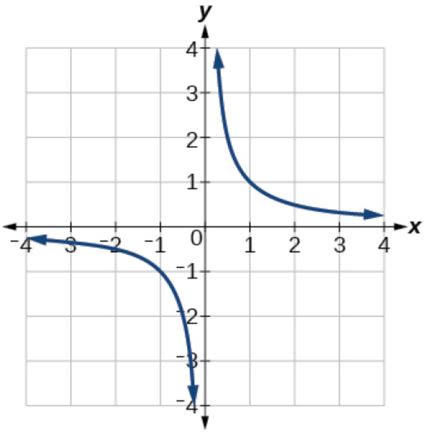
3 – Find the Asymptotes of a Rational Function

We use **arrow notation** to show that x or $f(x)$ is approaching a particular value.

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

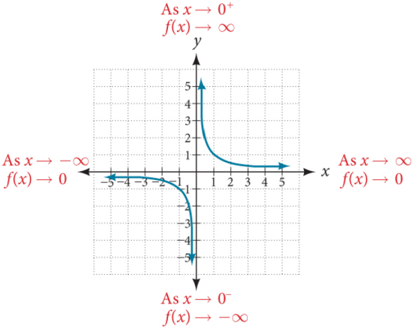
Example 4 – Analyzing the Graph of $f(x) = \frac{1}{x}$

Use arrow notation, the table from the previous and and the graph of $f(x)$ slide to discuss: $x \to 0^-$, $x \to 0^+$ and the End Behavior of $f(x)$.



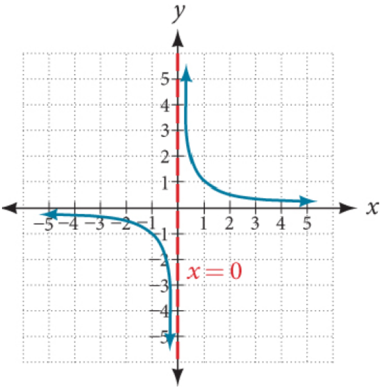
When we look at $x \to 0^-$, $x \to 0^+$ we are talking about the **local behavior** near some number. Here the number is 0.

In section 5.2/5.3 we discussed **End Behavior** of Polynomial Functions. We can discuss End Behavior for Rational Functions too!



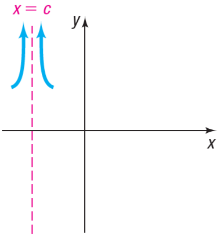
Vertical Asymptote

The local behavior of $f(x)$ at $x = 0$ creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this example, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero.

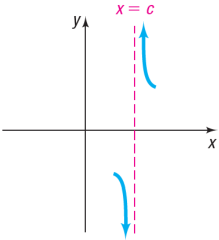


Definition – Vertical Asymptote

Let $R(x)$ be a function. If, as x approaches some number c, the values $R(x) \to \infty$ or $R(x) \to -\infty$, then the line $x = c$ is a **vertical asymptote** of the graph of $R(x)$.



As $x \longrightarrow c^-$, then $R(x) \longrightarrow \infty$; as $x \longrightarrow c^+$, then $R(x) \longrightarrow \infty$. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.



As $x \longrightarrow c^-$, then $R(x) \longrightarrow -\infty$; as $x \longrightarrow c^+$, then $R(x) \longrightarrow \infty$. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

Locating Vertical Asymptotes

The vertical asymptotes of a rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, are located at the real zeros of the denominator $q(x)$.

Theorem – Locating Vertical Asymptotes

The graph of $R(x) = \dfrac{p(x)}{q(x)}$, in *lowest terms*, has a vertical asymptote $x = c$ if c is a real zero of the denominator $q(x)$. That is, if $x - c$ is a factor of the denominator $q(x)$ of the rational function $R(x)$, in lowest terms, the graph of $R(x)$ has a vertical asymptote $x = c$.

Example 5 – Finding Vertical Asymptotes

Find the vertical asymptotes, if any, of the graph of each rational function.

(a) $F(x) = \dfrac{x + 3}{x - 1}$

(b) $R(x) = \dfrac{x}{x^2 - 4}$

(c) $H(x) = \dfrac{x^2}{x^2 + 1}$

(d) $G(x) = \dfrac{x^2 - 9}{x^2 + 4x - 21}$

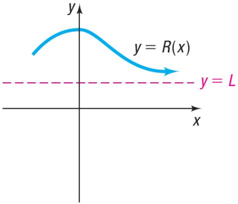
Solution:

Horizontal Asymptote

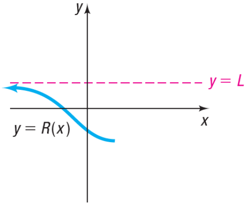
A **horizontal asymptote**, when it occurs,describes the **end behavior** of the graph as $x \to \infty$ or as $x \to -\infty$.

Definition – Horizontal Asymptote

Let $R(x)$ be a function. If, as $x \to \infty$ or as $x \to -\infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.



End behavior:
As $x \to \infty$, then $R(x) \to L$.
That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.



End behavior:
As $x \to -\infty$, then $R(x) \to L$.
That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

4 – Graph a Rational Function Using Transformations

In Math 195, we only study how to graph rational functions of the form $f(x) = \dfrac{ax+b}{cx+d}$. You will learn how to graph more complicated rational functions in Math 201.

Our process will be to use long division to express $f(x)$ as $Q(x) + \dfrac{R(x)}{D(x)}$. In that form we can view the graph of $f(x)$ as transformations of the graph of $\dfrac{1}{x}$. We simply identify the intercepts, the asypmtotes and then sketch the graph. Also we should make sure the state the domain and range of $f(x)$ using interval notation.

Example 6 – Using Transformations to Graph a Rational Function

Sketch the graph of $f(x) = \dfrac{3x+7}{x+2}$. Label all intercepts and asymptotes on your graph.

Solution:

Example 7 – Using Transformations to Graph a Rational Function

Sketch the graph of $f(x) = \frac{x-2}{x-3}$. Label all intercepts and asymptotes on your graph.

Solution: