5.5 – Zeros of Polynomial Functions

Learning Objectives

1. Use the Remainder Theorem.
2. Use the Factor Theorem.
3. Use the Rational Zero Theorem to find rational zeros.

1 – Using the Remainder Theorem

Theorem – The Remainder Theorem

Suppose $f(x)$ is a polynomial function. If $f(x)$ is divided by $x - k$, then the remainder is $f(k)$.

Example 1 – Using the Remainder Theorem

Find the remainder when $f(x) = x^3 - 4x^2 - 5$ is divided by

(a) $x - 3$ (b) $x + 2$

Solution:

2 – Using the Factor Theorem

An important and useful consequence of the Remainder Theorem is the *Factor Theorem*.

Theorem – The Factor Theorem

Suppose $f(x)$ is a polynomial function. Then $x - k$ is a factor of $f(x)$ if and only if $f(k) = 0$.

The Factor Theorem actually consists of two separate statements:

1. If $f(k) = 0$, then $x - k$ is a factor of $f(x)$.
2. If $x - k$ is a factor of $f(x)$, then $f(k) = 0$

One use of the Factor Theorem is to determine whether a polynomial has a particular factor.

Example 2 – Using the Factor Theorem

Use the Factor Theorem to determine whether $x - 1$ is a factor of the function $f(x) = 2x^3 - x^2 + 2x - 3$.

Solution:

Example 3 – Using the Factor Theorem to Find the Zeros of a Polynomial Expression

Show that $x + 2$ is a factor of $x^3 - 6x^2 - x + 30$. Find the remaining factors. Use the factors to determine the zeros of the polynomial.

Solution:

3 – Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. **The Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading coefficient of the polynomial.

RATIONAL ZERO THEOREM

If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients (where $a_n \neq 0$ and $a_0 \neq 0$), then every rational zero of P is of the form

$$\frac{p}{q}$$

where p and q are integers and

p is a factor of the constant coefficient a_0

q is a factor of the leading coefficient a_n

HOW TO

Given a polynomial function $f(x)$, use the Rational Zero Theorem to find rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of $f(\frac{p}{q})$.

FINDING THE RATIONAL ZEROS OF A POLYNOMIAL

1. **List Possible Zeros.** List all possible rational zeros, using the Rational Zeros Theorem.
2. **Divide.** Use synthetic division to evaluate the polynomial at each of the candidates for the rational zeros that you found in Step 1. When the remainder is 0, note the quotient you have obtained.
3. **Repeat.** Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

Example 4 – Listing All Possible Rational Zeros

List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.
Solution:

Example 5 – Using the Rational Zero Theorem to Find Rational Zeros

Find the rational zeros of $P(x) = x^3 - 3x + 2$.
Solution:

Example 6 – Using the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of $f(x) = 2x^3 + x^2 - 4x + 1$.

Solution:

Last Comment

In Math 195 – Precalculus, you must know how to use the four following theorems:

1. The Intermediate Value Theorem for Polynomials
2. The Remainder Theorem
3. The Factor Theorem
4. The Rational Zero Theorem

The final exam will test your knowledge on these theorems.