5.2 – Polynomial Functions and Their Graphs

Learning Objectives

1. Identify power functions and their end behavior
2. Identify polynomial functions.
3. Identify zeros and their multiplicities.
4. Graph polynomial functions
5. Use the Intermediate Value Theorem for Polynomial Functions

Power Function

A **power function** is a function that can be represented in the form $f(x) = kx^p$, where k and p are real numbers and k is known as a **coefficient**.

Example 1 – Identifying Power Functions

Which of the following functions are power functions?

$f(x) = 1$		Constant function
$f(x) = x$		Identify function
$f(x) = x^2$		Quadratic function
$f(x) = x^3$		Cubic function
$f(x) = \frac{1}{x}$		Reciprocal function
$f(x) = \frac{1}{x^2}$		Reciprocal squared function
$f(x) = \sqrt{x}$		Square root function
$f(x) = \sqrt[3]{x}$		Cube root function

Solution:

End Behavior of Power Functions

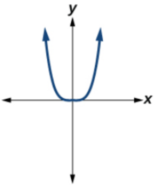
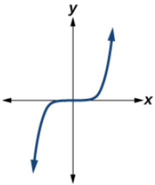
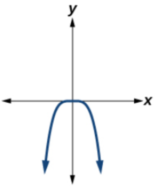
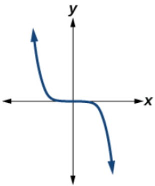
The behavior of the graph of a function as the input values get very small ($x \to -\infty$) and get very large ($x \to \infty$) is referred to as the **end behavior** of the function.

The table on the next slide shows the end behavior of power functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power and the constant.

HOW TO

Given a power function $f(x) = kx^n$ where n is a non-negative integer, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use the table on the next slide to identify the end behavior.

	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to \infty$	 $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to -\infty$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to -\infty$

Example 2 – Identifying the End Behavior of a Power Function

Find the end behavior of the graphs of $f(x) = x^8$ and g(x) $= -x^9$.
Solution:

Polynomial Functions

Let n be a non-negative integer. A **polynomial function of degree n** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

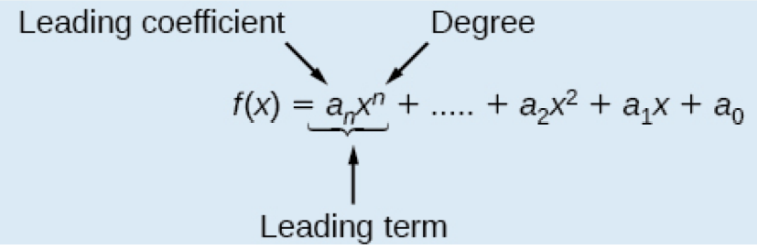
This is called the general form of a polynomial function. Each a_i is a coefficient and cna be any real number, but $a_n \neq 0$. Each expression $a_i x^i$ is a **term of a polynomial function**.

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form.

The **leading term** is the term containing the highest power of the variable, or the term with the highest degree.

The **leading coefficient** is the coefficient of the leading term.

Terminology of Polynomial Functions



Example 3 – Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$\begin{aligned} f(x) &= 3 + 2x^2 - 4x^3 \\ g(t) &= 5t^5 - 2t^3 + 7t \\ h(p) &= 6p - p^3 - 2 \end{aligned}$$

Solution:

End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph.

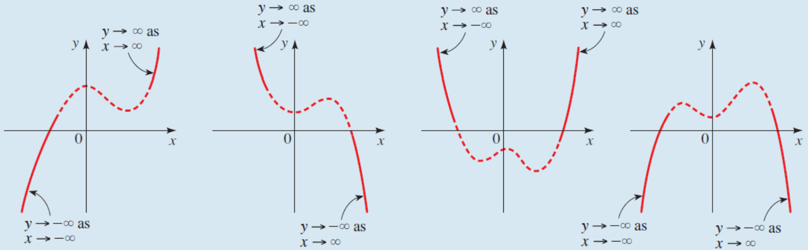
For any polynomial, the end behavior of the polynomial will match the end behavior of the power function consisting of the leading term.

End Behavior of Polynomial Functions

END BEHAVIOR OF POLYNOMIALS

The end behavior of the polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is determined by the degree n and the sign of the leading coefficient a_n, as indicated in the following graphs.

P has odd degree | **P has even degree**



Leading coefficient positive | Leading coefficient negative | Leading coefficient positive | Leading coefficient negative

Example 4 – Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution:

Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes.

Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing.

HOW TO

Given a polynomial function, determine the intercepts.

1. Determine the y-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the x-intercepts by solving for the input values that yield an output value of zero.

Example 5 – Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the *y*- and *x*-intercepts.

Solution:

Characteristics of a Graph of a Polynomial Function

The graph of a Polynomial Function is **continuous** and **smooth**.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper.

A **smooth curve** is a graph that has no sharp corners.

The turning points of the graph of a polynomial function must always occur at rounded curves.

A polynomial function of degree n will have, at most, n $x-$intercepts and $n-1$ turning points.

If P is a polynomial function, then c is called a **zero** of P if $P(c) = 0$.

Zeros of Polynomial Functions

If P is a polynomial and c is a real number, then the following are equivalent:

1. c is a zero of P.
2. $x = c$ is a solution of the equation $P(x) = 0$.
3. $x - c$ is a factor of $P(x)$.
4. c is an x-intercept of the graph of P.

To find the zeros of a polynomial function $P(x)$, we factor and then use the Zero-Product Property.

Example 6 – Finding the Zeros a Polynomial Function

Find the ($x-$intercepts) zeros of $h(x) = (x-2)^2(2x+3)$.
Solution:

Multiplicity

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercept h is determined by the power p. We say that $x = h$ has a zero of **multiplicity** p.

The graph of a polynomial function will touch the $x-$axis at zeros with even multiplicities.

The graph will cross the $x-$axis at zeros with odd multiplicities.

The sum of the multiplicities is the degree of the polynomial function.

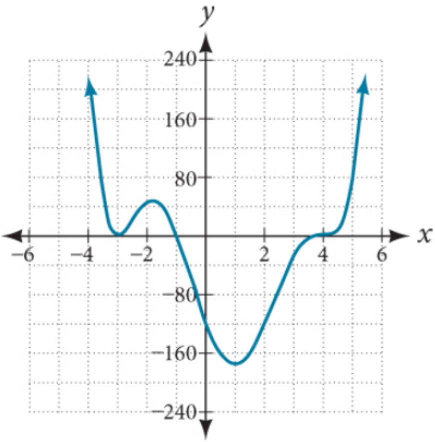
HOW TO

Given a graph of a polynomial function of degree n, identify the zeros and their multiplicities.

1. If the graph crosses the x-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the x-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is n.

Example 7 – Identifying Zeros and Their Multiplicities

Use the graph of the polynomial function of degree 6 shown below to identify the zeros of the function and their possible multiplicities.



Given a polynomial function, sketch the graph.

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y- axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x- intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.

GUIDELINES FOR GRAPHING POLYNOMIAL FUNCTIONS

1. **Zeros.** Factor the polynomial to find all its real zeros; these are the x-intercepts of the graph.
2. **Test Points.** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the x-axis on the intervals determined by the zeros. Include the y-intercept in the table.
3. **End Behavior.** Determine the end behavior of the polynomial.
4. **Graph.** Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

Example 8 – Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = (x+2)(x-1)(x-3)$.
Solution:

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.
Solution:

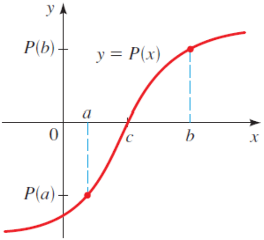
Example 10 – Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2x^4 - x^3 + 3x^2$.
Solution:

Theorem – Intermediate Value Theorem for Polynomial Functions

If $P(x)$ is a polynomial function and $P(a)$ and $P(b)$ have opposite signs, then there exists at least one value c between a and b for which $P(c) = 0$.

In other words, the Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the



Example 11 – Using the Intermediate Value Theorem

Show that the function $P(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Solution: