3.7 – Inverse Functions

Learning Objectives

1. Verify inverse functions.
2. Determine the domain and range of an inverse function.
3. Find or evaluate the inverse of a function.
4. Use the graph of a one-to-one function to graph its inverse function on the same axes.
5. Restrict the domain of a function to make it one-to-one.

Inverse Function

For any one-to-one function $y = f(x)$, a function $f^{-1}(x)$ is an **inverse function** if f if $f^{-1}(y) = x$.

In other words, to have an inverse function $f(x)$ must be one-to-one. Also if f has domain A and range B. f^{-1} has domain B and range A.

Think of f^{-1} as undoing what f has done.

Note: Keep in mind that $f^{-1}(x) \neq \dfrac{1}{f(x)}$.

Example 1 – Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$ what are the corresponding input and output values for the inverse function?

Solution:

Inverse Function Property

Inverse Function Property

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties:

1. $f^{-1}(f(x)) = x$ where x is in the domain of f
2. $f(f^{-1}(x)) = x$ where x is in the domain of f^{-1}

Any function f^{-1} satisfying these equations is the inverse of f

HOW TO

Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x)) = x$ or $g(f(x)) = x$.
2. If either statement is true, then both are true, and $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

Example 2 – Verifying That Two Functions Are Inverses

Show that $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{1}{x} - 2$ are inverses of each other.

Solution:

2 – Finding Domain and Range of Inverse Functions

Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

HOW TO

Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.

Example 3 – Finding the domain and range of Inverse Functions

Consider the functions $f(x) = \dfrac{2x+3}{x-1}$ and $f^{-1}(x) = \dfrac{x+3}{x-2}$. Find the domain and range of both functions.

Solution:

3 – Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Example 4 – Interpreting the Inverse of a Tabular Function

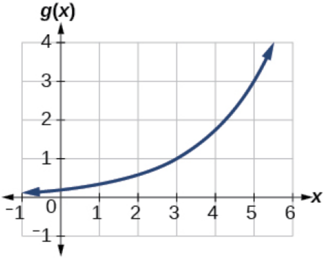
A function $f(t)$ is given in a table below. The table shows the distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Solution:

Example 5 – Evaluating a Function and Its Inverse from a Graph at Specific Points

The graph of a function $g(x)$ is given below. Find $g(3)$ and $g^{-1}(3)$.



Solution:

Finding the Inverse of a Function

Sometimes we will need to know an inverse function for all elements of its domain, not just a few.If the original function is given as a formula $y = f(x)$, then we can often find the inverse function by solving to get x as a function of y.

How to Find the Inverse of a One-to-One Function

1. Write $y = f(x)$
2. Swap x and y to get $x = f(y)$
3. Solve $x = f(y)$ for y
4. Write $y = f^{-1}(x)$

Find the inverse function of the function $f(x) = 3x - 2$.
Solution:

Example 7 – Finding the Inverse of a Function

Find the inverse function of the function $f(x) = \dfrac{x+1}{x+5}$.

Solution:

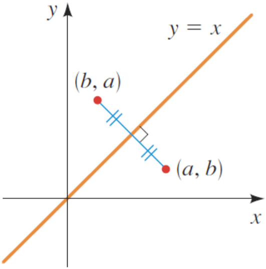
Example 8 – Inverting the Fahrenheit-to-Celsius Function

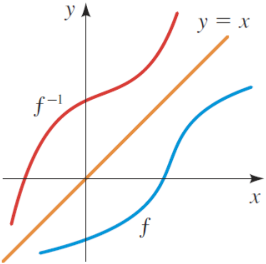
Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

Remember $C = \frac{5}{9}(F - 32)$

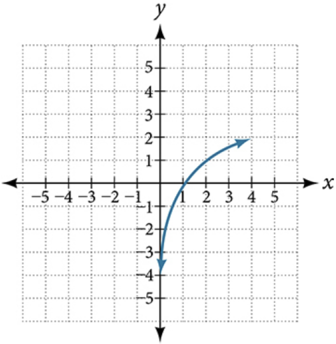
Solution:

The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.





Given the graph of $f(x)$ below, sketch a graph of $f^{-1}(x)$. State the domain and range for both functions.



5 – Restrict the Domain of a Function to make it One-to-One

If a function is not one-to-one, it has no inverse function. Sometimes, though, an appropriate restriction on the domain of such a function yields a new function that is one-to-one. Then the function defined on the restricted domain has an inverse function.

Example 10 – Finding the Inverse of a Domain-restricted Function

Find the inverse of $f(x) = x^2$ if $x \geq 0$. Graph f and f^{-1}.
Solution:

HOW TO

Given a polynomial function, restrict the domain of a function that is not one-to-one and then find the inverse.

1. Restrict the domain by determining a domain on which the original function is one-to-one.
2. Replace $f(x)$ with y.
3. Interchange x and y.
4. Solve for y, and rename the function or pair of function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

Example 11 – Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of f:

ⓐ $f(x) = (x-4)^2,\ x \geq 4$ ⓑ $f(x) = (x-4)^2,\ x \leq 4$

Solution:

Inverse of a Radical Function

Notice that the functions from previous examples were all polynomials, and their inverses were radical functions. If we want to find the inverse of a radical function, we will need to restrict the domain of the answer because the range of the original function is limited.

HOW TO

Given a radical function, find the inverse.

1. Determine the range of the original function.
2. Replace $f(x)$ with y, then solve for x.
3. If necessary, restrict the domain of the inverse function to the range of the original function.

Example 12 – Finding the Inverse of a Radical Function

Restrict the domain of the function $f(x) = \sqrt{x-4}$ and then find the inverse.

Solution: