Errata to
Derivatives and Integrals of Multivariable Functions

Introduction

This is a list of corrections to the text, *Derivatives and Integrals of Multivariable Functions* (Birkhauser, 2003). The corrections came about because Bruce Gould, of Chicago IL, having written a glowing review on Amazon.com's website, subjected the book to an astonishingly close reading. He thereby discovered a sequence of errors. Among them were numerous typos, more or less to be expected and not too harmful. Unfortunately, there were also several passages in which subtleties of reasoning were incompletely or incorrectly addressed (pages 43, 165, 227, 306) and one place where a flawed argument purported to prove a statement that is simply false (page 215).

I apologize to the reader for making these mistakes, failing to catch them, and neglecting to correct them until now. I hope that their appearance was not overly confusing to readers trying to learn the material. It is especially my hope that they did not work to dampen the interest of students in vector calculus; there are many places in the book where I intended my own enthusiasm for this body of knowledge to shine through.

In view of such humbling experience, I invite any reader who finds other mistakes, either in the book or in these corrections, to inform me at address aguzmath@gmail.com. I will include and acknowledge any such contributions in updated versions of this file.

Most of all, I owe a debt to Bruce Gould. He brought to his reading the most wonderfully exacting eye for mathematical rigor, a talent all the more impressive because his main interest is physics, not pure mathematics. I am grateful for the work he devoted to the study of mine.

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Corrections to the Text

p. 43, proof of Theorem 2

From the second paragraph, the proof looks at

$$g(t) := f(\mathbf{x}(t)),$$

in which

$$\mathbf{x}(t) := [1 - t]\mathbf{b} + t\mathbf{c}$$

is the point fraction t of the way from **b** to **c**. It says, "By the chain rule, g is a differentiable composite, ...". That is not a valid inference.

The call to the chain rule is unjustified under the stated hypotheses. The rule (pages 15-16) requires the components to be differentiable at the right places. The hypotheses do not come close to requiring f to be differentiable; they merely demand that certain directional derivatives exist. We want to allow f to be nondifferentiable at **b** and even nearby; compare Example 2 on p. 44 and the paragraph preceding it.

The conclusion remains true provided we make one boost to the hypotheses. We rewrite the theorem below with the stronger assumption and a valid proof.

Theorem 2. Assume that f is continuous in some neighborhood $N(\mathbf{b}, \varepsilon)$ [as opposed to merely at **b**]. Suppose that for each $\mathbf{x} \neq \mathbf{b}$ in $N(\mathbf{b}, \varepsilon)$, the derivative $\partial_{\mathbf{u}} f(\mathbf{x})$ in the direction of

$$\mathbf{u} = (\mathbf{x} - \mathbf{b})/\|\mathbf{x} - \mathbf{b}\|$$

exists. If every such **outward directional derivative** is nonnegative (respectively nonpositive, positive, negative), then $f(\mathbf{b})$ is a minimum (respectively maximum, strict minimum, strict maximum).

Proof. Fix $\mathbf{c} \neq \mathbf{b}$ in $N(\mathbf{b}, \varepsilon)$ and let

$$r := \|\mathbf{c} - \mathbf{b}\|, \qquad \mathbf{u} := (\mathbf{c} - \mathbf{b})/r.$$

Convince yourself that for every $\mathbf{x} \neq \mathbf{b}$ along the segment **bc**,

$$(\mathbf{x} - \mathbf{b})/\|\mathbf{x} - \mathbf{b}\| = \mathbf{u}.$$

Also, along the segment,

$$\mathbf{x}(t) := [1 - t]\mathbf{b} + t\mathbf{c} = \mathbf{b} + t(\mathbf{c} - \mathbf{b}) = \mathbf{b} + rt\,\mathbf{u}.$$

It is certainly true that

$$g(t) := f(\mathbf{x}(t)) \qquad 0 \le t \le 1$$

is a composite of *continuous* functions, so g is continuous. It has the further property that for $0 < t \le 1$, the **rightward derivative**

$$g'_+(t) := \lim_{h\to 0^+} \frac{g(t+h)-g(t)}{h}$$
$$= \lim_{h\to 0^+} \frac{f(\mathbf{b}+r[t+h]\mathbf{u})-f(\mathbf{b}+rt\mathbf{u})}{h}$$
$$= r\ \lim_{rh\to 0^+} \frac{f(\mathbf{b}+rt\mathbf{u}+rh\mathbf{u})-f(\mathbf{b}+rt\mathbf{u})}{rh}$$

matches, by the definition of directional derivative, $r\,\partial_{\mathbf{u}} f(\mathbf{b} + rt\mathbf{u})$. [Make sense of that: g squeezes the action of f along a segment of length r into the unit interval; it therefore changes r times as fast as f.]

Assume first that f has positive outward directionals. Pick $0 < s < 1$. On the interval $[s, 1]$, g is continuous, must therefore have a maximum value. That maximum cannot occur to the left of 1: If $s \le t < 1$, then

$$\lim_{h\to 0^+} \frac{g(t+h)-g(t)}{h} = r\,\partial_{\mathbf{u}} f(\mathbf{b} + rt\mathbf{u}) > 0;$$

accordingly, for small enough positive h,

$$g(t + h) - g(t) > 0,$$

and there are bigger values of g to the immediate right of t. Hence $g(1)$ is the maximum of g on the interval, and it is strict:

$$g(s) < g(1).$$

Since g is continuous at 0, it follows that

$$g(0) = \lim_{s\to 0^+} g(s) < g(1).$$

That says $f(\mathbf{b}) < f(\mathbf{c})$; $f(\mathbf{b})$ is a strict minimum. [Adapt the argument to prove the more general statement that a continuous function having positive rightward derivative on a closed interval must be strictly increasing.]

Assume instead that f has nonnegative outward directionals. Fix $\delta > 0$. Then

$$h(t) := g(t) + \delta t$$

is continuous and has rightward derivative

$$h'_+(t) = g'_+(t) + \delta > 0.$$

By the previous argument,

$$h(1) = g(1) + \delta \qquad \text{exceeds} \qquad h(0) = g(0).$$

In other words,

$$g(1) + \delta > g(0)$$

for every positive δ. It follows that $g(1) \ge g(0)$. In other symbols,

$$f(\mathbf{c}) \ge f(\mathbf{b});$$

$f(\mathbf{b})$ is a minimum.

Mirror-image arguments apply for the other cases. □

p. 164, line 11 from the bottom (sentence before Theorem 2)

The line refers to "extended definitions," but some of the extensions have not been made yet.

On p. 161, we note (just before Theorem 1) that "locally Archimedean set" is an extension of "Archimedean set." Nowhere, however, do we point out that the definitions of "locally integrable," "integrable," and "integral" on a locally Archimedean set are genuine extensions of integrability and integral on an Archimedean set. It would be proper to add this part to Exercise 6 on p. 169:

Exercise 5.4:6c) Suppose S is Archimedean and f is integrable on S in the sense of Chapter 4. Show that f is locally integrable on S, f is integrable on S in the sense of this section, and $\int_A f$ in the old sense matches the integral defined in this section. [Solution below, "*Corrections to the* Solutions to Exercises," under page 291.]

p. 165, line 5 from the bottom (within first paragraph in proof of ⇐)

The statement, "$\int_{A \cap T} f$ is a member of Itg(f, T)," is unjustified. The reason is that $A \cap T$ might not be a *closed* subset of T, even though A is assumed to be closed (and Archimedean).

Replace the paragraph and its follower by the text below.

⇐ Suppose conversely that f is integrable on T and $S - T$. Fix $\varepsilon > 0$, and take any $A \subseteq S$ [A understood to be closed and Archimedean]. We have

$$\int_A f = \int_{A \cap T} f + \int_{A \cap (S-T)} f \qquad \text{(set additivity),}$$

because the intersections are Archimedean (Exercise 6b) and do not overlap.

Because f is locally integrable on S, it is bounded on A, say $|f| \le M$. Recalling the definition of volume, let U be a union of nonoverlapping boxes interior to $A \cap T$ whose combined volume $V(U)$ exceeds $V(A \cap T) - \varepsilon/M$. We know

$$\int_{A \cap T} f = \int_U f + \int_{(A \cap T) - U} f.$$

On the right, the first integral belongs to Itg(f, T), because U is a closed Archimedean subset of T. The second integral has

$$\left| \int_{(A \cap T) - U} f \right| \le M\, V([A \cap T] - U) < \varepsilon.$$

As a result, the integral on $A \cap T$ satisfies

$$\inf \mathrm{Itg}(f, T) - \varepsilon < \int_{A \cap T} f < \sup \mathrm{Itg}(f, T) + \varepsilon.$$

That says the values of $\int_{A \cap T} f$ are bounded, independent of A. By similar reasoning, the values of $\int_{A \cap (S-T)} f$ are bounded. The same therefore goes for the values of $\int_A f$; f is integrable on S.

To prove additivity: Assume now that f is integrable on S, T, and therefore on $S - T$. From the last inequality above, we see that

$$\sup \{ \int_A f : A \subseteq S \} \le \sup \mathrm{Itg}(f, T) + \sup \mathrm{Itg}(f, S - T) + 2\varepsilon.$$

The left side is sup Itg(f, S), and we conclude that

$$\sup \mathrm{Itg}(f, S) \le \sup \mathrm{Itg}(f, T) + \sup \mathrm{Itg}(f, S - T).$$

On the other hand, if $B \subseteq T$ and $C \subseteq S - T$ are closed and Archimedean, then

$$\int_B f + \int_C f = \int_{B \cup C} f \le \sup \mathrm{Itg}(f, S).$$

Necessarily

$$\sup \{ \int_B f : B \subseteq T\} + \sup \{\int_C f : C \subseteq S - T\} \le \sup \mathrm{Itg}(f, S).$$

Therefore the suprema are additive. Similar reasoning establishes that the infima are likewise additive. Summing the corresponding pairs, we conclude that

$$\int_S f = \int_T f + \int_{S-T} f.$$

p. 215, Theorem 5(b)

The statement given as Theorem 5(b) is false.

The (⇐) half—if f is a function of just x_1, then $\int f \, dx_1$ is PI—is accurate. The proof given for that half is valid. The (⇒) half is false

The error in the proof of (⇒) is the statement that the union of local functions of x_1 is a global function of x_1. Bruce Gould suggested where to look for a counter-example to the statement: in a “C-shaped region.”

Example. Define a function everywhere in the plane but the nonnegative x-axis by

$$f(x, y) := \begin{cases} x^2 & \text{if } y > 0 \text{ or } x < 0, \\ -x^2 & \text{if } y < 0 \text{ and } x \ge 0. \end{cases}$$

The function is continuous on its open domain. On any convex open subset thereof—indeed, on any subset that excludes either the first quadrant or the fourth—f depends only on x. But no function $g(x)$ matches $f(x, y)$ globally, since

$$1 = f(1, 1) \ne f(1, -1) = -1.$$

The same $f(x, y)$ has PI dx_1-integrals. Hence it negates the (⇒) statement. Proving this PI claim is complicated with the tools currently at hand. It is wise to postpone proof until the exercise proposed below for p. 235.

Looking ahead to that next section, we might replace Theorem 5 as follows.

Theorem 5′. Let f be continuous on a connected open set O.
a′) $\int f \, ds$ is PI iff $f \equiv 0$.
b′) If $f(x_1, \ldots, x_n)$ is a function of just x_1, then $\int f \, dx_1$ is PI.

c′) If $\int f\,dx_1$ is PI, then f is "locally" a function of just x_1: For any convex open subset S of O, there exists a function $g_S(t)$ with the property that

$$f(x_1, \ldots, x_n) = g_S(x_1) \qquad \text{for all } (x_1, \ldots, x_n) \in S.$$

Proof. (a′) and (b′) are the statements given as part (a) and the (⇐) half of (b). The former was called "trivial" and remains so. The latter is proved in the text.

(c′) Let S be a convex subset of O.

Suppose there exist two places $\mathbf{u} := (u, u_2, u_3, \ldots, u_n)$ and $\mathbf{v} := (u, v_2, v_3, \ldots, v_n)$ in S where f takes different values, say w and $w + 3\varepsilon > w$, respectively. All along the segment L from $\mathbf{u}$ to $\mathbf{v}$, necessarily contained in S, x_1 is constant. Hence $dx_1 = 0$, and the integral $\int_L f\,dx_1$ is zero.

Let $\delta > 0$ be small enough to make three things true: 2δ is smaller than the (necessarily positive) distance from the segment to the complement of S; in the neighborhood $N(\mathbf{u}, 2\delta)$, f stays smaller than $w + \varepsilon$; and in the neighborhood $N(\mathbf{v}, 2\delta)$, f exceeds $w + 2\varepsilon$. Assume for definiteness that $u_2 \neq v_2$ (some u_k must differ from the corresponding v_k). Along the segment C from $\mathbf{u}$ to $(u + \delta, u_2, u_3, \ldots, u_n)$, f is less than $w + \varepsilon$, forcing

$$\int_C f\,dx_1 < (w + \varepsilon)\delta.$$

On the segment D from $(u + \delta, u_2, u_3, \ldots, u_n)$ to $(u + \delta, v_2, v_3, \ldots, v_n)$, $dx_1 = 0$, and

$$\int_D f\,dx_1 = 0.$$

Throughout the segment E from $(u + \delta, v_2, v_3, \ldots, v_n)$ to $(u, v_2, v_3, \ldots, v_n) = \mathbf{v}$, f exceeds $w + 2\varepsilon$ and dx_1 is negative. That makes

$$\int_E f\,dx_1 < (w + 2\varepsilon)(-\delta).$$

Accordingly, the broken line $F := C \cup D \cup E$ yields an integral

$$\int_F f\,dx_1 < (w + \varepsilon)\delta + 0 + (w + 2\varepsilon)(-\delta) < 0.$$

The integral from $\mathbf{u}$ to $\mathbf{v}$ is path-dependent.

We conclude that if f has PI dx_1-integrals, then for any points $\mathbf{U}$, $\mathbf{V}$ in S that have $U_1 = V_1$, we must have $f(\mathbf{U}) = f(\mathbf{V})$. Therefore

$$g_S(t) := f(t, x_2, \ldots, x_n)$$

defines, for all t in the projection of S onto the x_1-axis, a function that tracks the values of f. □

p. 223, line 10 (second line of (c))

The line invites division by zero. Change the first ≤ to <, to make the line read:
"the graph of $y = x^3 \sin(1/x)$, $0 < x \le 1/\pi$, ..."

p. 223, line 4 from the bottom

The ending "det[**T** **y** – **z**] > 0" should read:
"det[**T** **y** – **x**] > 0."

p. 225, definition of simply-connected set

The definition is not flawed, and it fits our purposes. However, it might have been wise to highlight that it applies just to bounded sets.

The restriction is important. Without it, you have to worry about the "point at infinity." Look in the plane at the points off the origin,

$$O := \{(x, y) \neq (0, 0)\}.$$

It is obviously an open set with a hole in the middle; we do not want to call it simply-connected. On the other hand, it fits the description "boundary is connected"; the boundary, under our rules, has only (0, 0). On the third hand, if you like to think of ∞ as a "point," then the boundary also has ∞; every neighborhood

$$N(\infty, r) := \{(x, y): x^2 + y^2 > r^2\}$$

has points of O. If we count ∞, then the boundary disconnected.

We can afford the restriction. Our interest is loops and other paths. Any one of those occupies only a bounded region within its range.

p. 227, last paragraph (second paragraph in the proof of ⇐)

This paragraph attempts to replace the given loop with a "step-loop" by working with Δx_j and Δy_j. The argument does not work at that level. It has to be carried out one level lower, at the loop's parametrization, working instead with Δt_j.

The key to the paragraph's argument is the (sort of average-value) relation

"$\int_D \mathbf{F} \bullet d\mathbf{s} = \mathbf{F}(\mathbf{x}_j^*) \bullet \Delta\mathbf{s}_j$ … for some $\mathbf{x}_j^*$ … along the [path]."

That might be false. Here is a counterexample:

Example. Look at $\int \mathbf{F} \bullet d\mathbf{s}$ for the field

$$\mathbf{F}(x, y) = (x + |x|)^2\mathbf{j}$$

on the semicircle given by

$$y = \sqrt{(1 - x^2)} \qquad \text{from } x = 1 \text{ to } x = -1.$$

In the first quadrant, $d\mathbf{s}$ has upward vertical component, which is the direction of **F**.

In the second quadrant, $\mathbf{F} = \mathbf{O}$. Hence $\int \mathbf{F} \bullet d\mathbf{s}$ is positive. But $\Delta\mathbf{s} = -2\mathbf{i}$ is orthogonal to every value $\mathbf{F}(\mathbf{x})$; $\mathbf{F}(\mathbf{x}) \bullet \Delta\mathbf{s} = 0$ for all $\mathbf{x}$.

To fix the paragraph, we need more than one [What were the odds?]. In view of the bulk the new paragraphs add to an already-long proof, we might choose to give their message as a separate lemma. However, it does not carry the general applicability of the lemma we wrote as Theorem 1 on page 218. We will accept the length of proof as the cost of an important result.

Reduction to the case of a step-loop. Suppose $\mathbf{g}: [a, b] \to L$ is a loop in O. In the first phase, we approximate the integral on L by one on a "step-loop" of horizontal and vertical pieces.

Start with a partition of $[a, b]$ containing the singularities of $\mathbf{g}(t) = (g_1(t), g_2(t))$. By assumption, on each subinterval, $\mathbf{g}'$ is (extendable to) a continuous function. (Because we will look at $\mathbf{g}'$one subinterval at a time, we will simply write $\mathbf{g}'$ for the continuous extension.) Therefore $\mathbf{g}'$ is bounded on each subinterval. The same is true on $[a, b]$, say $\|\mathbf{g}'\| \le M$. The loop, a continuous image of a compact set, is itself compact. It must lie at positive distance d from the boundary of O. Hence the set T of points whose distance from L is $d/2$ or less is a compact subset of O. Both P and Q must be bounded on T; we may assume that $|P| \le M$, $|Q| \le M$.

Now let $\varepsilon > 0$. On T, P and Q are uniformly continuous, so there exists $\delta < d/2$ such that

$$|P(\mathbf{u}) - P(\mathbf{v})| \text{ and } |Q(\mathbf{u}) - Q(\mathbf{v})| \quad \text{are less than} \quad \varepsilon/[4M(b - a)]$$

whenever $\mathbf{u}$ is within δ of $\mathbf{v}$ in T. Let $\{ a = t_0 < \ldots < t_J = b \}$ be any refinement (of the stated partition) with these properties:

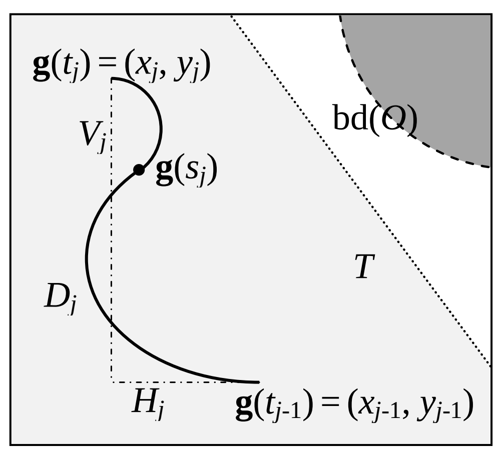
a) Each width $t_j - t_{j-1}$ is less than $\delta/(2M + 1)$.

b) On any subinterval,

$$\|\mathbf{g}'(t) - \mathbf{g}'(s)\| \le \varepsilon/[4M(b - a)]$$

for any s and t.

(Part (b) is a matter of uniform continuity of $\mathbf{g}'$. Observe that making $\mathbf{g}'$ vary over a narrow range amounts to looking at pieces of the path that are relatively straight.)



Look at the piece $D_j = \mathbf{g}([t_{j-1}, t_j])$ (heavy curve at right) of L. On that piece,

$$\int_{D_j} \mathbf{F} \bullet d\mathbf{s} \;=\; \int_{t_{j-1}}^{t_j} \left[P(\mathbf{g}(t)) g_1'(t) + Q(\mathbf{g}(t)) g_2'(t) \right] dt$$

(Theorem 6.1:1)

$$= P(\mathbf{g}(s_j))\, g_1'(s_j)\, (t_j - t_{j-1}) + Q(\mathbf{g}(s_j))\, g_2'(s_j)\, (t_j - t_{j-1})$$

(average value theorem)

for some s_j between t_{j-1} and t_j.

The horizontal span between the endpoints of D_j is the (directed) segment H_j (dashed horizontal segment in the figure) from

$\mathbf{g}(t_{j-1}) = (x_{j-1}, y_{j-1})$ to (x_j, y_{j-1}) (at the left end).

All of H_j lies within T: If (x, y_{j-1}) is on H_j, then

$$\|(x, y_{j-1}) - (x_{j-1}, y_{j-1})\| \;=\; |x - x_{j-1}|$$

$$= |g_1(t^*) - g_1(t_{j-1})|$$

(intermediate value theorem)

$$\le (\max g_1')\, |t^* - t_{j-1}|$$

$$< M\,\delta/(2M+1) \;<\; d/4.$$

Therefore P and Q are defined there. Along H_j,

$$\int_{H_j} \mathbf{F} \bullet d\mathbf{s} \;=\; \int_{x_{j-1}}^{x_j} P((x, y_{j-1}))\, dx$$

$$= P((x_j^*, y_{j-1}))(x_j - x_{j-1})$$ (average value theorem)

for some $x_j^* = g_1(r_j)$ (some r_j in the subinterval),

$$= P(\,(g_1(r_j), g_2(t_{j-1}))\,)\, g_1'(w_j)\, (t_j - t_{j-1})$$

(mean value theorem)

for some w_j in the subinterval. For the quantities highlighted in red and blue,

$$\left| \int_{D_j} P\, g_1' dt - \int_{H_j} P\, dx \right|$$

$$= \left| P(\mathbf{g}(s_j))\, g_1'(s_j) - P(\,(g_1(r_j), g_2(t_{j-1}))\,)\, g_1'(w_j) \right| (t_j - t_{j-1})$$

$$\le \left| P(\mathbf{g}(s_j)) - P(\,(g_1(r_j), g_2(t_{j-1}))\,) \right| \left| g_1'(s_j) \right| (t_j - t_{j-1}) +$$

$$\left| P(\,(g_1(r_j), g_2(t_{j-1}))\,) \right| \left| g_1'(s_j) - g_1'(w_j) \right| (t_j - t_{j-1}).$$

Of the two points where P is evaluated in the next-to-last line, the red one is on D_j, and the blue one is separated from the red by

$$\| \mathbf{g}(s_j)) - (g_1(r_j), g_2(t_{j-1})) \| \;\le\; |g_1(s_j) - g_1(r_j)| + |g_2(s_j) - g_2(t_{j-1})|$$

$$\le M\,|s_j - r_j| + M\,|s_j - t_{j-1}|$$

$$\le 2M\,\delta/(2M+1)$$

$$< \delta.$$

That means both points are in T. By the uniform continuity,

$$\left|P(\mathbf{g}(s_j)) - P(\,(g_1(r_j),\, g_2(t_{j-1}))\,)\right| \;<\; \varepsilon/[4M(b-a)].$$

By assumption (b),

$$\left|g_1'(s_j) - g_1'(w_j)\right| \;<\; \varepsilon/[4M(b-a)].$$

Consequently

$$\left|\int_{D_j} P\, g_1' dt - \int_{H_j} P\, dx\right|$$

$$< \varepsilon/[4M(b-a)]\, M\,(t_j - t_{j-1}) + M\, \varepsilon/[4M(b-a)]\,(t_j - t_{j-1})$$

$$= \varepsilon(t_j - t_{j-1})/[2(b-a)].$$

A similar estimate applies to the vertical segment V_j from (x_j, y_{j-1}) to (x_j, y_j):

$$\left|\int_{D_j} Q\, g_2' dt - \int_{V_j} Q\, dy\right| \;<\; \varepsilon(t_j - t_{j-1})/[2(b-a)].$$

For the piece in question,

$$\left|\int_{D_j} \mathbf{F} \bullet d\mathbf{s} \;-\; \int_{H_j + V_j} \mathbf{F} \bullet d\mathbf{s}\right| \;<\; \varepsilon(t_j - t_{j-1})/(b-a).$$

Daisy-chaining the H-V "steps" into the single "step-loop"

$$L^* := H_1 + V_1 + \ldots + H_J + V_J\,,$$

we get

$$\left|\int_L \mathbf{F} \bullet d\mathbf{s} - \int_{L^*} \mathbf{F} \bullet d\mathbf{s}\right| \;<\; \sum_j \varepsilon(t_j - t_{j-1})/(b-a) \;=\; \varepsilon.$$

In words, we can approximate $\int_L \mathbf{F} \bullet d\mathbf{s}$ by the like integral over a nearby loop consisting of horizontal and vertical pieces and having the same endpoints. Hence, we can reach our goal by proving that the work around every step-loop is zero.

p. 230, line 1

"$\mathbf{g}(C)$" should instead be "$\mathbf{G}(C)$."

p. 235, proposed new exercise

In view of the modification "Theorem 5′" proposed for p. 215, the exercise below is an appropriate addition at this place.

Exercise 6.3:7. Suppose $f(x_1, \ldots, x_n)$ is C^1 on the open set O.

a) Show that on any convex open subset S of O, f is a function of just x_1 iff

$$\frac{\partial f}{\partial x_2} = \ldots = \frac{\partial f}{\partial x_n} = 0 \qquad \text{throughout } S.$$

b) Show that convexity is essential in (a) and in what was proposed for p. 215 as Theorem 5′(c′): Give an example of a continuously differentiable function $f(x, y)$ for which $\int f\, dx$ is PI on an open set O, but there is no single function $g(x)$ that matches $f(x, y)$ globally on O.

[See below, "*Corrections to the* Solutions to Exercises," under page 308.]

p. 239, line 5 from the bottom (end of definition of *simple*)

You need to include the restriction on x:

$g_i(x) \le y \le G_i(x)$ and $a \le x \le b$ for the last two variables …

p. 243, Example 3

There is no reason to start off the example in $\mathbf{R}^n$. Clearly the example is talking about air (14.7 lbs/in^2) and water (62.5 lbs/ft^3) with $\mathbf{R}^3$ as the intended setting.

Corrections to the **Solutions to Exercises**

p. 291, solution to Exercise 5.4:6

The solution given in the text is for part (a) only. Below are solutions to part (b) and to the part (c) proposed here under page 164.

b) Again, A is required to be bounded. By (a), it is LA. By Exercise 5, $S \cap A$ is also LA. Since the intersection is bounded and LA, part (a) says it is Archimedean.

c) Assume that f is integrable on the Archimedean set S. (Let us agree to use "integrable," "integral," and the $\int$ symbol in the old sense; and 'g-integrable," "g-integral," and g-$\int$ for the generalized definitions in this section.) In that case, $|f|$ is also integrable on S (Theorem 5.2:1(d)).

We already know from part (a) that S is locally Archimedean. Suppose A is a closed Archimedean subset of S. Then f and $|f|$ are integrable on A (Theorem 4.4:5(a)). The former makes f locally integrable on the LA set S. Further,

$$\begin{aligned} \left|\int_A f\right| &\le \int_A |f| && \text{(triangle inequality)} \\ &\le \int_S |f| && \text{(set monotonicity).} \end{aligned}$$

Hence

$$l := \inf \{ \int_B f \} \qquad \text{and} \qquad u := \sup \{ \int_B f \}$$

($B \subseteq S$ closed and Archimedean) are both finite; that is, f is g-integrable on S.

Let $\varepsilon > 0$. There must exist a candidate B such that

$$\int_B f > u - \varepsilon.$$

Then

$$\begin{aligned} \int_{S-B} f &= \int_S f - \int_B f && \text{(set additivity)} \\ &< \int_S f - u + \varepsilon. \end{aligned}$$

Hence the infimum l satisfies

$$l < \int_S f - u + \varepsilon, \qquad \text{and} \qquad u + l < \int_S f + \varepsilon.$$

Consequently the g-integral g-$\int_S f := u + l$ has

$$\text{g-}\int_S f \le \int_S f.$$

At the same time, there must exist C such that

$$\int_C f < l + \varepsilon.$$

Then

$$\begin{aligned}\int_{S-C} f &= \int_S f - \int_C f \\ &> \int_S f - l - \varepsilon.\end{aligned}$$

Hence the supremum u satisfies

$$u > \int_S f - l - \varepsilon, \qquad \text{and} \qquad u + l > \int_S f - \varepsilon.$$

It follows that

$$\text{g-}\int_S f \ge \int_S f.$$

The new definition matches the old integral.

p. 306, solution to Exercises 6.2:4 and 5

The last line makes the argument circular. The previous line has evaluated the integral along a specific path, and path-independence has not been proven. Replace the last line by a new paragraph:

In view of that result, set

$$g(\mathbf{x}) := \int_{\|\boldsymbol{a}\|}^{\|\boldsymbol{x}\|} f(r) r \, dr.$$

Then

$$\frac{\partial g}{\partial x_j} = f(\|\mathbf{x}\|)\|\mathbf{x}\| \frac{\partial \|\mathbf{x}\|}{\partial x_j} = f(\|\mathbf{x}\|)\|\mathbf{x}\| \, x_j / \|\mathbf{x}\|.$$

That last expression is $f(\|\mathbf{x}\|)\, x_j$, the j-component of $f(\|\mathbf{x}\|)\mathbf{x} = \mathbf{F}$. Therefore $\mathbf{F} = \nabla g$, settling Exercise 4. The **a**-to-**b** integral we did establishes Exercise 5.

p. 308, solution to the proposed new Exercise 6.3:7

The exercise appears here under page 235.

7a) Suppose f is a function of just x_1 in a convex open $S \subseteq O$. For $\mathbf{a} = (a, b, \ldots)$ in S, we have $f(\mathbf{x}) = g(x_1)$ throughout a neighborhood of **a**. Then at **a**,

$$\begin{aligned}\partial f / \partial x_2 &:= \lim_{h \to 0} [\, (f(a, b+h, \ldots) - f(a, b, \ldots)) / h \,] \\ &= \lim_{h \to 0} [\, (g(a) - g(a)) / h \,] = 0;\end{aligned}$$

similarly with the other variables. For this half, the convexity of S is irrelevant.

Conversely, suppose that throughout the convex subset S,

$$\partial f / \partial x_2 = \ldots = \partial f / \partial x_n = 0.$$

If $\mathbf{a} = (a, a_2, \ldots, a_n)$ and $\mathbf{b} = (a, b_2, \ldots, b_n)$ are in S, then so is their line segment, and we may apply the mean value theorem. Accordingly,

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{x}^*) \bullet (\mathbf{b} - \mathbf{a}) \qquad \text{(some } \mathbf{x}^* \text{ on } \mathbf{ab})$$
$$= (?, 0, \ldots, 0) \bullet (0, b_2 - a_2, \ldots, b_n - a_n) = 0.$$

The value $f(\mathbf{x})$ depends only on x_1.

b) The example given under page 215,

$$f(x, y) := \begin{cases} x^2 & \text{if } y > 0 \text{ or } x < 0, \\ -x^2 & \text{if } y < 0 \text{ and } x \geq 0, \end{cases}$$

has the needed properties. It is continuously differentiable. (Check the only questionable place: $\partial f/\partial x$ is continuous at the negative y-axis.) It fits the requirement of part (a), $\partial f/\partial y = 0$. We saw that it is not a function of just x. It remains only to show that $\int f\, dx$ is PI.

Observe that the field

$$\mathbf{F}(x, y) := f(x, y)\mathbf{i}$$

is continuously differentiable, with

$$\partial Q/\partial x = \partial P/\partial y.$$

In fact, it is the gradient of

$$h(x, y) := \begin{cases} x^3/3 & \text{if } y > 0 \text{ or } x < 0, \\ -x^3/3 & \text{if } y < 0 \text{ and } x \geq 0 \end{cases}$$

on the complement of the non-negative x-axis, a set with no holes. (The set fits the standard definition of "simply-connected"—connected boundary—without the boundedness requirement.) Therefore **F** is PI, and for any path C,

$$\int_C \mathbf{F} \bullet d\mathbf{s} = \int_C f\, dx.$$

Corrections and Additions to the Index

counterclockwise	223, not 224
Euler's equation	239, not 240
field	add subheadings: *irrotational field*, *rotational field*, 221
flux	add subheading: *outward flux*, 235
incompressible fluid	239, not 240
integral, Riemann integral	add subheading: *coordinate integral*, 181
Jordan Curve Theorem	229, not 230
parametrization	61, 170

simple region	239, not 240
simply connected set	(better written “simply-connected set”) 225, not 226
singularity	The definition at p. 62 is needed for surfaces, but the description on p. 61 is easier to understand.
smooth curve, smooth surface	*See* curve *or* surface
work	202