4 Qualitative Methods

Dr. Frink: Here is an ordinary square…
Chief Wiggum: Woah! Woah! Slow down egghead!
From: *The Simpsons*

4.1 Introduction

In this chapter we discuss techniques for characterizing the behavior of the solution of an ODE without finding an explicit solution. Even when an explicit solution can be found the qualitative methods we will study often enable us to more readily answer certain questions about the solution, and usually with more insight as well. For instance, in many biological applications where the independent variable is time we might be interested in knowing:

i) how particular solutions behave as t increases?

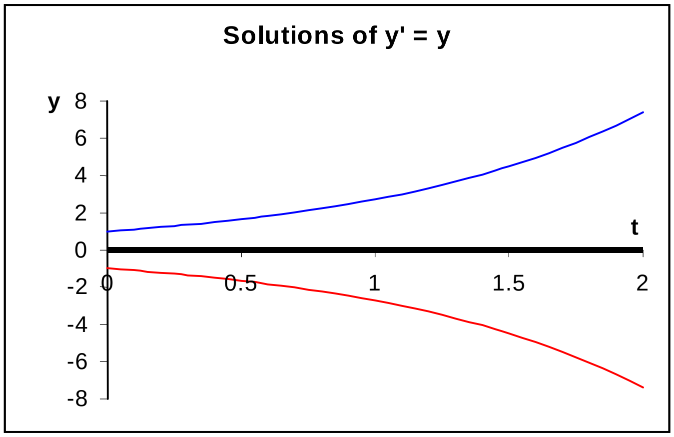
ii) how the behavior of solutions with respect to increasing t varies as the initial conditions are changed?

To illustrate these questions, consider a simple example in which the ODE has a straightforward explicit solution.

Example 4.1: Answer questions i) and ii) for the solutions of the equation $\frac{dy}{dt} = y$.

Solution:

We know from our work in Chapter 2 that the general solution of this equation is $y = Ce^t$. If the constant $C = y(0)$ is positive then the solution y increases without bound as $t \to \infty$. If, however, $C < 0$ the solution will become negative without bound as t increases, and finally, if $C = 0$ the solution remains zero for all time. Examples of the three sorts of solutions are pictured below.



■

The general case that we will consider in this chapter exhibits similarities to Example 4.1. In particular, the constant solutions of an ODE, which we have for the most part ignored, will play an important role in the analysis. Two theoretical ideas underlie the qualitative methods. The first of these is a familiar property of the derivative, while the second is a fundamental principle regarding the solutions of differential equations.

Theorem 4.1 (Monotonicity Theorem): If $f'(x_0) > 0$ (resp. $f'(x_0) < 0$) then $f(x)$ is monotone increasing (resp. decreasing) near x_0. ■

The reader is probably familiar with this result, as it is one of the primary tools for showing that a function is monotone. For example, a slightly more general statement forms the basis for the first derivative test for a relative maximum or minimum.

The other result we need is the so-called uniqueness theorem for differential equations.

Theorem 4.2 (Uniqueness Theorem): If $y_1(x)$ and $y_2(x)$ are both solutions of the same differential equation and $y_1(0) \neq y_2(0)$ then these solutions can never be equal at any value x_0. ■

Strictly speaking the result as stated is false (see exercise 34 for a counterexample) and some technical restrictions on the differential equation are necessary to ensure that the result holds. However, for the types of examples we need to consider the result is true. Basically, the result says that the differential equation and an initial condition determine a unique solution. Having played with Euler's method and its variants, the reader might certainly be willing to believe this, as these methods automatically generate an approximate solution once they are given an initial value problem. However, the Uniqueness Theorem (Theorem 4.2) states something else. Namely, that if two solutions start out with different initial values, the graphs of the solutions can never meet. This is surprisingly difficult to prove.

4.2 Steady State Solutions

For the remainder of this chapter we will take the independent variable to be time, t. In studying qualitative methods we need to restrict the class of differential equations.

Definition 4.1: A differential equation is called a*utonomous* if the right side depends only on the dependent variable y. In other words, the equation has the form $\frac{dy}{dt} = g(y)$. ■

For example, $\frac{dy}{dt} = 2y(1-y)$ is autonomous, whereas $\frac{dy}{dt} = yt$ is not. In an autonomous equation, the rate of change of the solution depends only on its current value and not the time at which that value is attained. Autonomous equations are typically encountered in many biological applications. In analyzing autonomous equations the steady state or equilibrium solutions play a pivotal role. These are defined by

Definition 4.2: A constant solution of the differential equation $\frac{dy}{dt} = g(y)$ is called a *steady state* or *equilibrium* solution.■

The terminology used in Definition 4.2 comes from dynamics. Imagine placing a marble on an interior wall of a deep bowl and then releasing it. If the marble is placed at the very bottom it remains there in a position of equilibrium or steady state. If the marble is released at a point above the bottom it will roll down the sides oscillating back and forth. Eventually, due to friction, its path $y(t)$ will approach the equilibrium position and the marble will come to rest at the bottom. We will see that the solutions of an autonomous ODE often behave in a similar, though simpler, manner.

Example 4.2: Find the steady state solutions of the differential equation $\frac{dy}{dt} = y(2-y)$.

Solution:

By definition a steady state solution is a constant solution. If $y \equiv c$ is a constant satisfying the differential equation then for the left side of the ODE we must have $\frac{dy}{dt} = 0$, since the derivative of a constant is zero. Substituting $y = c$ on the right side gives, $0 = c(2-c)$ and hence $c = 0$ or $c = 2$.■

Example 4.2 illustrates the key idea in finding the steady state solutions.

Theorem 4.3: The steady state solutions of the differential equation $\frac{dy}{dt} = g(y)$ are the values of y for which $g(y) = 0$.■

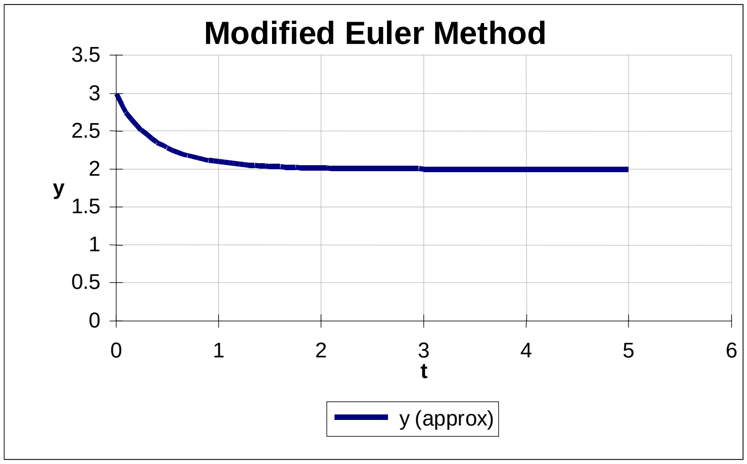
Knowing the steady state solutions allows us to answer question i) raised on page 49.

Example 4.3: What happens to the solution of the initial value problem $\frac{dy}{dt} = y(2-y)$, $y(0) = 3$ as t increases?

Solution:

We can use numerical methods to solve the equation and thereby formulate a reasonable conjecture as to the behavior. Having a plausible answer we can then try to explain it. Although mathematics is a deductive science in its reasoning, the process of discovery proceeds inductively, as in experimental sciences. Here we use a computer to carry out the necessary experiments.

The program *ode.xls* gives the following graph for the solution of the initial value problem.



The solution $y(t)$ appears to approach the horizontal line $y=2$ at t increases. Recall from Example 4.2 that $y=2$ is a steady state solution of this ODE. Thus, the behavior is reminiscent of the ball rolling down into the bowl, except the solution doesn't oscillate around its eventual equilibrium value. To understand why the picture is correct let's examine some of the numerical output from the spreadsheet. The left side of the table below gives the first few lines of output and the final output appears in the last three columns.

t	y_{approx}	y'_{approx}	t	y_{approx}	y'_{approx}
0	3.000	-3.000	4.85	2.000042	-0.000084
0.05	2.865	-2.477	4.9	2.000038	-0.000076
0.1	2.752	-2.070	4.95	2.000034	-0.000068
0.15	2.658	-1.748	5	2.000031	-0.000062

Recall that the column labeled y'_{approx} is obtained by evaluating the right side of the differential equation for the current values of t and y (of course only the y value matters since the equation is autonomous). For example, putting the initial condition $y=3$ into the right side $y(2-y)$ of the ODE gives the value $y'=-3$, as indicated in the first line. By the Monotonicity Theorem (Theorem 4.1) the y value must initially decrease and indeed for $t=0.05$ the value of y is smaller than it was at $t=0$. The derivative is still negative, though not as negative as initially, so the y value must continue to decrease. Looking at the right side of the table, we see that for t near five, y is still slightly larger than 2 and therefore the derivative $y'=y(2-y)$ remains negative, though barely so. The negative derivative slowly pushes the y value lower, though never enough to push its value below $y=2$. Numerically, the differential equation seems to behave like a delicately balanced feed-back loop: the current y value substituted in the right side gives the current rate of change of y, which in turn tells us how to change y over a short time period.

What's mysterious about the numerical behavior is why the solution never drops below $y = 2$. The Uniqueness Theorem (Theorem 4.2) provides the explanation. Since the solution $y(t)$ satisfies the initial condition $y(0) = 3$, the Uniqueness Theorem (Theorem 4.2) guarantees that it can never coincide with the constant solution $y \equiv 2$, which of course has the different initial condition $y(0) = 2$.

In summary the behavior of the solution $y(t)$ can be deduced from two facts.

a) Based on the initial condition the solution has a negative derivative and thus decreases, by the Monotonicity Theorem (Theorem 4.1).

b) Since a steady state solution lies below the initial value of y, the Uniqueness Theorem (Theorem 4.2) implies that $y(t)$ cannot dip below this steady state.

From a) and b) it is not difficult to show (though we omit the details) that $y(t)$ actually approaches the steady state value $y = 2$. ■

4.3 The Geometric Method

Example 4.3 shows that we need two pieces of information to analyze the behavior of the solution to an initial value problem for an autonomous ODE of the form $\frac{dy}{dt} = g(y)$. First, we need the sign of the derivative for the initial value y_0. This is of course obtained by substituting y_0 into the right side of the differential equation. In effect, the sign of $g(y_0)$ tells us whether the solution initially increases or decreases. We then need to know the location of the steady state solutions and how these solutions are disposed in relation to the given initial value. All of this information can be ascertained by graphing the function $g(y)$ that appears on the right side of the differential equation, thinking of y as an independent variable. We illustrate the method in the next example.

Example 4.4: Analyze the behavior of the solution of the initial value problem $\frac{dy}{dt} = y(2 - y)$, $y(0) = 1.5$.

Solution:

The graph of $g(y) = y(2 - y)$, with y as an independent variable, contains all the information we need. The graph is a parabola opening downward with horizontal intercepts at $y = 0$ and $y = 2$. Observe that these y values are precisely the equilibrium or steady state values of the ODE.

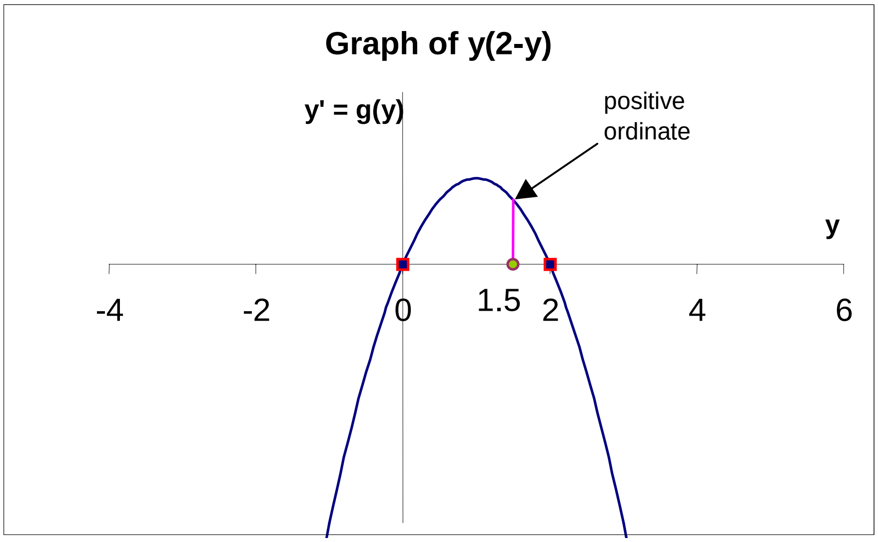


Figure 4.1

To analyze the behavior of the solution that has initial value $y(0) = 1.5$ we locate the latter value along the horizontal y axis. The ordinate, which is the value of $y' = g(y)$, is positive and therefore the solution $y(t)$ increases. However, the steady state value $y = 2$ lies above the initial value, so the solution must increase approaching this steady state, but never meeting or exceeding it. Hence, we obtain the following sketch for the solution $y(t)$. Note that the horizontal axis here represents time t and y is now a dependent variable plotted along the vertical axis. It is important to keep in mind the different role played by y in the two figures.

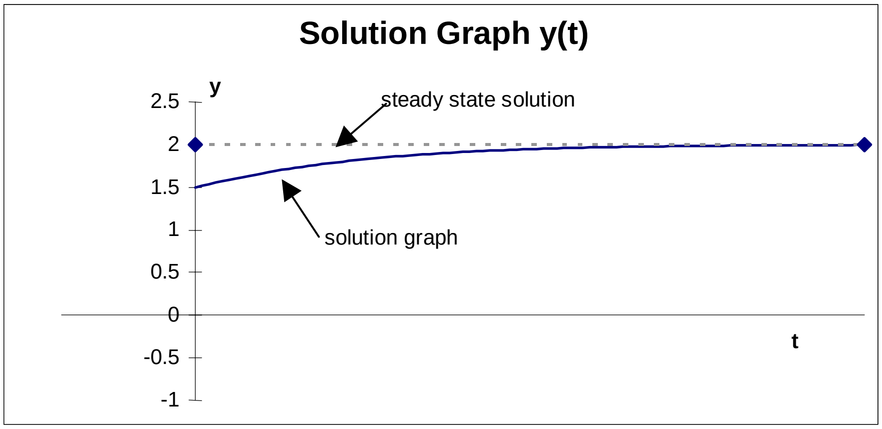


Figure 4.2

Although the above argument determines the shape of the solution graph, qualitative arguments give us no information at all regarding the time t at which the solution takes on a particular value. Hence, the t axis in Figure 4.2 contains no indication of scale. Only numerical methods or an explicit solution can answer specific questions related to the time domain.■

Although the graph in Figure 4.2 is correct, our methods are not refined enough to correctly answer questions about the concavity of the solution graph. Essentially, all we have discussed is whether or not a solution is increasing or decreasing and if so under what conditions it tends towards a limit. For example, if we take the equation $\frac{dy}{dt} = y(2-y)$ with the initial condition $y(0) = 0.5$ then, using the same Figure 4.1 above, we deduce that the solution increases towards the steady state value $y = 2$. However, the graph of $y(t)$ looks slightly different than the one sketched in Figure 4.2. Namely, the modified Euler method gives

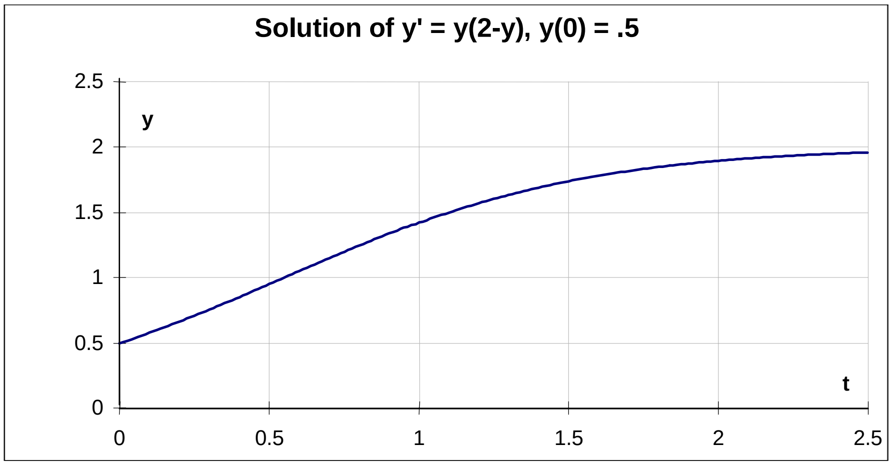


Figure 4.3

The graph begins concave up and then turns concave down as it approaches the asymptote $y = 2$. The arguments we have given so far do not allow us to distinguish this shape from the solution in Example 4.4, which was strictly concave down. We will now remedy this.

Theorem 4.4: If $y(t)$ is a solution of the autonomous equation $\frac{dy}{dt} = g(y)$ then at an ordinate y_0 on the graph of $y(t)$ vs. *t* the graph is concave up (resp. down) if $g'(y_0)g(y_0) > 0$ (resp. < 0).

Proof:

We know from calculus that the sign of the second derivative $y''(t)$ determines the concavity of the graph of $y(t)$. Using the differential equation we can find an expression for this second derivative. Namely,

$$y''(t) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(g(y)).$$

We can evaluate the last derivative using the chain rule, viewing $g(y)$ as a composite function in which $t \to y(t) = y \to g(y)$. Applying the chain rule to this composition (see Chapter 1) gives

$$\frac{d}{dt}(g(y)) = \frac{d}{dy}(g(y))\frac{d}{dt}(y).$$

The term $\frac{d}{dy}(g(y)) = g'(y)$, while by the differential equation $\frac{dy}{dt} = g(y)$. Putting everything together we obtain the formula $y''(t) = g'(y)g(y)$. Hence, if $y_0 = y(t_0)$ is an ordinate for which $g'(y_0)g(y_0) > 0$ then $y''(t_0) > 0$ and the graph will be concave up. A similar argument applies when the expression $g'(y_0)g(y_0)$ is negative.■

Theorem 4.4 might seem rather cumbersome to apply. Actually it is very easy. All the information needed to determine the sign of the product $g'(y)g(y)$ can be read from the graph of $g(y)$. Indeed, the sign of $g(y)$ is found from the vertical ordinate on this graph and the sign of $g'(y)$ is determined by whether the graph of $g(y)$ has a positive or negative slope. We illustrate using the initial value problem whose solution graph was computed in Figure 4.3.

Example 4.5: Sketch the graph of the solution to the initial value problem $\frac{dy}{dt} = y(2-y)$, $y(0) = 0.5$.

Solution:

We have re-drawn the graph in Figure 4.1, with emphasis on the particulars for this problem.

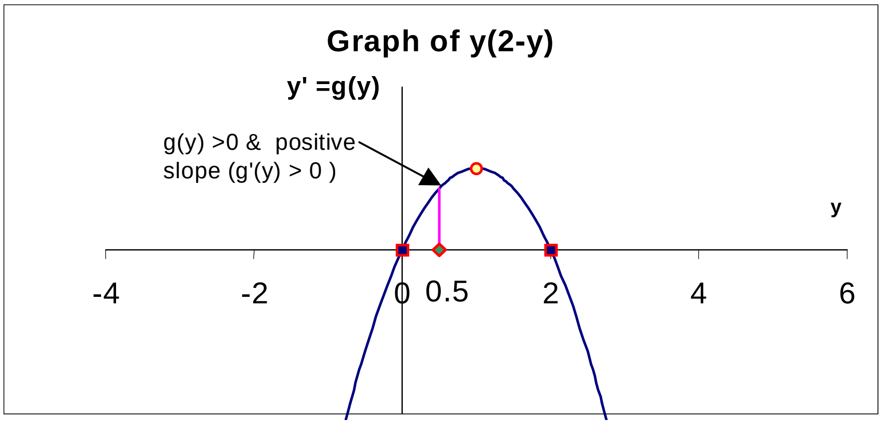
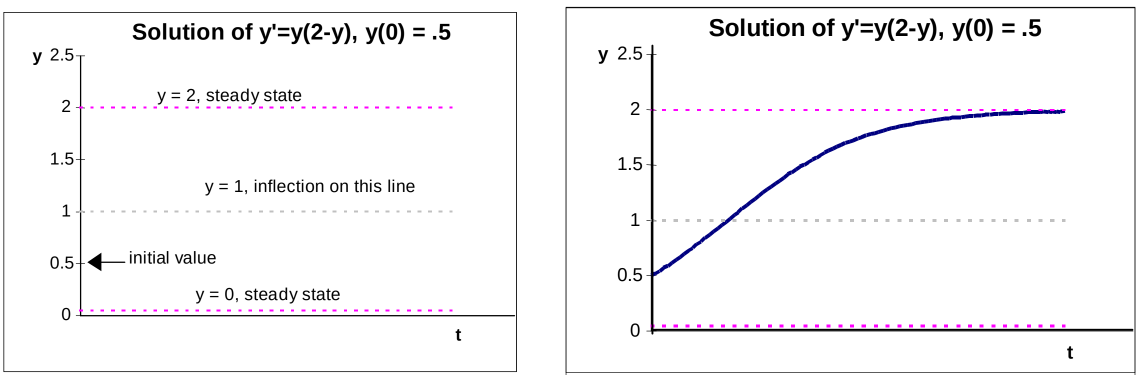


Figure 4.4

Initially, $y' = g(y) > 0$ so the solution increases. The Uniqueness Theorem (Theorem 4.2) prevents the solution from passing through the steady state solution $y = 2$ so $y(t)$ approaches that value asymptotically.

The concavity of the graph of $y(t)$ can be found using Theorem 4.4. Initially, as indicated in Figure 4.4, both $g(y)$ and $g'(y)$ are positive and therefore so is their product. Hence the graph of $y(t)$ starts out concave up. When the y value exceeds one, the slope $g'(y)$ of the graph of $g(y)$ becomes negative and the product $g'(y)g(y)$ is also negative. Consequently, the graph of $y(t)$ turns concave down. As there is a change in concavity of the graph of $y(t)$, there must be a point of inflection on this graph. This point of inflection must occur where y'' vanishes. Since $y'' = g'(y)g(y)$, at a point of inflection either $g'(y) = 0$ or $g(y) = 0$. The latter cannot happen, since y is not a steady state solution. Thus, a point of inflection occurs when $g'(y) = 0$. From the geometry of the parabola (or by direct computation of $g'(y) = 2 - 2y$) the solution of $g'(y) = 0$ is $y = 1$. At that y value the graph of $y(t)$ switches from concave up to concave down.

We can generate the sketch of the graph of the solution $y(t)$ in two stages.



a) b)

Figure 4.5

First in Figure 4.5a) we sketch the steady state solutions, the horizontal line(s) along which any inflection points may occur and mark the initial condition along the vertical y axis. Using the graphical analysis based on Figure 4.4 we then add the sketch of the solution graph, as in Figure 4.5b). Note again that no scale can be placed on the t axis using qualitative methods. In particular, it is not possible to say at what value of t the graph reaches its inflection point.■

It is worthwhile extracting from the above discussion the criterion for an inflection point.

Theorem 4.5: If $y(t)$ is a non-equilibrium solution of the differential equation $y' = g(y)$ then the graph of $y(t)$ has an inflection point at an ordinate y_0, provided $g'(y_0) = 0$ and the sign of g' is different above and below y_0. In effect, the graph of $g(y)$ has a relative maximum or minimum at y_0.

Proof:

From the proof of Theorem 4.4 we know that $y'' = g'(y)g(y)$. At an inflection point we must have $y'' = 0$. Since by assumption $y(t)$ is not a steady state solution, $g(y)$ is never zero. Hence at an inflection point y_0 we must have $g'(y_0) = 0$. Furthermore, y'' must change sign at an inflection point, so therefore by the same reasoning so must $g'(y)$. From the first derivative test of calculus, a relative maximum or minimum occurs at any value y_0 where $g'(y_0) = 0$ and g' changes sign.■

4.4 Stability

Successful biological systems are stable, to at least some extent. Your heart rate speeds up when you exercise, but it returns to its normal rhythm soon after your exercise ceases. A tree branch laden with ice and snow bends, but usually recovers its original shape when the ice melts and the snow is blown away. A persistent state that is able to recover from modest disturbances is called stable. When we use differential equations to describe biological or physical systems the persistent states correspond to steady state or equilibrium solutions. Such a state is called stable if solutions that begin near it return to the original state over time. More formally,

Definition 4.3: A steady state solution $y \equiv c$ of $\dfrac{dy}{dt} = g(y)$ is called *stable* if any solution of the ODE whose initial value is close to c tends towards c as $t \to \infty$.■

A steady state solution that is not stable is called *unstable*, although there are some situations where a notion of semi-stable is appropriate (see exercise 25.) A steady state that is unstable will never actually persist in nature for very long, since even random fluctuations will move the system away from the unstable equilibrium.

The graphical method of Section 4.3 can be used to classify the equilibria according to their stability. In effect, the analysis of stability provides an answer to question (ii) that we raised at the beginning of this chapter. We illustrate the method with our by now familiar example.

Example 4.6: Classify each of the steady state solutions of the ODE $y' = y(2 - y)$ as to their stability.

Solution:

We use the graph of the right side of the ODE, $g(y) = y(2 - y)$. To analyze the stability of the two steady state solutions, we must consider solutions of the ODE whose initial values are both larger and smaller than the values $y = 0$ and $y = 2$. These points are marked on the horizontal axis in Figure 4.6 below. The specific value of these selected points is not important.

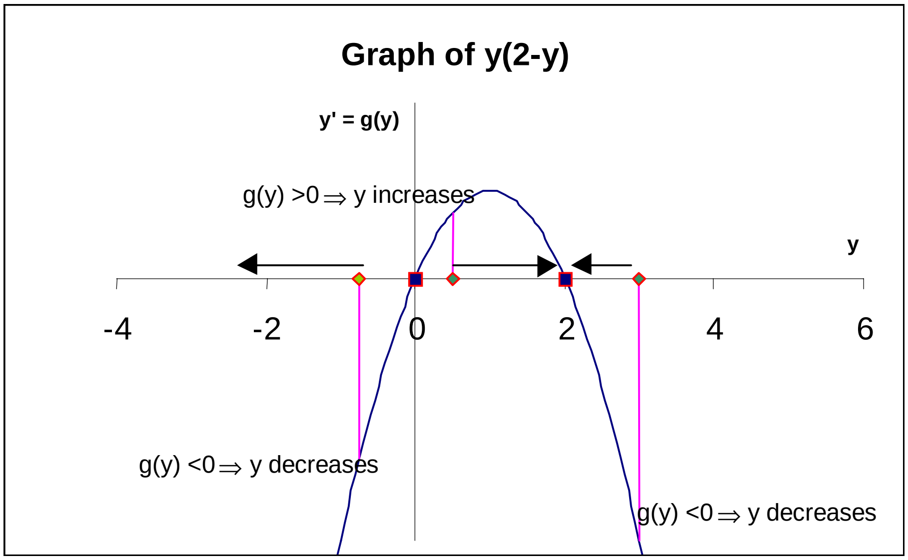
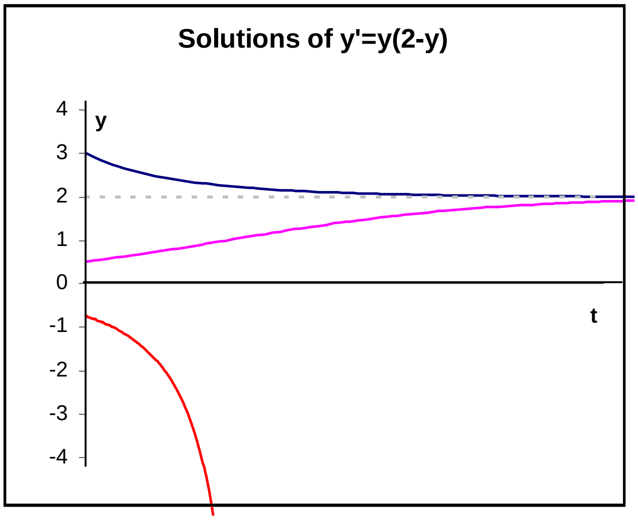


Figure 4.6

For each of the three initial values the arrows in Figure 4.6 show the direction in which the corresponding solution evolves as t increases. We call this figure a *stability diagram*. Based on these arrows we can assert that the equilibrium value $y = 2$ is stable, since solutions that start nearby are drawn towards it. The steady state $y = 0$ is unstable, since solutions that start with initial values nearby are drawn away from that state. If desired, we can sketch the solutions for the three selected initial values, although this is not necessary for the purposes of the stability analysis.



Note that the solution that begins with a negative initial value does not approach any limiting value as t increases. In fact, in this case the solution is not actually defined for all $t > 0$, but reaches a vertical asymptote rather quickly. However, this fact cannot be ascertained from qualitative analysis alone. (See exercise 33 for a related discussion.)■

Given a steady state solution, the graphical method can be used to derive an analytical procedure that classifies the solution according to its stability. The reader should consult exercise 32 for the details.

4.5 Summary

To analyze the behavior of solutions of an autonomous differential equation $\frac{dy}{dt} = g(y)$:

a) First find the steady state or equilibrium solutions by solving the equation $g(y) = 0$.

b) Sketch the graph of $y' = g(y)$ vs. y. The steady state solutions are the horizontal intercepts on this graph.

c) If a specific initial value problem is given, locate the specified initial value y_0 on the horizontal axis of the graph you drew in b). Based on the sign of $g(y_0)$ determine if the corresponding solution $y(t)$ is increasing or decreasing. Determine if a limiting value exists based on the location of the nearest equilibrium values.

d) Perform a concavity analysis using the sign of the product $g'(y)g(y)$.

e) Using the results of c) and d) sketch a graph of the solution $y(t)$ vs. t. For reference, include the graphs of the steady state solutions.

f) To conduct a stability analysis, return to the graph in b) and select initial values around each equilibrium point. Based on the sign of $g(y_0)$ determine the direction in which each solution will evolve and indicate on the graph using arrows (stability diagram). Use the stability diagram to classify each steady state solution according to its stability properties.

4.6 Exercises

In each of exercises 1 - 10 find the steady state solutions and graph each solution on a t, y coordinate system for $t \geq 0$.

1. $\frac{dy}{dt} = 3 - y$

2. $\frac{dy}{dt} = 1 + y$

3. $\frac{dy}{dt} = (y+1)(3-y)$

4. $\frac{dy}{dt} = y(y-4)$

5. $\frac{dy}{dt} = y^2(y-4)$

6. $\frac{dy}{dt} = y(4-y^2)$

7. $\frac{dy}{dt} = y(3-y)+10$

8. $\frac{dy}{dt} = \frac{2-y}{1+y^2}$

9. $\dfrac{dy}{dt} = \dfrac{2+y^2}{1+y^2}$

10. $\dfrac{dy}{dt} = 2e^{-y} - e^{2y}$

In each problem 11 - 20 sketch the solution curve(s) of the stated initial value problem(s) $\dfrac{dy}{dt} = g(y)$, $y(0) = y_0$. Your answer should include

i) The graph of $y' = g(y)$ vs. y, if not given .

ii) The solution curve(s) for the initial value problem(s) drawn on a t, y coordinate system with a clear indication of any asymptotic behavior, concavity and points of inflection.

Give an explanation as to how the graph i) is used to derive the solution graph(s) ii).

11. $\dfrac{dy}{dt} = 3 - y$, $y(0) = 2$, $y(0) = 4$. (See exercise 1.)

12. $\dfrac{dy}{dt} = 1 + y$, $y(0) = 0$, $y(0) = -2$. (See exercise 2.)

13. $\dfrac{dy}{dt} = (y+1)(3-y)$, $y(0) = 0$, $y(0) = 4$. (See exercise 3.)

14. $\dfrac{dy}{dt} = y(y-4)$, $y(0) = -1$, $y(0) = 5$. (See exercise 4.)

15. $\dfrac{dy}{dt} = y^2(y-4)$, $y(0) = -1$, $y(0) = 1$. (See exercise 5.)

16. $\dfrac{dy}{dt} = y(4-y^2)$, $y(0) = 1$, $y(0) = 3$. (See exercise 6.)

17. $\dfrac{dy}{dt} = y(3-y) + 10$, $y(0) = 0$, $y(0) = 2$. (See exercise 7.)

18. $\dfrac{dy}{dt} = \dfrac{2-y}{1+y^2}$, $y(0) = -2$. (See exercise 8 and Figure 4.7 below.)

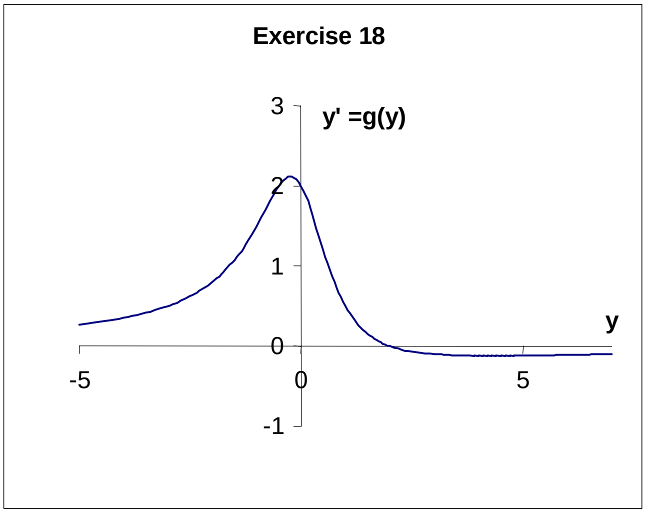


Figure 4.7

19. $\dfrac{dy}{dt} = \dfrac{2+y^2}{1+y^2}$, $y(0) = -1$. (See exercise 9 and Figure 4.8 below.)

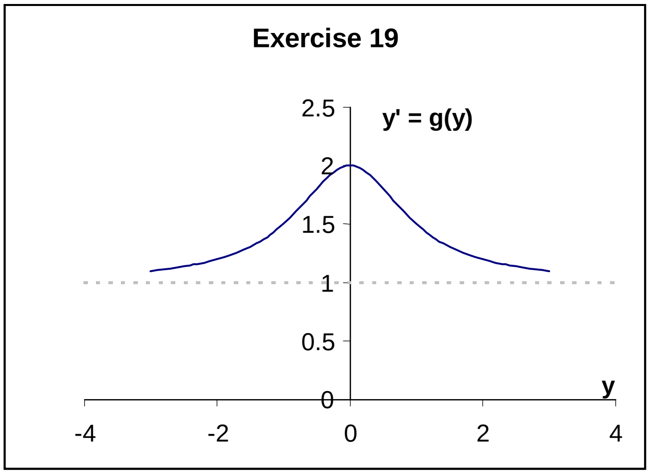


Figure 4.8

20. $\dfrac{dy}{dt} = 2e^{-y} - e^{2y}$, $y(0) = 0$. (See exercise 10 and Figure 4.9 below.)

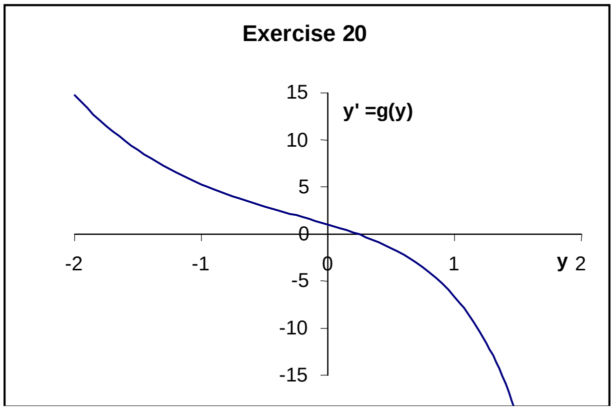


Figure 4.9

In exercises 21 - 29 determine the stability of each steady state solution for the given differential equation $\frac{dy}{dt} = g(y)$. Include a stability diagram (see Figure 4.6).

21. $\frac{dy}{dt} = 3 - y$, See exercises 1 & 11

22. $\frac{dy}{dt} = 1 + y$, See exercises 2 & 12

23. $\frac{dy}{dt} = (y+1)(3-y)$, See exercises 3 &13

24. $\frac{dy}{dt} = y(y-4)$, See exercises 4 & 14

25. $\frac{dy}{dt} = y^2(y-4)$, See exercises 5 &15

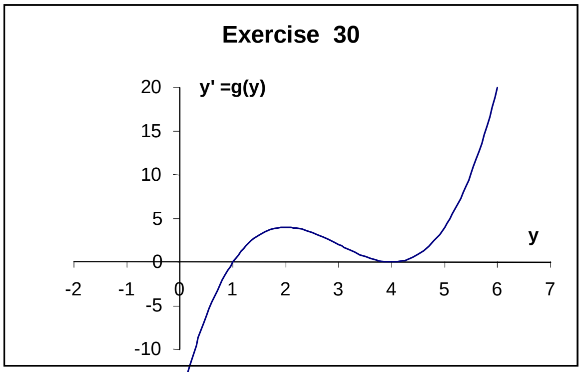
26. $\frac{dy}{dt} = y(4-y^2)$, See exercises 6 & 16

27. $\frac{dy}{dt} = y(3-y)+10$, See exercises 7 & 17

28. $\frac{dy}{dt} = \frac{2-y}{1+y^2}$, See exercises 8 & 18

29. $\frac{dy}{dt} = 2e^{-y} - e^{2y}$, See exercises 10 & 20

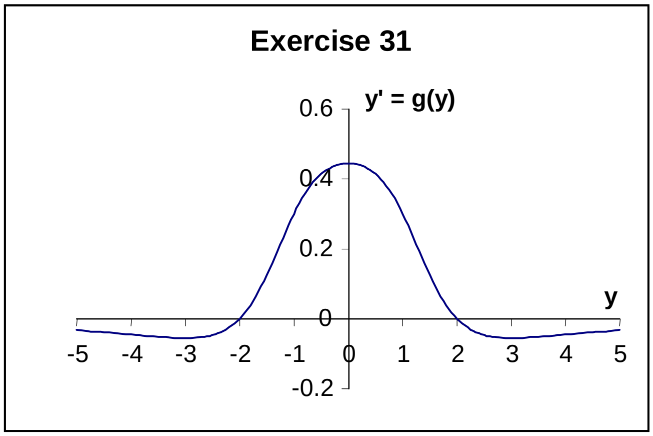
30. Consider the differential equation $\frac{dy}{dt} = g(y)$, where $g(y)$ has the graph shown below.



Sketch on the same axes solution curves $y(t)$ for each of the following initial values. Your graph should show the behavior for large t as well as inflection points, if any.

a) $y(0) = 1.5$ b) $y(0) = 3$ c) $y(0) = 5$

31. Consider the differential equation $\frac{dy}{dt} = g(y)$, where $g(y)$ has the graph shown below.



Sketch on the same axes solution curves $y(t)$ for each of the following initial values. Your graph should show the behavior for large t as well as inflection points, if any.

a) $y(0) = -1.5$ b) $y(0) = 1.5$ c) $y(0) = 4$

32. Using a graphical argument explain why the following tests for stability are true.

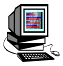
a) If y_0 is a steady state solution of the ODE $\frac{dy}{dt} = g(y)$ and $g'(y_0) < 0$ then y_0 is stable.

b) If y_0 is a steady state solution of the ODE $\frac{dy}{dt} = g(y)$ and $g'(y_0) > 0$ then y_0 is unstable.

Note: If $g'(y_0) = 0$ the test is inconclusive, as the reader can deduce from the examples $g(y) = y^2$, $g(y) = y^3$ and $g(y) = -y^3$, each with steady state solution $y_0 = 0$.

33. a) Use qualitative methods to sketch the solution of the initial value problem $\frac{dy}{dt} = y^2$, $y(0) = 1$.

b) Use separation of variables to solve the initial value problem in a) and observe that the solution has a vertical asymptote at $t = 1$. This phenomenon is called *blow-up*, because the solution becomes infinite in a finite time. The blow-up cannot be predicted by qualitative methods.



c) Use the workbook *ode.xls* to solve the initial value problem in a) using the modified Euler method. Print a graph that is suggestive of the blow-up phenomenon.

34. Show that each of the following functions is a solution of the differential equation $\frac{dy}{dt} = -y^{1/3}$.

Include a sketch of each solution for $t \geq 0$.

a) $y \equiv 0$ (steady state solution)

b) $y = \begin{cases} (1-\frac{2}{3}t)^{3/2} & \text{if } t \leq 3/2 \\ 0 & \text{if } t > 3/2 \end{cases}$

(In b)), why can't the first formula be used for $t > 3/2$? Why does the function in b) have a derivative at $t = 3/2$?)

c) What initial value problem (at $t = 0$) is solved by each of these functions? In this case although both solutions begin with different initial values, they become equal for $t \geq 3/2$. This is an example of a differential equation for which the Uniqueness Theorem (Theorem 4.2) is false.