2 A Differential Equations Primer

Homer: Keep your head down, follow through. [Bart putts and misses] Okay, that didn't work. This time, move your head and don't follow through.

From: *The Simpsons*

2.1 Introduction

An ordinary differential equation (ODE) is an equation involving the derivative of an unknown function of a single variable. In working with such equations our objective is to understand something about the function(s) that satisfy the equation. The most obvious goal might be to find explicit formulas for the solutions. We have already tackled that problem for the simplest ODEs in Chapter 1, although from a slightly different perspective.

Example 2.1: Find all functions y that satisfy the differential equation $\dfrac{dy}{dx} = x^2 + 1$.

Solution:

In effect, the differential equation provides a formula for the derivative of the unknown function. We have to find all functions with the given derivative. This is just the antidifferentiation problem we addressed in Chapter 1. Integrating the right side of the ODE we obtain that $y = \dfrac{x^3}{3} + x + C$.■

Example 2.1 was particularly simple because the ODE expressed the derivative of the unknown function in terms of the independent variable. In general, the differential equations we will encounter will not be of such a simple form. A more typical and important example is

Example 2.2: Find all solutions of the differential equation $\dfrac{dy}{dx} = y$. (2.1)

Solution:

In words, this equation says that differentiating the unknown function returns the original function. From our differentiation review in Chapter 1 we know that $y = e^x$ has this property since $(e^x)' = e^x$. So $y = e^x$ is certainly a solution of this differential equation. However, in Example 2.1 we found infinitely many solutions of the ODE as the constant C in the solution formula varies. We would like to find a similar family of solutions to (2.1). Trying the obvious, we consider $y = e^x + C$ and ask whether or not this satisfies (2.1). ***To test whether an expression y satisfies a differential equation we find $\dfrac{dy}{dx}$ and see if it is equal to the right side of the differential equation, when the latter is expressed in terms of the independent variable.***

Here we find that $\frac{dy}{dx} = \frac{d}{dx}(e^x + C) = e^x$ and since the right side is just $y = e^x + C$, the differential equation is not satisfied except when $C = 0$. However, if we instead consider $y = Ce^x$ then we find that $\frac{dy}{dx} = \frac{d}{dx}(Ce^x) = Ce^x = y$, so that these functions are solutions of the differential equation.■

These two examples illustrate two characteristics of ODEs. First, the equation will express $\frac{dy}{dx}$ in terms of either the independent or dependent variable or both. Second, the equation will usually have an infinite number of solutions expressible by varying some arbitrary constant. This family of solutions is called the ***general solution*** of the differential equation. To single out a unique solution we must specify some additional information, as in the next example.

Example 2.3: Find the solution of $\frac{dy}{dx} = y$ that satisfies $y(0) = 10$.

Solution:

We use the general solution $y = Ce^x$ found in Example 2.2 (a systematic method for finding these solutions will be discussed in the next section). Since when $x = 0$ we are supposed to have $y = 10$, we can substitute this information into the general solution to determine C:

$$10 = y(0) = Ce^0 = C.$$

Thus, the specific solution we want is $y = 10e^x$.■

The data consisting of a differential equation together with a numerical value of the unknown function is called an ***initial value problem***. Quite often, as in Example 2.3, the function value will be the value of y at $x = 0$, but a value at some other x value may be given instead. This piece of information is referred to as the ***initial condition***. For most problems there is a unique solution of the differential equation that satisfies the given initial condition.

2.2 Separation of Variables

At this point the only type of ODE we can solve is one of the form $y' = f(x)$, provided we can compute the antiderivative of $f(x)$. In this section we develop a technique that can be used to solve many differential equations. The method applies to what are called ***separable equations*** - those for which the right side can be factored into a product or quotient of expressions, each of which involves only the independent or dependent variable.

Example 2.4: Determine which of the following differential equations is separable.

a) $\dfrac{dy}{dx} = xy + x$

b) $\dfrac{dy}{dx} = \dfrac{y+1}{x+1}$

c) $\dfrac{dy}{dx} = y(y+1)$

d) $\dfrac{dy}{dx} = x + y$

Solution:

a) The right side may be factored as $x(y+1)$, which meets the condition for separability.

b) The right side is the quotient of a function of y divided by a function of x. Therefore, this equation is separable.

c) The right side can be thought of as a product $y(y+1)\times 1$, where the constant factor 1 can be viewed as a function of x. Therefore, the equation is separable.

d) The right side cannot be factored as a product of a function of x and a function of y. Therefore, this equation is not separable.■

The idea behind the method of separation of variables is to multiply or divide both sides of the separable differential equation by suitable factors so that all the y terms wind up on the left side, together with the unknown derivative, and all the x terms wind up on the right side. Following that step, we then have two big hurdles to overcome. First, we must be able to integrate both sides of the transformed equation. Second, even after integrating, we will not usually have an explicit formula for the solution but rather an implicit one. Some tricky algebra may be needed to obtain an explicit solution. Often, an explicit solution cannot be found. We illustrate with some examples, beginning with a reprise of Example 2.2.

Example 2.5: Use the method of separation of variables to find the general solution of $\dfrac{dy}{dx} = y$.

Solution:

Although we will eventually work with the differential notation, to understand the method it is better to write the equation as $y' = y$. Following the instructions in the previous paragraph, we divide both side of the equation by y. ***(It is worth emphasizing that division and/or multiplication must be used to move factors from one side to the other, never addition or subtraction.)*** We then obtain

$$\frac{y'}{y} = 1.$$

Now we integrate both sides of the latter equation with respect to x, the independent variable. This gives

$$\int \frac{y'}{y}\,dx = \int 1\,dx .$$

Although we don't know the function y that appears in the integrand on the left side, we can use the general integration formula discussed in Chapter 1 for functions u of x, $\int u'/u\,dx = \ln|u| + C$, to evaluate the left side. The right side can be explicitly integrated. We get

$$\ln|y| + C = x + D .$$

Since C and D are arbitrary constants, we combine them on the right side as $D - C$ obtaining the so-called implicit solution of the differential equation,

$$\ln|y| = x + D - C . \tag{2.2}$$

The constant $D - C$ on the right is again an arbitrary constant. In order to avoid a proliferation of letters, we will simply call this C again, as we have no interest in keeping track of the original value of all the constants. In general, in these types of manipulations, if an arbitrary constant is modified to produce a constant of a slightly different form, we will usually continue labeling the new variant with the same letter. Note that here we could have avoided this notational annoyance had we only used a single constant of integration in deriving (2.2). In the future we shall include only one constant, on the right side of the integrated equation.

We must now solve the implicit equation (2.2) for y. In order to extract the y term on the left we use the identity $e^{\ln w} = w$, valid for any positive w. This identity is simply a restatement of the inverse function relationship that holds between the exponential and logarithm functions. Thus from (2.2) (with $D - C$ replaced by C) we get

$$|y| = e^{\ln|y|} = e^{x+C} = e^{C}e^{x} .$$

Since $|y| = \pm y$, we can write the last equation as $y = \pm e^{C}e^{x}$. The quantity $\pm e^{C}$ is just a constant, so following the convention mentioned earlier, we again denote its value by C. Thus we obtain the final solution $y = Ce^{x}$. ■

We can simplify the mechanics of the solution process by using the dy/dx form of the differential equation. In this procedure we not only separate the x and y terms, but we also separate the dx and dy differentials, placing each with the corresponding variable. This method of doing separation of variables is similar in spirit to using the differential to do substitutions in integration problems. The manipulations are easier to carry out, though the mathematical reasoning behind them is somewhat obscured.

Example 2.6: Solve the initial value problem $\frac{dy}{dx} = (y+1)x$, and $y(0) = 4$.

Solution:

The first step is to produce the general solution of the differential equation. The ODE has separable variables. We separate the variables by dividing both sides by $y+1$ and multiplying by dx. This leads to

$$\frac{dy}{y+1} = x\,dx.$$

We now integrate both sides, treating each side as if the variable on that side were an independent variable.

$$\int \frac{dy}{y+1} = \int x dx = \frac{x^2}{2} + C.$$

The integral $\int dy/(y+1)$ is evaluated using the substitution $u = y+1$ and yields $\ln|y+1|$, ignoring the constant of integration. From this we obtain the implicit form of the solution.

$$\ln|y+1| = \frac{x^2}{2} + C.$$

We solve for y by exponentiation of both sides. This yields, (after writing C for e^C)

$$|y+1| = Ce^{x^2/2}.$$

The solution y is unwrapped from the absolute value as we did in Example 2.5, producing the general solution

$$y = Ce^{x^2/2} - 1.$$

Having found the general solution, we can solve the initial value problem. Substituting $x = 0$ and $y = 4$ in the last equation gives $4 = Ce^0 - 1 = C - 1$, so $C = 5$ and the solution of the initial value problem is $y = 5e^{x^2/2} - 1$.■

Example 2.7: Find the general solution of $\frac{dy}{dx} = \frac{\sqrt{2y+1}}{x}$ and determine the solution for which $y(1) = 4$.

Solution:

We can separate the variables yielding

$$\frac{dy}{\sqrt{2y+1}} = \frac{dx}{x} .$$

We now integrate both sides.

$$\int \frac{dy}{\sqrt{2y+1}} = \int \frac{dx}{x} \tag{2.3}$$

On the left side of (2.3) we have

$$\int \frac{dy}{\sqrt{2y+1}} = \int (2y+1)^{-1/2}\,dy .$$

Making the substitution $u = 2y+1,\ du = 2\,dy$ the last integral becomes $\frac{1}{2}\int u^{-1/2}du = u^{1/2} = \sqrt{2y+1}$ (ignoring the constant of integration). Therefore (2.3) evaluates to

$$\sqrt{2y+1} = \ln|x| + C , \tag{2.4}$$

which is the implicit solution for y. To solve for y we square both sides of (2.4). This gives $2y+1 = (\ln|x|+C)^2$ or

$$y = \frac{(\ln|x|+C)^2 - 1}{2} .$$

Note that constants such as -1 and 2 that appear in the final formula are not arbitrary. They are specific pieces of the general solution and must not be omitted or changed.

To find the value of C it is more convenient to substitute the initial conditions into (2.4) rather than into the final expression for y. When we set $x=1$ and $y=4$ in (2.4) we get $C=3$ (assuming the square root is the positive one). The solution of the initial value problem is therefore $y = ((\ln x + 3)^2 - 1)/2$. ■

2.3 Money

What is the derivative of a function $y = f(x)$? In Chapter 1 we discussed how to compute derivatives and in this chapter we learned some methods for finding a function from information about its derivative. However, to understand how differential equations arise we need to review the fundamental idea of a derivative as an instantaneous rate of change. You may recall the standard definition of the derivative in calculus:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{2.5}$$

This definition is useful for doing algebraic exercises and proving the various differentiation rules, but there is an older, less fashionable notation that captures more of the flavor of the concept. Instead of using h for the increment in the independent variable x, we use Δx (read "delta x"). The symbol Δx represents a single number, not a product. You can think of Δx as an abbreviation for the "difference in x". The numerator in (2.5) can then be written as $f(x+\Delta x) - f(x) = \Delta y$ and represents the difference Δy in the y value when x changes from x to $x + \Delta x$. Using the differential notation for the derivative we can rewrite (2.5) as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{2.6}$$

The fraction on the right side of (2.6) represents the average rate of change of the function $y = f(x)$ over the interval from x to $x + \Delta x$. The derivative is the limit of this average rate of change as the increment Δx approaches zero. Therefore, it is often referred to as the instantaneous rate of change of y.

Differential equations arise when we know something about the rate at which a function changes and want to deduce the behavior of the function itself. In physics, because of Newton's law relating force to acceleration (which is the rate of change of velocity), differential equations are at the core of the subject. We will see in Chapters 5and 6 that many problems concerning biological populations can be given reasonable formulations as differential equations. Here we want to consider a different sort of aggregate, but variable quantity, - money. Unlike populations, for which our mathematical descriptions represent a simplification of reality, an exact mathematical process is usually at the bottom of many financial transactions. We consider the question of continuous compounding of interest and show how equation (2.6) leads us to a description of this process as a differential equation.

Example 2.8: Suppose you have \$1000 in a bank account paying 10% annual interest, compounded daily. After 30 days what would be your total accumulation?

Solution:

The daily compounding of interest means that on any day the interest rate would be 1/365$^{\text{th}}$ of the annual rate or 0.10/365. The amount of interest earned in a day, I, is the current principal P_{old} multiplied by this daily rate. The new principal is then given by $P_{new} = P_{old} + I$ from which we can derive the next day's interest. The table below shows the results obtained at the beginning and end of the 30-day period. The interest is rounded to three decimal places. The change in principal from one day to the next is just the interest earned during the day in question. Notice that the daily interest does change, though it only increases slightly over the entire 30-day period.

Day	Principal ($)	Interest ($)
1	1,000.00	0.274
2	1,000.27	0.274
3	1,000.55	0.274
4	1,000.82	0.274
5	1,001.10	0.274
6	1,001.37	0.274

Day	Principal ($)	Interest ($)
25	1,006.60	0.276
26	1,006.87	0.276
27	1,007.15	0.276
28	1,007.42	0.276
29	1,007.70	0.276
30	1,007.98	0.276

After 30 days the total principal would be $\$1,007.98 + 0.276 \approx \$1,008.26$.■

What will happen if we compound the interest even more frequently? We can work out the framework of Example 2.8 in abstract terms. If r is the annual interest rate and the time period is represented as a fraction of the year, Δt (which is $1/365$ in the example), then the interest rate during that time period will be $r \times \Delta t$. If P denotes the principal at time t then the interest earned during the time Δt from t to $t + \Delta t$ will be $P \times r\Delta t$. This interest accounts for the change in the principal ΔP, so we have

$$\Delta P = P(r\Delta t). \tag{2.7}$$

We want to let $\Delta t \to 0$. Of course, this says that the time period goes to zero. Hence, so does the rate and therefore also the interest earned - not very surprising in view of the interest listed in the table above. In such situations we can learn something by considering the rate at which the interest goes to zero, namely the ratio $\Delta P / \Delta t$. By (2.7) this ratio is rP. However according to (2.6), as $\Delta t \to 0$ the ratio $\Delta P / \Delta t$ approaches the derivative, the instantaneous rate of change of the function $P(t)$. Thus we arrive at the differential equation

$$\frac{dP}{dt} = rP, \tag{2.8}$$

which we take as the definition of continuous compounding of interest.

Definition 2.1: Money earns interest at an annual rate r compounded continuously if the principal satisfies equation (2.8).■

Although we can't physically compound the interest continuously, we can solve the differential equation (2.8) and thereby figure out what the principal would be if that idealized process could be achieved. The formula turns out to be quite simple and can be used to directly compute the principal.

Example 2.9: Find the general formula for the principal $P(t)$ when money is compounded continuously at an annual interest rate r, assuming an initial principal of P_0.

Solution:

We must solve (2.8) with initial condition $P(0) = P_0$. The method of separation of variables can be applied. Separating the variables gives

$$\frac{dP}{P} = rdt\,.$$

Integrating both sides of the latter equation yields

$$\ln|P| = rt + C\,.$$

Solving for P by exponentiation (see Example 2.5) we get

$$P = Ce^{rt}\,.$$

Substituting $P_0 = P(0)$ and putting $t = 0$ we have $P_0 = Ce^0 = C$ so the formula for the principal becomes

$$P(t) = P_0 e^{rt}\,. \tag{2.9}$$

■

There are a number of interesting financial lessons that can be learned from (2.9).

Example 2.10:

a) What is the effective annual interest rate if money is compounded continuously at an annual rate r?

b) How many years are needed to double any starting principal if money earns interest at an annual rate r compounded continuously?

Solution:

a) By definition, the effective annual interest rate is the relative change in the principal in one year. In other words, it is the quantity

$$\frac{P(1) - P(0)}{P(0)} = \frac{P_0 e^r - P_0}{P_0} = e^r - 1\,.$$

When r is small (as it usually is when we are dealing with interest rates) this quantity is just slightly larger than the nominal rate r. The increase is due to the compounding. The table below shows this for some representative interest rates. For example, at 7% annual interest you would actually earn 7.25% over the course of a year.

r	.03	.05	.07	.10	.15
Effective Annual Rate	.0304	.0513	.0725	.1052	.1618
Doubling Time(years)	23.1	13.9	9.9	6.9	4.6

Table 2.1

b) We need to find how long it takes for the principal to reach double its initial amount. We must find the time t for which $2P_0 = P_0 e^{rt}$. Canceling the term P_0, we are left with the equation $2 = e^{rt}$. To solve this equation we take the natural logarithm of both sides obtaining

$$\ln 2 = rt \ln e = rt ,$$

since $\ln e = 1$. Denoting the solution by t_2, we have

$$t_2 = \frac{\ln 2}{r} . \qquad (2.10)$$

Some representative values of the doubling time are given in Table 2.1. We can derive a useful rule of thumb to estimate the doubling time. Observe first that $\ln 2 \approx .693 \approx .70$. Therefore, multiplying the numerator and denominator in (2.10) by 100 we have

$$t_2 = \frac{\ln 2 \times 100}{r \times 100} \approx \frac{70}{\%\text{ interest rate}} .$$

For example, if the interest rate is $r = .07 = 7\%$, the estimate gives $70/7 = 10$ years for the doubling time, compared to the more exact value of 9.9. In finance texts, this rule is often called the Rule of 70.■

Money might seem far removed from biology. Yet the same mathematical arguments that allow us to analyze the growth of money in the bank can be used to model the growth of populations. In exercises 18 to 21 we describe how the continuous interest model we have described in this section can be extended to quantify a variety of common financial transactions. In Chapter 5 we will see that the same differential equations are related to problems of harvesting and conservation in population biology.

2.4 Summary

A ***differential equation*** expresses a relation between the derivative $\frac{dy}{dx}$ of an unknown function and values of the independent variable x and dependent variable y. The collection of all solutions to the differential equation is called the ***general solution***. This family of solutions will usually contain an arbitrary constant C, whose value may be determined by giving additional information about the unknown function. Quite often this extra information is in the form of an ***initial condition***, $y(0) = y_0$, specifying the value of the unknown function at $x = 0$.

Differential equations are solved using integration, but the equation may have to be manipulated algebraically before this can be done. In a ***separable differential equation***, we can separate the dependent and independent variables on either side of the equation and solve the equation by integrating each side. An important example of this type with many applications is the equation $\frac{dy}{dx} = y$ or more generally $\frac{dy}{dx} = ry$, where r is a constant. The solution of the latter is given by $y = y_0 e^{rx}$.

2.5 Exercises

1. By computing $\frac{dy}{dx}$ verify that for any constant C the expression $y = xe^x + Ce^x$ is a solution of the differential equation $\frac{dy}{dx} = e^x + y$. (See Example 2.2.)

2. Verify that for any constant C the expression $y = \sqrt{x^2 - 2x + C}$ is a solution of the differential equation $\frac{dy}{dx} = \frac{x-1}{y}$.

3. a) Verify that for any constant C the expression $y = x - 1 + Ce^{-x}$ is a solution of the differential equation $\frac{dy}{dx} = x - y$.

   b) Find the solution of the differential equation in 3a) that satisfies $y(0) = 2$.

4. a) Verify that for any constant C the expression $y = \frac{Ce^x - 1}{1+x}$ is a solution of the differential equation $\frac{dy}{dx} = \frac{1+xy}{1+x}$.

   b) Find the solution of the differential equation in 4a)that satisfies $y(1) = 2$.

5. a) Verify that for any constant $C \geq 0$ the expression $y = -x + \sqrt{2x^2 + C}$ is a solution of the differential equation $\frac{dy}{dx} = \frac{x-y}{x+y}$.

   b) Find the solution of the differential equation in 5a) that satisfies $y(1) = 1$.

6. Which, if any, of the differential equations in exercises 1 - 5 are separable?

7. Which, if any, of the following differential equations are separable?

a) $\frac{dy}{dx} = \frac{xy}{1+x}$

b) $\frac{dy}{dx} = \frac{xy}{y+x}$

c) $\frac{dy}{dx} = (3y+1)^2$

d) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Use separation of variables to solve each of the differential equations in exercises 8 - 16 below:

8. $\frac{dy}{dx} = xy$, $y(0) = 2$.

9. $\frac{dy}{dx} = 2y^2 x$, $y(0) = 2$

10. $\frac{dy}{dt} = \frac{t}{\sqrt{y+1}}$, $y(1) = 3$

11. $\frac{dy}{dt} = \frac{t}{e^{2y}}$, $y(0) = 1$.

12. $\frac{dy}{dt} = 3y+1$, $y(1) = 2$.

13. $y' = y(x+1)$, $y(0) = 2$.

14. $\frac{dy}{dx} = \frac{2y}{x+1}$, $y(0) = 3$.

15. $y' = \frac{2(t+1)}{y}$, $y(0) = 2$.

16. $\frac{dy}{dx} = \frac{1+2y}{1+x}$

17. After 5 years, which of the following accounts will have a larger principal? After 10 years? Justify your answers.

   a) An account earning 5% annual interest compounded continuously and starting out with $1000.

   b) An account earning 4% annual interest compounded continuously and starting out with $1100.

18. a) Suppose you want to put away a certain amount of money so that in 10 years the principal will be $10,000. If you can invest at 6% annual interest compounded continuously, what must be your initial principal to meet your 10-year objective?

   b) Referring to18a), if the best investment available to you will only pay 5% annual interest, show that your initial principal must be increased by more than 10% over the answer to a) in order to reach the $10,000 objective in 10 years.

19. a) If you deposit $5000 in an account earning 5% annual interest compounded continuously, how long will it take for the principal to reach $7500?

   b) If the deposit of $5000 earns 10% interest, how long will it take for the principal to reach $7500?

20. A 5-year investment of \$10,000 begins at 6% annual interest compounded continuously. The investment agreement calls for the interest rate to be evaluated after 2½ years, when it can move up or down by at most 2% for the remaining 2½ years. Determine the minimum and maximum principal you can receive at the end of 5 years.

21. For each of the interest rates listed in Table 2.1, find the number of years for an initial principal to triple.

22. a) Suppose you deposit an initial principal of \$1000 in an account paying 6% annual interest compounded continuously. Every year, including the first, you make additional deposits of \$500 spread out uniformly throughout the year. By repeating the arguments that led to (2.8), show that the principal $P(t)$ satisfies the ODE

$$\frac{dP}{dt} = .06P + 500 . \tag{2.11}$$

b) Find the solution of (2.11) satisfying the initial condition $P(0) = 1000$.

c) After 10 years what will be the total principal in the account as described in a)? How much of this total has come from deposits? How much from interest?

23. a) Consider the investment scenario described in22a), except that your initial principal is zero. Each year you want to make a deposit of K dollars spread out uniformly throughout the year, so that at the end of 25 years the account will have a principal of \$50,000. By repeating the arguments that led to equation (2.8), show that the principal $P(t)$ satisfies the ODE

$$\frac{dP}{dt} = .06P + K . \tag{2.12}$$

b) Find the solution of (2.12) satisfying $P(0) = 0$. (The as yet unknown constant K will appear in your formula)

c) Using the formula in b) find the value of K that will meet your objective as stated in a). How much of the final \$50,000 consists of deposits and how much consists of interest?

24. a) A woman retires and receives a lump sum of \$200,000 from her company's retirement account. Suppose she deposits this in an account paying 7% continuously compounded annual interest and withdraws \$20,000 a year spread out uniformly throughout the year. By repeating the arguments that led to equation (2.8), show that the principal $P(t)$ satisfies the ODE

$$\frac{dP}{dt} = .07P - 20, \tag{2.13}$$

where we have used units of \$1000.

b) Find the solution of (2.13) satisfying the initial condition $P(0) = 200$.

c) For how many years can the woman continue withdrawing the \$20,000 (usually referred to as an annuity)?

25. a) A man wants to retire at age 65 and be able to draw \$10,000 a year from retirement savings to supplement his Social Security income and employee retirement benefits. At retirement he anticipates investing his savings in government securities that should return 5% a year compounded continuously. By repeating the arguments that led to equation (2.8), show that after retirement the principal $P(t)$ in his account satisfies the ODE

$$\frac{dP}{dt} = .05P - 10, \qquad (2.14)$$

where we have used units of \$1000.

b) Determine the general solution of (2.14) and find a formula for the solution with, as yet undetermined, initial value P_0.

c) How large must the man's initial retirement savings be so that he can continue drawing his \$10,000 income (annuity) for 20 years?

26. Exercises 22 - 25 dealt with ODEs of the form $\frac{dP}{dt} = rP + b$. Find a formula for the solution of this equation with initial condition $P(0) = P_0$. Check that when $b = 0$ your answer reduces to (2.9).