

## 17 Discrete Deterministic Modeling

*Dr. Frink:* You've got to listen to me. Elementary chaos theory tells us that all robots will eventually turn against their masters and run amok in an orgy of blood and the kicking and the biting with the metal teeth and the hurting and shoving.

*Scientist:* How much time do we have professor?

*Dr. Frink:* Well according to my calculations, the robots won't go berserk for at least 24 hours. (*The robots go berserk.*)

*Dr. Frink:* Oh, I forgot to er, carry the one.

From: *The Simpsons*

### 17.1 Introduction

In Part I we used differential equations to construct mathematical models for biological systems that change continuously. However, for many problems in population biology such an assumption is unwarranted and applying continuous modeling techniques cannot be expected to yield satisfactory results. For example, in temperate climates the life cycles of many insects are seasonal; births occur at approximately the same time of year in a given location and are not spread out uniformly throughout the time period under study. The same is true of many species of birds as well as some larger mammals that depend on seasonal food supplies.

In cases such as those mentioned above it is mathematically more realistic to consider taking a snapshot of the population at a specific time during the year, say right after the breeding season (or birthing season) has ended. In this way we get a set of discrete population values from one year to the next, which we denote by  $N_0, N_1, N_2, \dots$ . In most of our problems the time period implicit between successive values of  $N$  will be one year, though it could be smaller or longer depending on the reproductive cycle. The goal in discrete modeling is to find some mathematical relationship between successive population values that will enable us to better understand how the population changes. Typically these relationships have the form  $N_{t+1} = f(N_t)$ , where  $f$  is some function that enables us to compute the value of  $N$  in time period  $(t+1)$ ,  $N_{t+1}$ , from its value  $N_t$  in the preceding time period. A relationship of this sort is called a (first-order) *difference equation* or a *recurrence equation*.

In this chapter we will study certain aspects of the theory of difference equations. Our construction of various model equations will often proceed via arguments similar to the ones we used in constructing continuous population models in chapter 5. However, the resulting equations, though analogous to some differential equations we have studied, often exhibit solutions that show much more complicated behaviors. In fact, many ideas associated with “chaos theory” originated in this study. Similar types of behavior are possible in the continuous case involving systems of three or more differential equations. The discovery that certain simple mathematical models can produce random-like results has profoundly influenced thinking in population biology, physiology, meteorology, and even finance. We will consider the origins of this phenomenon in section 17.5.

Although many discrete models may produce random-like outcomes, the inputs to the model involve no randomness at all. As such, these models are called *deterministic*. In chapter 18 we will consider how we can incorporate randomness into the inputs as well, leading to what are known as *stochastic models*.

## 17.2 Discrete Exponential Growth

We first consider the discrete analog of continuous exponential growth. Suppose the per capita change in the population from one time period to the next has a constant value  $r$ . This means that if the population has  $N_t$  members at time  $t$  then when the count is made at time  $t+1$  the population will have increased (or decreased if  $r$  is negative) by  $rN_t$ . This can be written more concisely in the form

$$\Delta N = N_{t+1} - N_t = rN_t. \quad (17.1)$$

This is quite analogous to the equation developed in Chapter 5 (section 5.1) for the change in a continuously varying population over a short interval of time  $\Delta t$ ,

$$N(t + \Delta t) - N(t) = (\Delta t)rN.$$

In equation (17.1) however,  $\Delta t = 1$  and we cannot divide by  $\Delta t$  and take a limit to obtain a differential equation. All we can do is rewrite the equation so that it is solved for  $N_{t+1}$  in terms of  $N_t$ , giving

$$N_{t+1} = (1 + r)N_t. \quad (17.2)$$

**Example 17.1:** Find the first 10 population values  $N_1, N_2, \dots, N_{10}$  for the difference equation (17.2) if the initial value  $N_0 = 1.5$  and  $r = 0.2$ .

*Solution:*

Since  $r = 0.2$  the difference equation is simply

$$N_{t+1} = 1.2N_t. \quad (17.3)$$

Just as for a differential equation, in order to find numerical solutions we must provide an initial value for the numerical sequence  $N_t$ . However, unlike differential equations no special mathematical ideas, such as Euler's method, are needed to compute successive values. Indeed letting  $t = 0$  in (17.2) gives

$$N_1 = 1.2N_0 = 1.2(1.5) = 1.8.$$

Then letting  $t = 1$  in (17.2) yields  $N_2 = 1.2N_1 = 1.2(1.8) = 2.16$ .

Continuing in this way (which as the tech notes at the end of the chapter show can be implemented very easily on a spreadsheet) yields the following list. The values of  $N_t$  have been rounded to three decimal places.

$t$	0	1	2	3	4	5	6	7	8	9	10
$N_t$	1.5	1.800	2.160	2.592	3.110	3.732	4.479	5.375	6.450	7.740	9.288

■.

In principle, Example 17.1 shows that computing the solution of a difference equation is a straightforward task (actually, as we will see later, even this is not really true.) However, having a mass of numbers does not give much insight into the solution. We consider next a graphical means of representing the computed solution. This graph is called the *trajectory* of the solution.

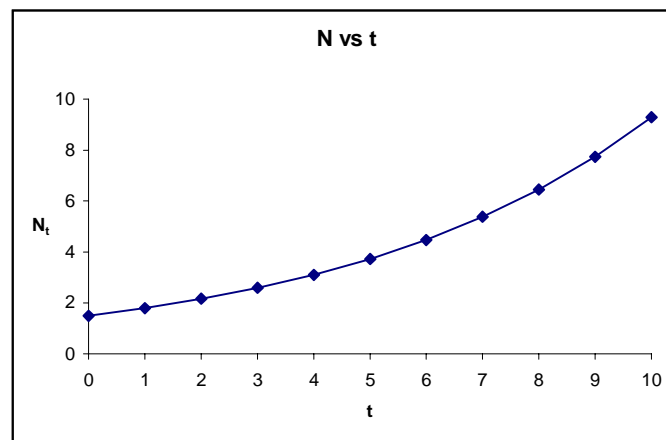
**Definition 17.1:** If  $N_0, N_1, \dots$  is the solution of a difference equation then the points  $(0, N_0), (1, N_1), \dots, (t, N_t), \dots$  constitute the *trajectory* of the solution. ■

Often we use line segments to connect the discrete points of the trajectory. These lines are for visual purposes and are not part of the trajectory.

**Example.17.2:** Draw the trajectory of the 10 values of  $N_t$  obtained in Example 17.1.

*Solution:*

Using *Excel* the graph below is easily constructed from the table obtained in Example 17.1 (see tech notes).



The trajectory is useful for visualizing the computed solution. It suggests in this case that the values of  $N_t$  will continue to increase. We would like to be more certain of this and there are two ways to confirm this guess. In this section we will find an explicit formula for the solution that will enable us to ascertain its long-term behavior. In the next section we will examine a graphical technique known as “cobwebbing” that allows us to answer such questions in more complicated cases.

**Theorem.17.1:** If  $N_t$  is a solution of the difference equation  $N_{t+1} = (1+r)N_t$  then

$$N_t = (1+r)^t N_0, \quad (17.4)$$

where  $N_0$  is the given initial value of the sequence.

*Proof:*

If we set  $t=0$  in the equation  $N_{t+1} = (1+r)N_t$  we obtain  $N_1 = (1+r)N_0$ , which is equation (17.4) for  $t=1$ . Letting  $t=1$  in the recurrence formula  $N_{t+1} = (1+r)N_t$  gives  $N_2 = (1+r)N_1$ . Now substitute in this the expression for  $N_1$  and you obtain

$$N_2 = (1+r)N_1 = (1+r)\overbrace{(1+r)N_0}^{N_1} = (1+r)^2 N_0$$

which is precisely formula (17.4) for  $t=2$ . We can continue in this fashion, for example,

$$N_3 = (1+r)N_2 = (1+r)\overbrace{(1+r)^2 N_0}^{N_2} = (1+r)^3 N_0.$$

The reader who is familiar with mathematical induction will see how to formalize the argument into a complete proof. ■

**Example 17.3:** Compute  $N_{25}$  for the sequence described in Example 17.1.

*Solution:*

Using formula (17.4) we have  $N_{25} = (1.2)^{25}(1.5) \approx 143.09$ . ■

From Theorem.17.1 we easily deduce the long-term behavior of the solution to  $N_{t+1} = (1+r)N_t$  when the quantity  $N_t$  is the size of a population, and therefore of necessity non-negative.

**Theorem.17.2:** If a population satisfies  $N_{t+1} = (1+r)N_t$  and  $N_0 > 0$  then if  $r > 0$  the values  $N_t \rightarrow \infty$  as  $t$  increases and if  $-1 < r < 0$  then  $N_t \rightarrow 0$  as  $t$  increases.

N.B. Since  $N_t \geq 0$  for all  $t$  a value  $r < -1$  is impossible. We leave it to the reader to describe what happens when  $r = 0$  or  $r = -1$ .

*Proof:*

Since by Theorem.17.1 we have  $N_t = (1+r)^t N_0$ , the sequence  $N_0, N_1, \dots, N_t, \dots$  exhibits exponential growth with base  $(1+r)$ . If  $(1+r) > 1$  the powers  $(1+r)^t \rightarrow \infty$ . Thus the conclusion when  $r > 0$ . Similarly, when  $0 < (1+r) < 1$ , the powers  $(1+r)^t \rightarrow 0$  as  $t \rightarrow \infty$ . But  $0 < (1+r) < 1 \Leftrightarrow -1 < r < 0$ , which is the second assertion. ■

Thus, a population growing according to the discrete model  $N_{t+1} = (1+r)N_t$  will grow exponentially if the per capita growth rate  $r$  is positive, and will decrease exponentially towards zero if the per capita growth rate is negative. This is pretty much the same conclusion we found for the continuous growth Malthus model. **One source of possible confusion that should be clarified arises because the difference equation  $N_{t+1} = (1+r)N_t$  is sometimes rewritten as  $N_{t+1} = kN_t$ , where  $k = 1+r$  is a constant.** The behavior of the solution is often expressed in terms of the quantity  $k$ , rather than  $r$ , as in the following theorem, the details of which are left to the reader. Note that in this theorem we do not impose the restriction that  $N_t \geq 0$  and therefore we allow both positive and negative values of  $k$ .

**Theorem.17.3:** If  $k$  is an arbitrary constant, the solution of the difference equation  $N_{t+1} = kN_t$  is given by  $N_t = k^t N_0$ . If  $|k| < 1$  the solution tends towards zero as  $t$  increases. If  $|k| = 1$  the solution is either constant or oscillates between  $N_0$  and  $-N_0$ . If  $|k| > 1$  (and  $N_0 \neq 0$ ) the solution either approaches  $\infty$ ,  $-\infty$ , or oscillates towards  $\pm\infty$ . ■

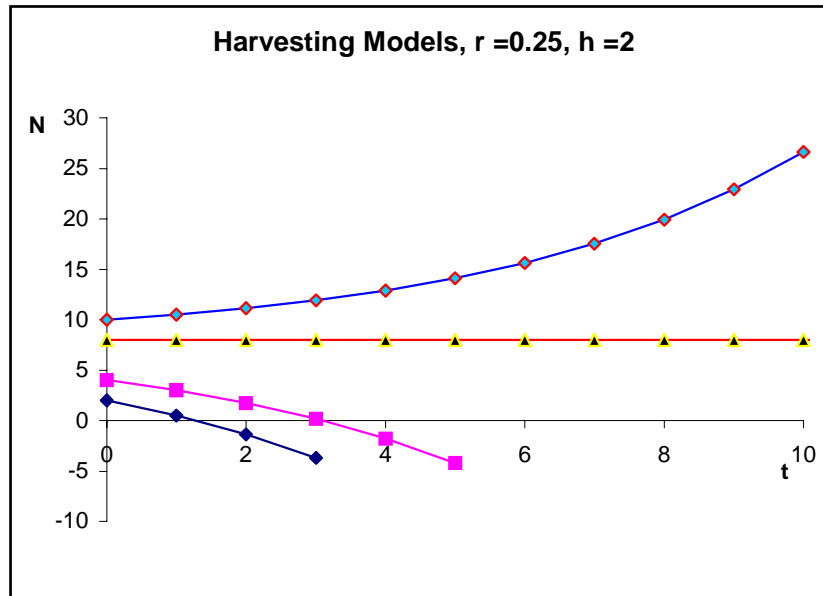
By way of contrast with Theorem.17.3, the solution of the analogous differential equation  $\frac{dN}{dt} = kN$  is  $N = N_0 e^{kt}$  (See Chapter 2, example 2.9, with a slight change in notation). Here the solution will tend to a limit of zero when the growth rate parameter  $k < 0$ . If  $N_0 \neq 0$  and  $k > 0$  the solution will tend towards  $\pm\infty$  depending on the sign of  $N_0$ .

### 17.3 Steady States

Suppose a population of insects is growing according to a discrete difference equation  $N_{t+1} = (1+r)N_t$ , with  $r > 0$ . If we attempt to control the population by applying some pesticide we can consider this as a type of harvesting. For simplicity, suppose the pesticide application kills a fixed number  $h$  of insects between time  $t$  and  $t+1$ . We can then write a new difference equation for the population growth:

$$N_{t+1} = (1+r)N_t - h \quad (17.5)$$

Figure 17.1 below shows some trajectories for solutions of (17.5). It appears that the population collapses, grows unbounded as  $t$  increases or remains constant.



**Figure 17.1**

Figure 17.1 is suggestive but provides little quantitative value in determining the relationship between  $r$ ,  $h$  and  $N_0$  that leads to one of the three types of outcomes depicted. There are two ways around this. The first is to make use of an explicit formula for the solution of (17.5). This result is more complicated than the formula stated in Theorem.17.3 for the case when there is no harvesting. We leave the derivation of the result to the exercises, but indicate how we can use it to address the question of the relationship of the parameter values ( $r$ ,  $h$  and  $N_0$ ) to the qualitative system behavior.

**Theorem 17.4:** The solution of the difference equation  $N_{t+1} = (1+r)N_t - h$ , with initial condition  $N_0$  and  $r \neq 0$  is given by

$$N_t = \left(N_0 - \frac{h}{r}\right)(1+r)^t + \frac{h}{r}. \quad \blacksquare \quad (17.6)$$

Although more complicated than the simple exponential growth formula, (17.6) can be easily applied to characterizing the parameter relations that lead to the three behaviors indicated in Figure 17.1

**Example 17.4:** Use (17.6) to determine when the solution of the difference equation  $N_{t+1} = (1+r)N_t - h$  will, (i) remain constant for all  $t$ , (ii) tend towards infinity as  $t$  increases, or (iii) eventually reach zero.

*Solution:*

Observe that if  $N_0 = \frac{h}{r}$  the coefficient of  $(1+r)^t$  is zero and therefore the value of  $N_t = \frac{h}{r}$ , a constant. This provides the answer to (i). If  $N_0 > \frac{h}{r}$  the term  $N_0 - \frac{h}{r} > 0$  and therefore the value of  $N_t \rightarrow \infty$  as  $t$  increases, answering (ii). Finally, if the initial value  $N_0 < \frac{h}{r}$  the solution  $N_t$  will eventually assume negative values since the term  $(N_0 - \frac{h}{r})(1+r)^t \rightarrow -\infty$  in this case.

In the example illustrated in Figure 17.1 we have  $r = .25$  and  $h = 2$  so the initial value  $N_0$  that yields a constant solution is  $N_0^* = \frac{h}{r} = 8$ . If  $N_0$  exceeds that value the insect population will still increase with the given level of pesticide use, while if the initial population is below  $N_0^*$  the population will eventually be wiped out. ■

The analysis in Example 17.4 shows the importance of constant solutions of the difference equation. These play a critical role in understanding the behavior of difference equations, as their analogs did for differential equations.

**Definition 17.2:** A solution  $N_t$  of a difference equation that has the same value for all  $t$  is called a *steady state solution*. ■

In principle it is easy to find steady state solutions. In practice the work may be formidable or impossible if exact formulas are desired. The method is based on the following theorem, which is the analog of Theorem 4.3 in Chapter 4.

**Theorem.17.5:**  $N_t = N^*$  is a steady state solution of the difference equation  $N_{t+1} = f(N_t)$  if and only if  $N^*$  satisfies the equation

$$N^* = f(N^*) \tag{17.7}$$

*Proof:*

If  $N_t = N^*$  is a constant solution of the difference equation then we must also have  $N_{t+1} = N^*$ . Therefore,  $N^* = N_{t+1} = f(N_t) = f(N^*)$  as asserted. ■

If we have a function  $f(x)$  and a number that satisfies  $x = f(x)$  we say that  $x$  is a *fixed point* of the function. According to Theorem.17.5 **a steady state solution of a difference equation is a fixed point of the right side of the difference equation.**

Let's use Theorem.17.5 to find the steady state solution for (17.5).

**Example.17.5:** Assuming  $r \neq 0$ , find the steady state solution for  $N_{t+1} = (1+r)N_t - h$ .

*Solution:*

Using  $N$  as our variable we can write this difference equation as  $N_{t+1} = f(N_t)$ , where  $f(N) = (1+r)N - h$ . We must solve the equation  $N = f(N)$  or

$$N = (1+r)N - h.$$

Expanding the right side and solving for  $N$  gives the steady state solution  $N^* = \frac{h}{r}$ . ■

#### 17.4 Stability & Cobwebs

**Definition 17.3:** A steady state solution  $N^*$  of the difference equation  $N_{t+1} = f(N_t)$  is said to be *stable* if any solution that has initial value  $N_0$  near  $N^*$  approaches  $N^*$  as  $t \rightarrow \infty$ . If there are solutions that have initial value arbitrarily close to  $N^*$  (but not equal to it), yet do not approach  $N^*$  as  $t$  increases, the steady state is called *unstable*. ■

From the analysis provided in Example 17.4 the steady state  $N^* = \frac{h}{r}$  for the difference equation

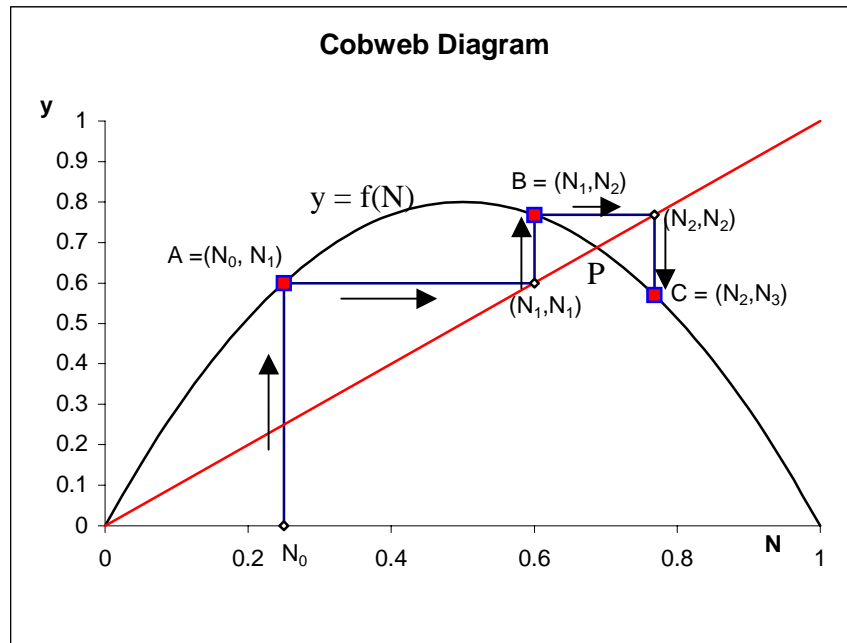
$N_{t+1} = (1+r)N_t - h$  is unstable, for in this case solutions that begin with initial value  $N_0$  near  $N^*$  tend to diverge from this steady state. (See also Figure 17.1). In this section we will learn how to make a stability determination without using an explicit formula for the solutions. The key idea is to construct a graphical scheme to follow the generation of successive terms in the solution of a difference equation. The scheme is called a *cobweb diagram*. Here is how it works.

If the difference equation has the form  $N_{t+1} = f(N_t)$  we draw the graph of the function  $y = f(N)$  together with the straight line  $y = N$ . A typical example is shown in Figure 17.2 below:

We locate the initial value  $N_0$  along the horizontal axis. A vertical line connects this to the graph of  $y = f(N)$  at the point labeled A. The ordinate of this point is  $f(N_0) = N_1$ . We now slide horizontally until we intersect the line  $y = N$ . During a horizontal slide the ordinate does not change and, since on the line  $y = N$  both coordinates are equal, we arrive at the point  $(N_1, N_1)$ . Now we repeat our vertical movement to the graph of  $y = f(N)$  arriving at point labeled B, whose



coordinates are  $N_1$  and  $y = f(N_1) = N_2$ . The steps are repeated as many times as are desired. The abscissas of the points (A, B, C, etc.) obtained on the graph of  $y = f(N)$  show the successive values of the solution of the difference equation  $N_{t+1} = f(N_t)$ .



**Figure 17.2**

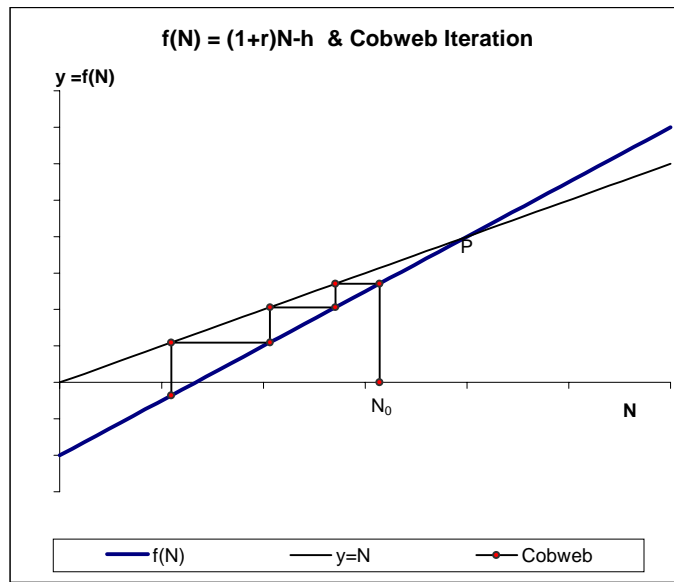
The cobweb diagram also shows the steady state solution(s). By Theorem.17.5 these are the values of  $N$  where  $N = f(N)$  and therefore equal the coordinates of the points where the graph of  $y = f(N)$  meets the line  $y = N$ . In Figure 17.2 the point labeled P has as its coordinates the value of the steady state solution of the difference equation.

Let's apply these ideas to the harvesting model that we considered in the last section.

**Example.17.6:** Construct a cobweb diagram for the difference equation  $N_{t+1} = (1+r)N_t - h$  and use the diagram to assess the stability of the steady state solution.

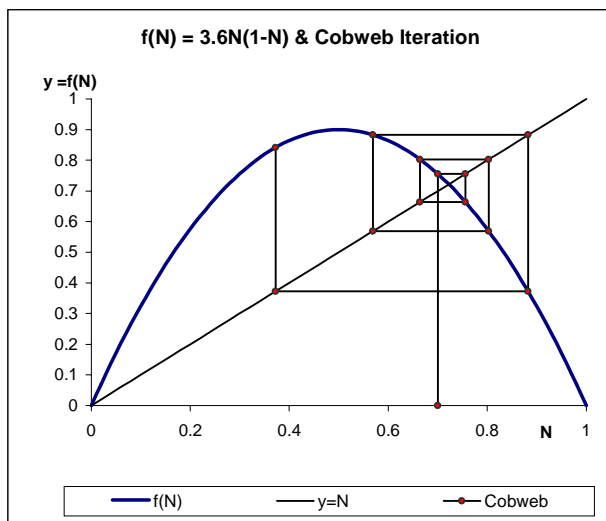
*Solution:*

The graph of the equation  $f(N) = (1+r)N - h$  is a straight line of slope  $1+r > 0$  and intercept  $-h$ . This line crosses the graph of  $y = N$  at the point P with  $N = \frac{h}{r}$ . The graph below shows a typical configuration.

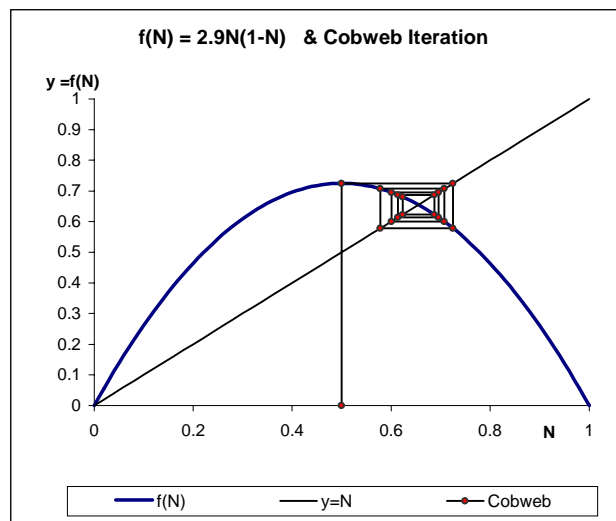


For the initial point selected  $N_0$  the successive iterates steadily diminish towards zero. If the initial value had been selected above  $P$  then the successive iterates would have become unbounded. In both cases we clearly see the instability of the steady state solution. ■

In Chapter 4 we discussed a geometric method for analyzing the stability of steady state solutions of a differential equation  $\frac{dN}{dt} = g(N)$ . Recall that this involved graphing the right side  $g(N)$ , and examining the behavior of the latter graph near its intercepts. The cobweb diagram provides a similar graphical tool for analyzing difference equations. However, for a cobweb diagram to be useful it must be drawn fairly accurately. Consider the following examples:



An Unstable Fixed Point



A Stable Fixed Point

Figure 17.3

The graphs of the iteration functions  $f(N)$  in Figure 17.3 appear quite similar, but the stability of the fixed point is different in the two cases. In the left panel the steady state is unstable, the points spiraling outward from the steady state, while in the right panel it is stable. The following theorem explains why these two graphs give such different qualitative behavior and why we need an accurate graph to draw reliable conclusions from a cobweb diagram.

**Theorem 17.6** Suppose  $N^*$  is a steady state of the difference equation  $N_{t+1} = f(N_t)$ . If  $|f'(N^*)| < 1$  then  $N^*$  is stable, while if  $|f'(N^*)| > 1$  then  $N^*$  is unstable.

*Proof:*

We sketch the main idea of the proof, omitting some technical details. According to the differential approximation (Theorem 3.1) when  $N$  is close to  $N^*$  we may approximate the function  $f(N)$  using its tangent line. More precisely, letting  $\Delta x = \Delta N = N - N^*$  in (3.1), we may write

$$f(N) \approx f(N^*) + f'(N^*)(N - N^*). \quad (17.8)$$

The essence of (17.8) is that when  $N$  is close to  $N^*$  we can replace  $f(N)$  in the difference equation  $N_{t+1} = f(N_t)$  by the right side of (17.8). Using that  $N^*$  is a fixed point of  $f$  and so  $f(N^*) = N^*$ , and letting  $r = f'(N^*)$ , the resulting difference equation is  $N_{t+1} = N^* + r(N_t - N^*)$ , or

$$N_{t+1} - N^* = r(N_t - N^*). \quad (17.9)$$

We can solve the difference equation (17.9) in a similar way that we developed the solution of the difference equation  $N_{t+1} = rN_t$  in Theorem.17.1. Starting with an initial value  $N_0$  close to the steady state  $N^*$ , we find that  $N_1 - N^* = r(N_0 - N^*)$ , then  $N_2 - N^* = r^2(N_0 - N^*)$  and in general

$$N_t - N^* = r^t(N_0 - N^*) \quad (17.10)$$

From (17.10) we have  $|N_t - N^*| = |r|^t |N_0 - N^*|$ . When  $|r| < 1$  the quantity  $|r|^t \rightarrow 0$  as  $t$  increases and therefore  $N_t$  gets closer and closer to  $N^*$ . In other words, the steady state is stable. If  $|r| > 1$  the quantity  $|r|^t \rightarrow \infty$ , so  $N_t$  moves farther away from  $N^*$  and the steady state is unstable. ■

**Example 17.7:** Find the fixed points for the iteration functions in Figure 17.3 and using Theorem 17.6 classify each fixed point according to its stability.

*Solution:*

The two functions are  $f(N) = 3.6N(1 - N)$  and  $f(N) = 2.9N(1 - N)$ .

Fixed point and stability analysis for  $f(N) = 3.6N(1 - N)$ :

The solutions of  $f(N) = 3.6N(1 - N) = N$  are  $N = 0$  and (after canceling  $N$  from both sides)  $N = \frac{2.6}{3.6} \approx .722$ . Since  $f'(N) = 3.6 - 7.2N$ , we have  $f'(.722) \approx -1.6$ . By Theorem 17.6 the fixed point is unstable.

Fixed point and stability analysis for  $f(N) = 2.9N(1 - N)$ :

The solutions of  $f(N) = 2.9N(1 - N)$  are  $N = 0$  and  $N = \frac{1.9}{2.9} \approx .655$ . Since  $f'(N) = 2.9 - 5.8N$  we obtain that  $f'(.655) \approx -.89$ . Therefore, by Theorem 17.6 the fixed point is stable. ■

### 17.5 The Logistic Map and Deterministic Chaos

In discussing population models in Chapter 5 we used the idea of density dependent growth as a mathematical mechanism that might account for stable population limits. It is reasonable to explore such a mechanism in the case when the population changes occur in discrete time units. Choosing our time unit so that the time between successive reproductive periods is  $\Delta t = 1$ , we denote the successive population sizes by  $N_0, N_1, \dots, N_t, \dots$ . To convert a continuous model of

the form  $\frac{dN}{dt} = f(N)$  to a discrete difference model we think of the derivative as the change per unit time  $\frac{\Delta N}{\Delta t}$ , then take  $\Delta t = 1$ . This gives the model  $\Delta N = N_{t+1} - N_t = f(N_t)$  or  $N_{t+1} = N_t + f(N_t)$ . From the logistic differential equation we arrive at the discrete logistic model:

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right) \quad (17.11)$$

Equation (17.11) can be put into a more convenient form by factoring  $N_t$  from the first terms on the right. This gives  $N_{t+1} = N_t\left(1 + r\left(1 - \frac{N_t}{K}\right)\right) = N_t\left(1 + r - \frac{rN_t}{K}\right)$ . Factoring  $1 + r$  we obtain

$$N_{t+1} = (1 + r)N_t\left(1 - \frac{r(1 + r)N_t}{K}\right).$$

The latter equation resembles the discrete exponential model (17.2). The per capita growth factor  $1 + r$  is moderated by the term  $1 - \frac{r(1 + r)N_t}{K}$ , which acts to decrease the relative growth as  $N$  increases. We can simplify the study of this map (without losing any mathematical essentials) by selecting the constant  $K$  equal to  $r(1 + r)$ . This results in the standard discrete logistic map

$$N_{t+1} = (1 + r)N_t(1 - N_t) = kN_t(1 - N_t), \quad (17.12)$$

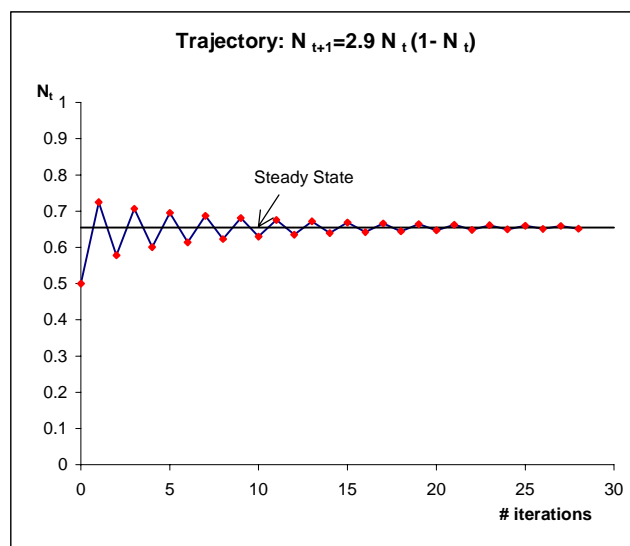
where  $k = 1 + r$ . The values of  $N_t$  in (17.12) must be non-negative if they are to represent population values. From the form of (17.12) the requirement that  $N_t \geq 0$  for all  $t$  necessitates as well that  $N_t \leq 1$ . It is not difficult to show then that the parameter  $k$  must satisfy  $0 \leq k \leq 4$ .

**Example 17.8:** If  $0 \leq k \leq 4$  find the steady state solutions of the discrete logistic map  $N_{t+1} = kN_t(1 - N_t)$  and determine whether these steady states are stable or unstable.

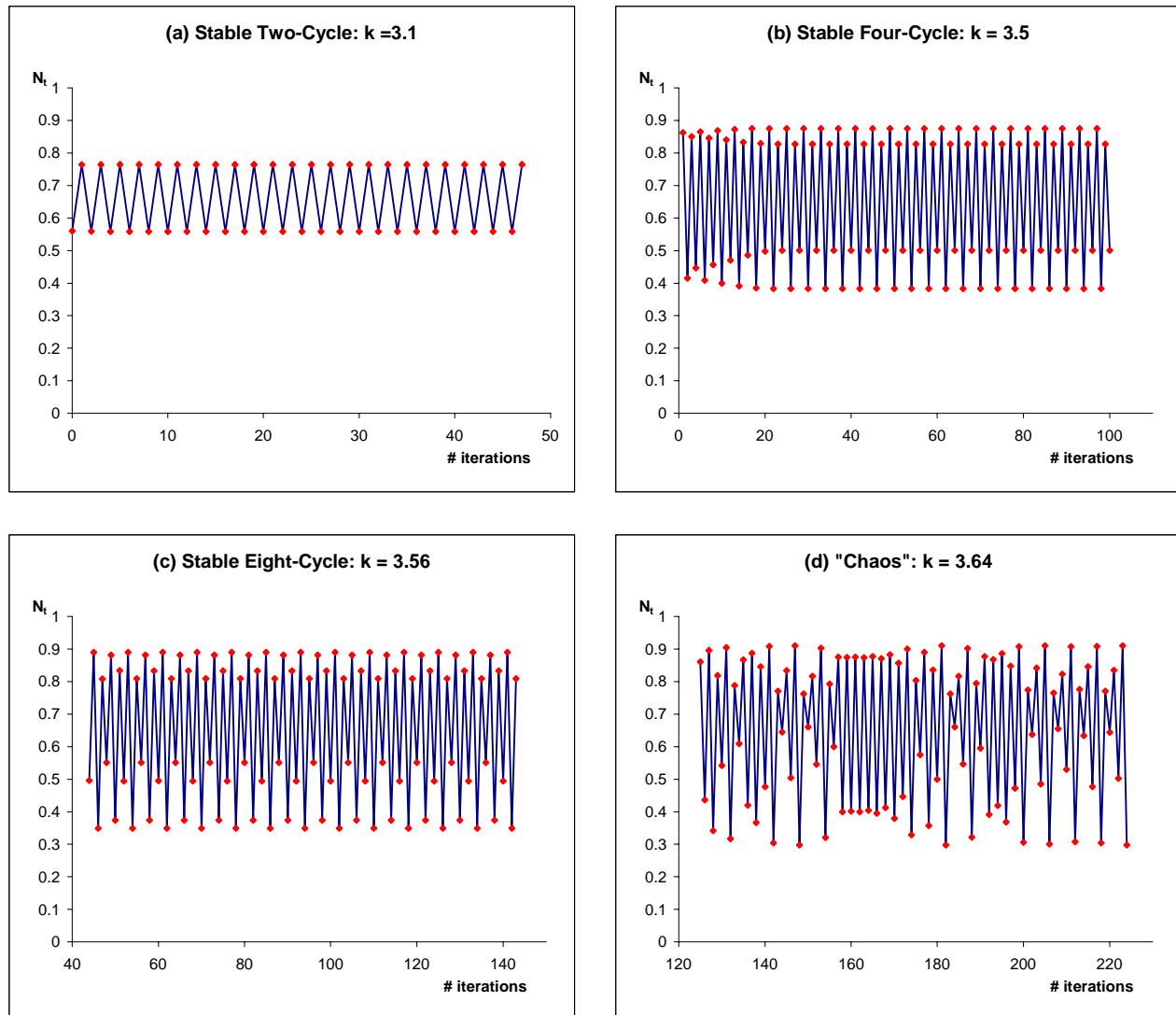
*Solution:*

To find the fixed points we must solve the equation  $f(N) = N$ , where  $f(N) = kN(1 - N)$ . The solutions are  $N = 0$  and  $N = \frac{k-1}{k}$ . Since  $f'(N) = k - 2kN$  we find  $f'(0) = k$  and  $f'(\frac{k-1}{k}) = 2 - k$ . According to Theorem 17.6 the steady state  $N = 0$  will be stable when  $0 \leq k < 1$  and unstable when  $1 < k \leq 4$ . Using the same theorem we can classify the other steady state. We will have  $|2 - k| < 1$  provided  $-1 < 2 - k < 1$  or  $1 < k < 3$ . Thus, for those values of  $k$  the steady state  $\frac{k-1}{k}$  will be stable. For the remaining  $k$  the steady state is unstable. Notice that the two functions examined in Example 17.1 illustrate both behaviors for the second fixed point. ■

The result of the previous example contrasts with the behavior of the steady state solutions to the logistic differential equation  $\frac{dN}{dt} = rN(1 - \frac{N}{K})$ . For the latter, when  $r$  and  $K$  are positive the steady states  $N \equiv 0$  and  $N \equiv K$  are unstable and stable, respectively. Moreover, in this case all solutions approach the steady state  $N \equiv K$  monotonically. For the discrete logistic map the dynamics are more complicated. We have already seen in Figure 17.3 that the trajectory of a solution that approaches a steady state may oscillate around the steady state as it gets closer to it. For the difference equation illustrated in that figure, namely  $N_{t+1} = 2.9N_t(1 - N_t)$ , we show the oscillating trajectory below.



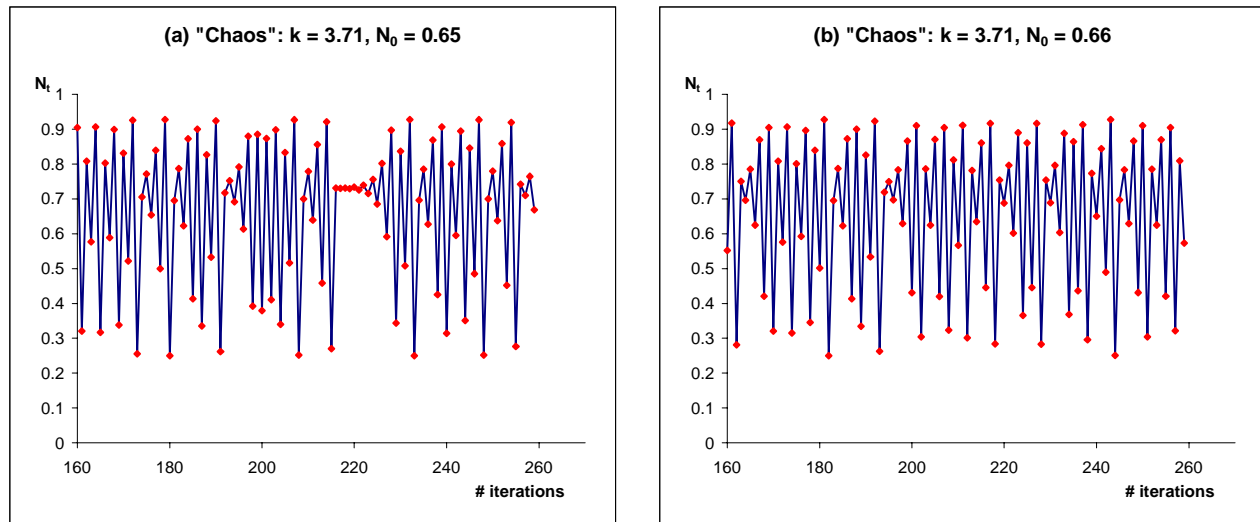
However, when the parameter  $k$  in the equation  $N_{t+1} = kN_t(1 - N_t)$  becomes larger than three altogether unexpected behavior emerges in the solution of the difference equation. Some of these patterns are illustrated below. In each case the initial value  $N_0 = 0.56$ .



**Figure 17.4 Some Dynamics of the Discrete Logistic Map**

In each of the situations illustrated in Figure 17.4 since  $k > 3$  there is an unstable steady state at  $N^* = k^{-1}/k$ . In (a) through (c), although the solutions tend away from the unstable steady state they still display a regular behavior. In (a) the solutions oscillate around two definite values, which we call a *stable two-cycle*. In (b) and (c) the solution of the difference equation becomes periodic with periods of 4 and 8 respectively. Notice as in (c) that ***the periodic limiting behavior may only become evident after a large number of iterations.*** From the population viewpoint these correspond to populations that in the long-term show periodic cycles in their size. Finally (d)

illustrates so-called chaotic behavior. The solutions of the difference equation appear to follow a random pattern, although they are generated by a perfectly definite process. Even more surprising in this case is that changing the initial value by a small amount may drastically affect the long-term behavior of the solution. This is usually characterized by saying that the difference equation exhibits sensitive dependence on initial conditions. Figure 17.5 illustrates this behavior.



**Figure 17.5 Sensitive Dependence on Initial Conditions**

The discovery that a relatively simple mathematical system such as the discrete logistic map can produce enormously varied solution patterns has had a profound impact on the modeling of complex systems. On the one hand, it shows that simple deterministic mechanisms can produce observable results with great complexity. On the other hand, even knowing the mechanism behind such systems, we may be unable to accurately predict their long-term behavior. Indeed, we have seen above that chaotic systems exhibit sensitive dependence on initial conditions that over time will greatly magnify small errors in initial measurements. Nevertheless, as might be expected, even amidst the chaotic surface some statistical regularity persists. See Exercise 22.

Although a thorough analysis of the discrete logistic map is beyond the level of this book, we can give some idea as to the mathematical issues involved. We consider the problem of finding two-cycles for the logistic map.

**Definition 17.4:** If the repeating sequence  $N_0, N_1, N_0, N_1, \dots$  (with  $N_1 \neq N_0$ ) is a solution of a difference equation  $N_{t+1} = f(N_t)$  then we say that the pair  $(N_0, N_1)$  forms a *two-cycle*. ■

Whereas a steady state solution is a constant solution of a difference equation, a two-cycle is a pair of numbers that alternate in a solution sequence. Thus the condition for a pair to be a two-cycle is that

$$N_1 = f(N_0) \text{ and } N_0 = f(N_1). \quad (17.13)$$

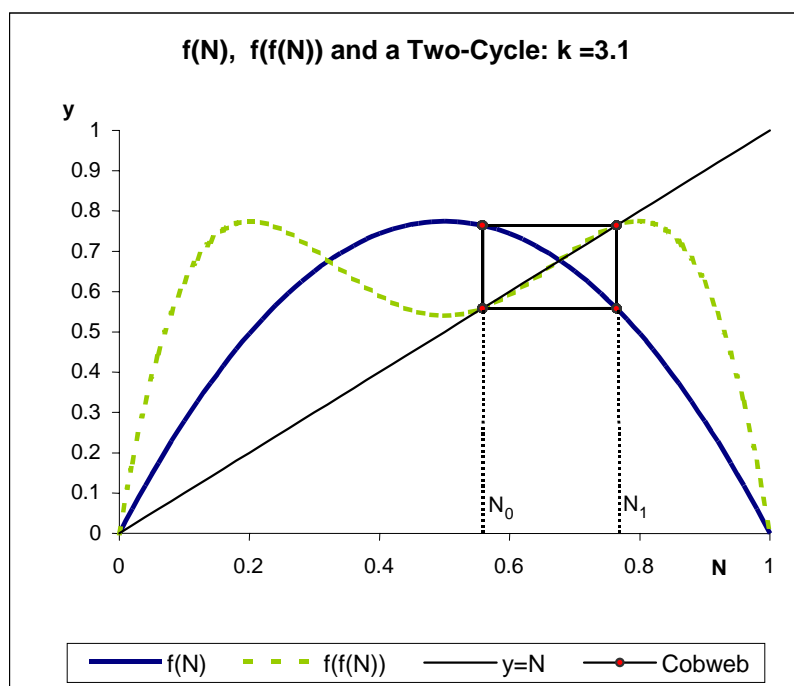
Substituting the first equation in (17.13) into the second gives that

$$N_0 = f(f(N_0)). \quad (17.14)$$

Conversely, if  $N_0$  satisfies (17.14) and is not a fixed point of  $f$  then defining  $N_1 = f(N_0)$  the pair  $(N_0, N_1)$  satisfies (17.13) and therefore forms a two-cycle. The condition (17.14) says that  $N_0$  is a fixed point for the composite function  $f(f(N))$ . We have thus arrived at the following characterization of a two-cycle:

**Theorem 17.7:** A pair  $(N_0, N_1)$  forms a two-cycle for the difference equation  $N_{t+1} = f(N_t)$  if and only if  $N_0$  is a fixed point of the function  $f(f(N))$ , but not a fixed point of  $f$ , and  $N_1 = f(N_0)$ . ■

We illustrate below the graphical content of Theorem 17.7 by showing the cobweb diagram related to the two-cycle shown in Figure 17.4a. In this graph we include the composite function  $f(f(N))$  as well as the graphs of  $f(N)$  and  $y = N$ . The two-cycle occurs at abscissa values  $N_0$  and  $N_1$  that are not fixed points of  $f$ , but where the graph of the composite function  $f(f(N))$  meets the line  $y = N$ .



**Figure 17.6**

Theorem 17.7 provides the key to finding the two-cycles for the logistic difference equation. The interested reader can consult some of the references on non-linear dynamics for additional details.



In the exercises we explore some of the longer cycle structures that exist for the logistic difference equation. If a population is governed by such dynamics then it may be vulnerable to extinction through random events during periods when its numbers are low. We will consider such scenarios in Chapter 18.

### 17.6 Tech Notes

A spreadsheet provides a convenient environment in which to calculate values of a difference equation. We illustrate this by displaying the setup needed to produce the values in Example 17.1.

**Example 17.9:** Use *Excel* to generate the first five values of the difference equation  $N_{t+1} = 1.2N_t$ , if  $N_0 = 1.5$ .

*Solution:*

	A	B
1	1.5	Initial N
2	1.8	
3	2.16	
4	2.592	
5	3.1104	

	A	B
1	1.5	Initial N
2	=1.2*A1	
3	=1.2*A2	
4	=1.2*A3	
5	=1.2*A4	

**Table 17.1**


The left side of Table 17.1 shows the entries in the spreadsheet as they actually appear. The initial value of  $N$  is labeled in column B. On the right side are the actual formulas. In practice, only the formula in cell A2 has to be explicitly typed. The remaining formulas in column A are obtained by copying the formula in cell A2 into as many cells as one wishes. ■

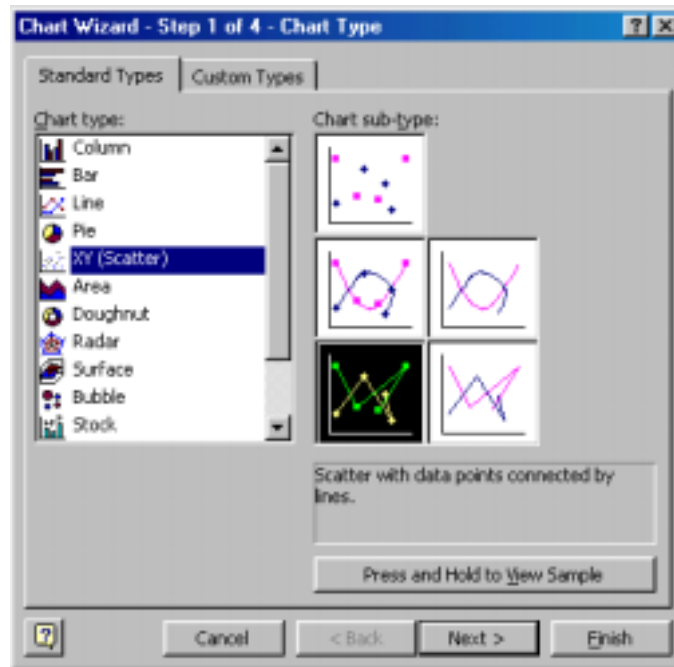
Once we have generated numerical values for the solution of a difference equation it is a straightforward process to produce a trajectory of the sort shown in Example.17.2.

**Example 17.10:** Produce a plot of the trajectory for the difference equation in Example 17.9.

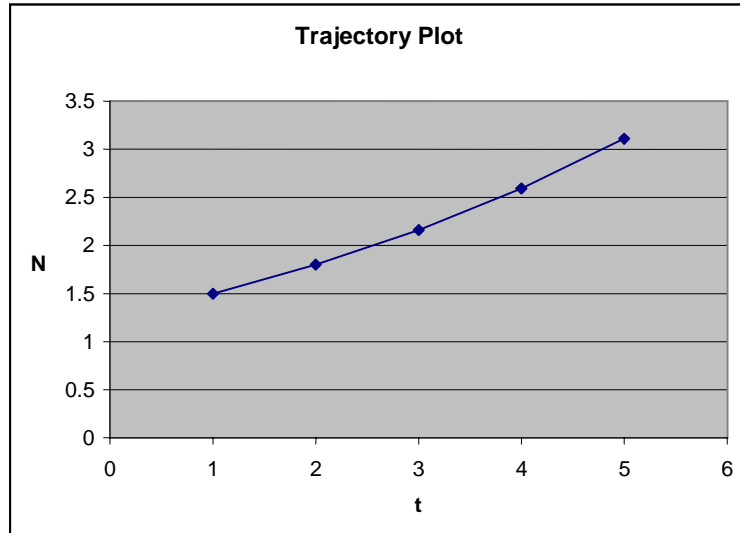
*Solution:*

The method is similar to the preparation of scatter plots discussed in the Tech Notes of Chapter 8.

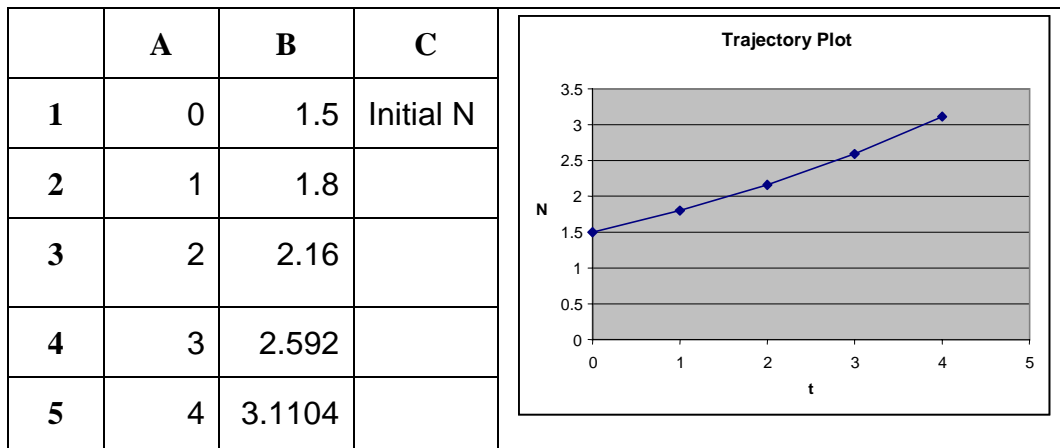
Select the data you wish to plot and then click on the chart wizard icon . The first screen of the chart wizard should be set as shown below, using a scatter plot with the points connected by straight lines.



The remaining steps are done as in our earlier work with bivariate scatter plots. After giving a title, labeling the axes and eliminating the legend we obtain the graph shown below.



Note that we did not specifically give *Excel* any data for the horizontal time axis. In this case the program simply marks the values at 1, 2, 3, etc. If this is not satisfactory then an additional column of time values should be inserted to the left of the values of the difference equation. This may be done initially when the spreadsheet is first setup or it may be inserted later using the Insert menu. When creating the trajectory using the Chart Wizard we then select the time column and the values of  $N$  as the data we want to plot. The figure below shows the relevant modifications to the Table 17.1 and the resulting relabeled trajectory graph.



As shown above, the computation of the solution to a difference equation is in principle extremely easy to carry out. However, as we have noted in the text, some difference equations are very sensitive to initial conditions. *Since computations on a computer inevitably involve some error due to round off we must be extremely careful in accepting the accuracy of a long series of computations.*

The *Excel* file *cobweb.xls* provides a convenient tool for studying families of difference equations, such as the logistic map. One can also use it to study trajectory and cobweb maps for a specific difference equation of the user's choice. Note that in this file the variable  $x$  is used for the population, rather than  $N$  as we have done in the text. Nonetheless, the reader is urged to practice the computational and graphical techniques outlined above so that he or she will be prepared to handle more complicated difference schemes (for example, 2<sup>nd</sup> order difference equations) that are important in applications but have not been discussed in the text.

### 17.7 Summary

Difference equations of the form  $N_{t+1} = f(N_t)$  play an important role in the study of systems that change at discrete time intervals. This contrasts with differential equation models of the form  $\frac{dN}{dt} = g(N)$  that are used to model many continuously varying phenomena. The table below summarizes some of the similarities and differences in the mathematics associated with these approaches.

	Differential Equation $\frac{dN}{dt} = g(N)$	Difference Equation $N_{t+1} = f(N_t)$
<i>Steady States</i>	Constant solution : $g(N^*) = 0$	Constant solution: $f(N^*) = N^*$
<i>Stability Analysis I</i>	Use ordinates on graph of $g(N)$	Use cobwebbing on graph of $f(N)$
<i>Stability Analysis II</i>	$g'(N^*) < 0$ stable (See exercise 32, chapter 4)	$ f'(N^*)  < 1$ stable
<i>Stability Analysis III</i>	$g'(N^*) > 0$ unstable	$ f'(N^*)  > 1$ unstable
<i>Trajectories</i>	Solutions are all increasing or decreasing towards or away from steady states.	Solutions may exhibit oscillations, cycling or chaotic behavior.

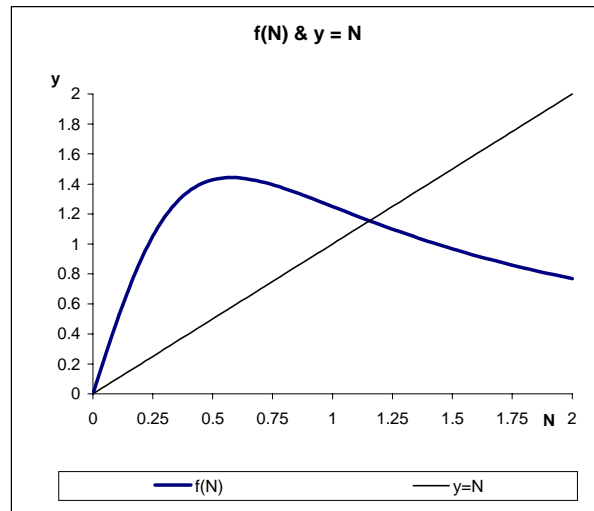
### 17.8 Exercises

In each of problems 1 – 5 find the terms  $N_1, N_2, N_3, N_4, N_5$  for the given difference equation. You may round numerical answers to three decimal places

1.  $N_{t+1} = 2N_t + 1, N_0 = 1$
2.  $N_{t+1} = 10N_t e^{-N_t}, N_0 = 5$
3.  $N_{t+1} = N_t^2 - 2, N_0 = -1.1$
4.  $N_{t+1} = 1.25N_t^2(2 - N_t), N_0 = .75$
5.  $N_{t+1} = \frac{3N_t^2}{1 + N_t^2}, N_0 = 0.5$

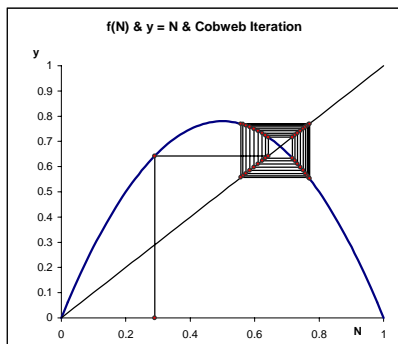
6. Given the graph of  $f(N)$  below:

- a) Estimate the steady state solutions of the difference equation  $N_{t+1} = f(N_t)$
- b) Use cobwebbing to determine the behavior of the solution to the difference equation that has initial value
  - i)  $N_0 = 0.5$  and
  - ii)  $N_0 = 1.5$ .
- c) How would you classify the stability of the steady states that you found in a)?

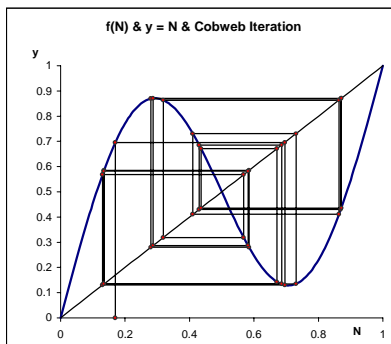


**Exercise 6**

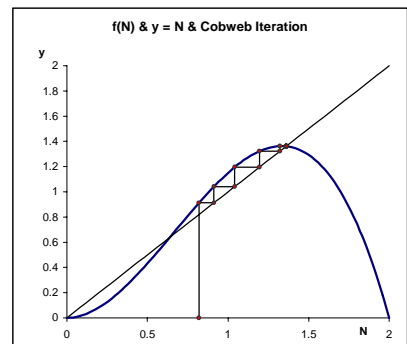
7. In the array below, match the cobweb diagram in the first row with the corresponding trajectory given in the second row.



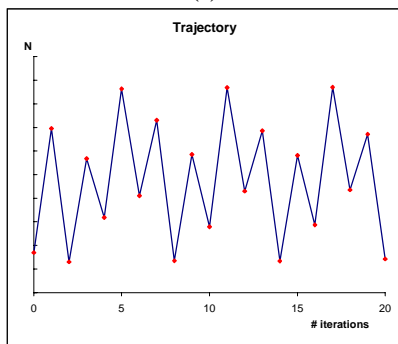
(I)



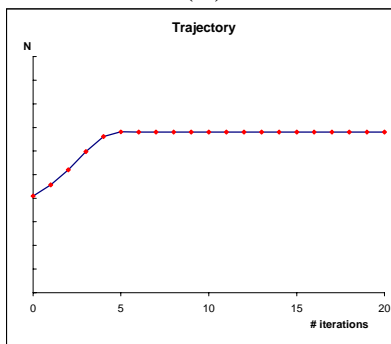
(II)



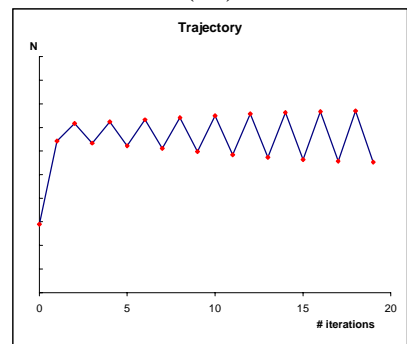
(III)



(a)



(b)

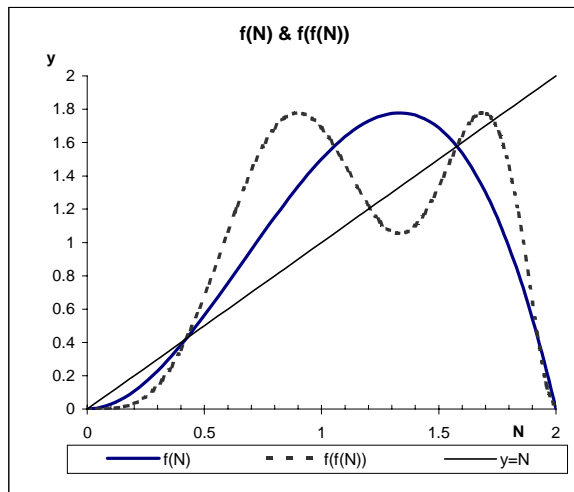


(c)

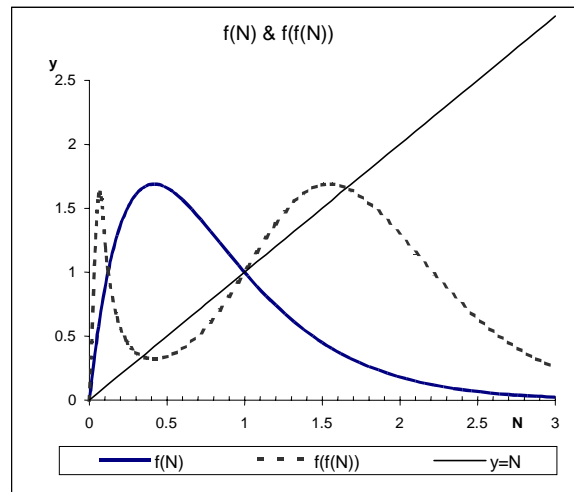
8. a) Suppose a bank pays 5% interest at the end of each year on principal that has been left on deposit the entire year. If an account is opened on January 1<sup>st</sup> with a balance of \$1000, how much will be in the account after 10 years (assuming no additional deposits except interest.)

- b) With the scenario as described in 8a) how many years are needed to double the initial principal?
- c) Compute the corresponding quantities requested in parts a) and b) if the interest rate of 5% is compounded continuously.
- 9 In this exercise we derive the solution of the difference equation  $N_{t+1} = (1+r)N_t - h$  that was discussed in Theorem 17.4.
- a) If  $N_t$  is an arbitrary solution of the difference equation  $N_{t+1} = (1+r)N_t - h$  show that the quantity  $M_t = N_t - \frac{h}{r}$  satisfies the equation  $M_{t+1} = (1+r)M_t$ .
- b) Using a) and Theorem.17.1 deduce a formula for  $M_t$  and then find the formula for  $N_t$  given in Theorem 17.4.
- c) Use a similar “trick” as in a) to reduce the solution of the differential equation  $\frac{dN}{dt} = rN - h$  to the simpler equation for exponential growth  $\frac{dy}{dt} = ry$ .
- 10 The difference equation  $N_{t+1} = (1+r)N_t - h$  provides a precise description of the outstanding balance for car loans and mortgages. In such agreements an initial amount  $N_0$  is lent to the borrower at an annual interest rate  $r$ , which we will assume is fixed for the duration of the loan. The loan is paid back in equal monthly installments  $h$ . Each month the outstanding balance increases by the interest owed  $N_t \frac{r}{12}$  (the monthly interest rate is a 12<sup>th</sup> of the annual interest rate  $r$ .) and is reduced by the monthly payment  $h$ .
- a) Show that the balance remaining at month  $t$ , denoted by  $N_t$ , satisfies the difference equation  $N_{t+1} = (1 + \frac{r}{12})N_t - h$ .
- b) Letting  $p = \frac{r}{12}$  in the equation in part a) use Theorem 17.4 to show that  $N_t = (1+p)^t (N_0 - \frac{h}{p}) + \frac{h}{p}$ .
- c) If the loan is to be repaid in  $t$  months, in other words after  $t$  months the balance  $N_t$  is zero, show that the monthly payment  $h$  is given by the formula
- $$h = \frac{p(1+p)^t N_0}{(1+p)^t - 1} = \frac{pN_0}{1 - (1+p)^{-t}}.$$
- d) Suppose the annual interest is 6%, the initial loan is for \$20,000 and the payment period is 3 years. Using the result in c) find the required monthly payment. Repeat the calculation if the repayment period is 4 years.
- e) For each of the payment scenarios given in d) determine the total payments over the life of the loan and the total interest paid in each case.

11. Find all steady states and classify their stability for the difference equation  $N_{t+1} = 2N_t + 1$  (see Exercise 1).
12. Find all steady states and classify their stability for the difference equation  $N_{t+1} = 10N_t e^{-N_t}$  (see Exercise 2).
13. Find all steady states and classify their stability for the difference equation  $N_{t+1} = N_t^2 - 2$  (see Exercise 3).
14. Find all steady states and classify their stability for the difference equation  $N_{t+1} = 1.25N_t^2(2 - N_t)$  (see Exercise 4).
15. Find all steady states and classify their stability for the difference equation  $N_{t+1} = \frac{3N_t^2}{1 + N_t^2}$  (see Exercise 5).
16. Based on the graphs of  $f(N)$  and  $f(f(N))$  below determine whether the associated difference equation  $N_{t+1} = f(N_t)$ 
  - a) Has a two-cycle.
  - b) If so, estimate the points  $N_0$  and  $N_1$  in the two-cycle
  - c) If there is a two-cycle, use cobwebbing to assess its stability. (Stability means that trajectories beginning at values near the two-cycle are eventually drawn into it. Equivalently, each point in the two-cycle is a stable fixed point of  $f(f(N))$ .)



(I)



(II)

17. The discrete logistic model  $N_{t+1} = (1+r)N_t(1 - \frac{N_t}{K})$  (where  $r > 0$  and  $K > 0$ ) has the obvious flaw for a population model that if  $N_t$  exceeds  $K$  the value of  $N_{t+1}$  becomes negative. One way to avoid this is via the Ricker model (used in fishery management) which has the form  $N_{t+1} = (1+r)N_t e^{(1-\frac{N_t}{K})}$ .

a) Show that if  $N_t$  satisfies the Ricker equation then the quantity  $M_t = \frac{N_t}{K}$  satisfies the equation  $M_{t+1} = e(1+r)M_t e^{-M_t} = sM_t e^{-M_t}$ , where  $s = e(1+r) > e$ . (See Exercise 12.) If  $K$  represents the carrying capacity, what interpretation can one place on the quantity  $M_t$ ?

b) Show that the adjusted Ricker model  $M_{t+1} = sM_t e^{-M_t}$  has steady states of  $M^* = 0$  and  $M^* = \ln s$ .

c) Show that the steady state  $M^* = 0$  is unstable for any  $s$  (remember  $s > e$ ). Show that the steady state  $M^* = \ln s$  is stable if  $s < e^2 \approx 7.389$  and unstable if  $s > e^2$ .



d) Using the parametric family  $f(x) = sx e^{-x} = sx \exp(-x)$  in the *cobweb.xls* file, show that when  $s > 7.38$  the difference equation  $M_{t+1} = sM_t e^{-M_t}$  has a two-cycle. Using the same file estimate the points in this two-cycle when  $s = 8$ .



e) The two-cycle you found in d) when  $s = 8$  is stable, in the sense that trajectories beginning at values near the two-cycle are eventually drawn into it. As the parameter  $s$  increases show that the two-cycle eventually becomes unstable and instead a stable 4-cycle appears. At approximately what value of  $s$  do we first see a 4-cycle?



f) Show that as  $s$  increases the stable 4-cycle eventually becomes unstable and a stable 8-cycle emerges. This process of “bifurcation” (splitting in two) continues with progressively smaller gaps between the parameter values at which the new period doubling cycle appears. Eventually the trajectories appear chaotic. At approximately what value of  $s$  does this occur?



g) Within the so-called chaotic range for the parameter  $s$  found in f) there are values of the parameter for which the difference equation  $M_{t+1} = sM_t e^{-M_t}$  has cycles whose period is not of length 2, 4, 8, etc. Find two examples of such parameter values and the periods of the relevant cycles. Based on the graphical evidence do you think these cycles are stable or unstable?



18. a) When  $k > 3$  the steady state  $N^* = \frac{k-1}{k}$  of the logistic difference equation  $N_{t+1} = kN_t(1 - N_t)$  is unstable and we have seen in section 17.5 that cycle structures emerge. Using the file *cobweb.xls* find approximate values for the stable two-cycle when  $k = 3.25$ ..





b) As the parameter  $k$  increases show that the two-cycle eventually becomes unstable and a stable 4-cycle appears. At approximately what value of  $k$  does the 4-cycle appear? When a stable 4-cycle appears, is it possible to still display the two-cycle on the computer generated cobweb diagram? Do so, or explain why it can't be done.



c) When a stable 4-cycle appears, is it possible to still display the two-cycle on the computer generated cobweb diagram? Do so or explain why it can't be done.



d) Show that as  $k$  increases the stable 4-cycle eventually becomes unstable and an a stable 8-cycle emerges. This process of “bifurcation” (splitting in two) continues with progressively smaller gaps between the parameter values at which the new period doubling cycle appears. Eventually the trajectories appear chaotic. At approximately what value of  $k$  does this occur?



e) Within the so-called chaotic range for the parameter  $k$  found in d) there are values of the parameter for which the difference equation  $N_{t+1} = kN_t(1 - N_t)$  has cycles whose period is not of length 2, 4, 8, etc. Find two examples of such parameter values and the periods of the relevant cycles. Based on the graphical evidence do you think these cycles are stable or unstable?



19. a) Theorem 17.6 gives no information about the stability of a steady state for which  $|f'(N^*)| = 1$ . For the discrete logistic difference equation  $N_{t+1} = kN_t(1 - N_t)$  such a steady state occurs when  $k = 3$ , where  $N^* = \frac{2}{3}$ . Using the file *cobweb.xls* and the logistic family determine graphically whether this steady state is stable or unstable.



b) The difference equation  $N_{t+1} = N_t^2(2 - N_t)$  has a steady state solution  $N^* = 1$ . Does Theorem 17.6 provide a classification of this steady state? If not, use the file *cobweb.xls* to assess the stability.

20. Suppose a population follows the discrete model  $N_{t+1} = f(N_t)$  with initial value  $N_0$ . Assuming  $N_1 = f(N_0)$  exceeds the initial population we can harvest the difference. Thus, letting  $h = f(N_0) - N_0$  the harvested population follows the model (see Exercise 17, Chapter 5 for the continuous analog).

$$N_{t+1} = f(N_t) - h. \quad (17.15)$$

a) Show that (17.15) (with  $h = f(N_0) - N_0$ ) has  $N_0$  as a steady state solution. The harvest  $h$  is said to be *sustainable*.

b) Show that the *maximum sustainable yield*  $h_{\max}$  occurs when the initial population  $N_0$  satisfies  $f'(N_0) = 1$ .

c) Suppose the unharvested population follows a logistic growth equation  $N_{t+1} = 3.5N_t(1 - N_t)$ . Using b) find the exact value of the maximum sustainable yield  $h_{\max}$ .



- d) Investigate the dynamics of the population model  $N_{t+1} = 3.5N_t(1 - N_t) - h_{\max}$ , where  $h_{\max}$  is the maximum sustainable yield found in c). In particular, characterize the stability of the steady state. What aspect of the stability might make it risky to harvest the maximum sustainable yield?

21. Difference equation models of the form  $N_{t+1} = f(N_t)$  assume that the population at time  $t+1$  is determined solely by the population at time  $t$ . In some situations, however, a delay effect may be appropriate so that the population at time  $t+1$  is determined by the populations at time  $t$  and at time  $t-1$ . These models are called *delay difference equations*. In this exercise we study a delay form of the (adjusted) Ricker model  $N_{t+1} = sN_t e^{-N_t}$  (see Exercise 17), namely

$$N_{t+1} = sN_t e^{-N_{t-1}} \quad (17.16)$$

- a) If  $s = 6$ ,  $N_0 = 3$ ,  $N_1 = 4$ , compute the values  $N_2, N_3, N_4$  and  $N_5$  determined by (17.16).  
 b) A steady state solution of a delay difference equation is a constant solution of the equation. Show that the delay model (17.16) has the steady states  $N^* = 0$  and  $N^* = \ln s$ . (See Exercise 17b)



- c) Set up an *Excel* spreadsheet to generate 100 values for the delay difference equation (17.16) with the values of  $s$ ,  $N_0$ , and  $N_1$  as given in a). Create a plot of the trajectory. Compare the behavior with the corresponding behavior of the standard Ricker model  $N_{t+1} = sN_t e^{-N_t}$  with the same  $s$  and  $N_0$ . (N.B. In order to allow for conveniently changing the value of  $s$ , place it in its own cell and refer to it in formulas via an absolute reference.)



- d) Using your spreadsheet model in c) with  $s = 6$ , determine whether the steady state  $N^* = \ln s$  is stable or unstable for the delay Ricker equation. Compare with the stability for the corresponding standard Ricker equation with the same  $s$ .



- e) Change  $s$  in your spreadsheet model using some values from 3 to 10 with various initial conditions (values of  $N_0$  and  $N_1$ ). Describe the population dynamics that result.

22. Examine the trajectory outputs for the logistic system  $N_{t+1} = 3.71N_t(1 - N_t)$  with  $N_0 = 0.65$  and with initial value  $N_0 = 0.66$ . (The trajectory values can be found on the calculation sheet of the file *cobweb.xls* or they can be generated easily using the methods described in the Tech Notes section.). Show via histograms, means, medians, etc. that the two data sets have virtually the same distribution of values.

