

# 1 Calculus Review

*Ms. M:* So  $y = r^3 / 3$ . And if you determine the rate of change in this curve correctly, I think you'll be pleasantly surprised.

*Class:* [chuckles]

*Ms. M:* Don't you get it, Bart?  $dy = 3r^2 / 3 dr$ , or  $r^2 dr$ , or  $r dr r$ . Har-de-har-har, get it?

From: *The Simpsons*

## 1.1 Differentiation

In this chapter we review some basic computational techniques from calculus. Of course, it takes more than mechanical skills to properly use these tools. One needs some understanding of the ideas behind the manipulations. We will review those in later chapters as the need develops. Let's begin by recalling some fundamental techniques of differentiation. We make no attempt here to justify these procedures.

The computation of derivatives relies on two components:

- *formulas* for the derivatives of “elementary” functions
- *rules* explaining how to differentiate functions that are put together from the elementary ones

Let's start with the formulas. In fact for our purposes we need only the following three:

(1.1) $\frac{d}{dx}(x^k) = kx^{k-1}$	(1.2) $\frac{d}{dx}(e^x) = e^x$	(1.3) $\frac{d}{dx}(\ln x) = \frac{1}{x}$
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Here we have used the differential notation to indicate the derivative. Sometimes it is more convenient to use the “prime” notation  $(e^x)' = e^x$  as shorthand for the derivative. In (1.1) it is important to realize that the exponent  $k$  can be any real number, not just a positive integer.

**Example 1.1:** Find the derivative of  $\frac{1}{\sqrt[3]{x}}$ .

*Solution:*

The expression is first converted to pure exponential form  $\frac{1}{\sqrt[3]{x}} = x^{-1/3}$ . We now use (1.1) to carry

out the differentiation:  $\frac{d}{dx}(x^{-1/3}) = -\frac{1}{3}x^{-4/3} = -\frac{1}{3x^{4/3}}$ . The last conversion is optional. Generally it is more convenient to work algebraically with a fractional form having no negative exponents, but this depends on the final objectives. ■

To proceed further requires the basic rules of differentiation.

**Rule 1.1:** The derivative of a sum is the sum of derivatives. In symbols:  $(f(x) + g(x))' = f'(x) + g'(x)$ . A similar result applies to sums of three or more functions. ■

**Rule 1.2:** If  $c$  is a constant then  $(c f(x))' = c f'(x)$ . In words, the derivative of a constant times a function is the same constant times the derivative of the function. ■

These rules allow one to differentiate polynomials as well as other combinations of the elementary functions listed above.

**Example 1.2:** Find the derivative of each of the following expressions:

a)  $x^3 - 4x^2 + 1$                       b)  $\frac{2}{x^2} + 3 \ln x$                       c)  $\frac{3\sqrt{t} - 2t^3 + 1}{t^2}$

*Solution:*

- a) The derivative computation is a direct application of Rules 1.1 and 1.2. This is sometimes referred to as term-by-term differentiation.

$$\frac{d}{dx}(x^3 - 4x^2 + 1) = \frac{d}{dx}(x^3) - 4 \frac{d}{dx}(x^2) + \frac{d}{dx}(1).$$

Equation (1.1) (using  $k = 0$ ) and Rule 1.2 imply that the derivative of any constant is zero. The derivatives of the other terms are obtained directly from (1.1). Thus,

$$\frac{d}{dx}(x^3 - 4x^2 + 1) = \frac{d}{dx}(x^3) - 4 \frac{d}{dx}(x^2) + \frac{d}{dx}(1) = 3x^2 - 8x.$$

- b) Convert the fractional term to exponent form as we did in Example 1.1; then differentiate term-by-term.

$$\frac{d}{dx} \left( \frac{2}{x^2} + 3 \ln x \right) = \frac{d}{dx}(2x^{-2}) + 3 \frac{d}{dx}(\ln x) = -4x^{-3} + \frac{3}{x}.$$

The last answer may be combined into a single fraction with common denominator of  $x^3$ .

$$-4x^{-3} + \frac{3}{x} = -\frac{4}{x^3} + \frac{3}{x} = \frac{-4 + 3x^2}{x^3}.$$

- c) Here the independent variable is  $t$  so we need to compute  $d/dt$ . Although the expression is a fraction, since the denominator is a pure power we can convert the fraction into a sum by dividing each term in the numerator by the denominator  $t^2$ . Note that this is exactly the reverse of the simplification process we carried out at the conclusion of b).

$$\frac{3\sqrt{t} - 2t^3 + 1}{t^2} = \frac{3\sqrt{t}}{t^2} - \frac{2t^3}{t^2} + \frac{1}{t^2} = 3t^{-3/2} - 2t + t^{-2}.$$

The last expression can now be differentiated term-by-term.

$$\frac{d}{dt}(3t^{-3/2} - 2t + t^{-2}) = -\frac{9}{2}t^{-5/2} - 2 - 2t^{-3} = -\frac{9}{2t^{5/2}} - 2 - \frac{2}{t^3}.$$

If desired, the answer may be expressed as a single fraction. The reader should show that the final result is

$$-\frac{9t^{1/2} + 4t^3 + 4}{2t^3}. \blacksquare$$

Rules 1.1 and 1.2 are quite straightforward. To handle products and quotients more elaborate rules are necessary, since unfortunately the derivative of a product is emphatically not the product of the derivatives.

### Rule 1.3: Product Rule

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \blacksquare$$

### Rule 1.4: Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \blacksquare$$

The Product Rule has a form that is easy to recall and allows an immediate generalization. Each term on the right side contains the derivative of one of the factors on the left, with other factors left untouched. These are then added. The same principle works for three or more factors. For example,

$$(f(x)g(x)h(x))' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

The Quotient Rule is not so simple to remember. The numerator resembles the right side of the Product Rule. However, because of the subtraction we must be careful not to switch the terms. Doing so will give an incorrect sign for the answer.

**Example 1.3:** Compute the derivative of each of the following expressions:

a)  $(e^x + \ln x + 1)(x^5 - 2)$

b)  $e^{2x}$

c)  $e^{-x}$

d)  $\frac{e^x + e^{-x}}{e^x - e^{-x}}$

*Solution:*

- a) We use the Product Rule. (Alternatively, we could expand the expression into a sum and differentiate term-by-term. This is usually a bad idea as it destroys any benefit that might accrue from the factored structure of the original expression.)

$$\frac{d}{dx}((e^x + \ln x + 1)(x^5 - 2)) = \left(e^x + \frac{1}{x}\right)(x^5 - 2) + (e^x + \ln x + 1)5x^4.$$

- b) We can write  $e^{2x} = e^x e^x$ . Now we use the Product Rule.

$$\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x)e^x + e^x \frac{d}{dx}(e^x) = e^x e^x + e^x e^x = 2e^{2x}$$

The same result will be obtained below using the Chain Rule. (See Example 1.5)

- c) We can write  $e^{-x} = \frac{1}{e^x}$ . Now we use the Quotient Rule.

$$\frac{d}{dx}(e^{-x}) = \frac{\frac{d}{dx}(1)e^x - 1 \frac{d}{dx}(e^x)}{e^{2x}} = \frac{-e^x}{e^{2x}} = -\frac{1}{e^x} = -e^{-x}$$

The result may also be obtained more easily using the Chain Rule as in Example 1.5.

$$d) \quad \frac{d}{dx} \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right) = \frac{(e^x + e^{-x})'(e^x - e^{-x}) - (e^x + e^{-x})(e^x - e^{-x})'}{(e^x - e^{-x})^2} \quad (1.4)$$

We use the result in c) (together with Rule 1.1) to differentiate the terms in the numerator. We have  $(e^x + e^{-x})' = e^x - e^{-x}$  and  $(e^x - e^{-x})' = e^x + e^{-x}$ . Substituting these in the right side of (1.4) gives

$$\frac{d}{dx} \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right) = \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2}.$$

This answer can be considerably simplified. Squaring the terms in the numerator gives

$$(e^x - e^{-x})^2 = e^{2x} - 2e^x e^{-x} + e^{-2x} = e^{2x} - 2 + e^{-2x},$$

since  $e^x e^{-x} = e^0 = 1$ . Similarly,

$$(e^x + e^{-x})^2 = e^{2x} + 2e^x e^{-x} + e^x = e^{2x} + 2 + e^{-2x}.$$

On subtracting we get  $(e^x - e^{-x})^2 - (e^x + e^{-x})^2 = -4$ .

This gives as our final answer

$$\frac{d}{dx} \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right) = -\frac{4}{(e^x - e^{-x})^2}. \blacksquare$$

Our final rule, the Chain Rule, is the most powerful and also the most difficult to apply and state. Suppose we want to find the derivative of  $y = (3x^2 - 2)^5$ . Consider first how this expression is evaluated. The evaluation process involves two steps:  $x \rightarrow (3x^2 - 2) = u \rightarrow u^5 = y$ . In effect, the variable  $u$  serves as a link between the independent variable  $x$  and the dependent variable  $y$ . The linking variable  $u$  is a dependent variable in the defining equation  $u = 3x^2 - 2$  and an independent variable in the equation  $y = u^5$ . If we know how to differentiate each of these links, the Chain Rule shows us how to find  $dy/dx$ .

**Rule 1.5: Chain Rule**

Suppose  $y$  is a function of a variable  $u$ , and  $u$  is in turn a function of the independent variable  $x$ . Then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}. \blacksquare$$

Observe the powerful symbolism. The derivatives that are multiplied on the right side appear to symbolically yield  $dy/dx$  through cancellation of the expression  $du$ . Although not mathematically accurate, this is a useful fiction for remembering and generalizing the result (See Example 1.4d below).

**Example 1.4:** Find the derivatives of the following expressions.

a)  $y = (3x^2 - 5)^5$

b)  $y = e^{x^2}$

c)  $y = \ln(x^2 + 1)$

d)  $y = e^{-\sqrt{\ln x}}$

*Solution:*

- a) Introduce the intermediary variable  $u = 3x^2 - 5$ , as discussed above. We can write  $y = u^5$ , and  $u = 3x^2 - 5$ . The Chain Rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 5u^4 (6x) = 5(3x^2 - 5)^4 6x = 30(3x^2 - 5)^4 x.$$

Note that in the expression  $dy/du$  it is necessary to substitute for  $u$  its dependence on  $x$ . The final answer should only involve the variable  $x$ .

b) Introduce the intermediary variable  $u = x^2$ . We then have  $y = e^u$  and  $u = x^2$ . The Chain Rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = e^u 2x = 2xe^{x^2}.$$

c) We have  $y = \ln u$ , where  $u = x^2 + 1$ . From the Chain Rule  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} 2x = \frac{2x}{x^2 + 1}$ .

d) We have  $y = e^u$ , where  $u = -\sqrt{w}$ , and  $w = \ln x$ , a chain with two links. The Chain Rule extends to multiple links. As indicated above, the notation guides you to the correct result.

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dw} \frac{dw}{dx} = e^u \left( -\frac{1}{2} w^{-1/2} \right) \frac{1}{x} = -\frac{e^u}{2x\sqrt{w}} = -\frac{e^{-\sqrt{\ln x}}}{2x\sqrt{\ln x}}. \blacksquare$$

In Example 1.4 we followed fairly closely the pattern stipulated in the statement of the Chain Rule. In practice, introducing intermediary variables explicitly often makes the work unwieldy. To make it easier to perform these calculations without having recourse to the extra variables, it is useful to keep in mind some special cases of the Chain Rule. Once one masters these special patterns the general situation will seem quite transparent.

### Rule 1.6: Special Chain Rule

a) If  $u$  is a function of  $x$  then  $\frac{d}{dx}(u^k) = ku^{k-1}u'$ .

b) If  $u$  is a function of  $x$  then  $\frac{d}{dx}(e^u) = e^u u'$ .

c) If  $u$  is a function of  $x$  then  $\frac{d}{dx}(\ln u) = \frac{u'}{u}$ . ■

These of course all follow from the Chain Rule. For example, to demonstrate c) write  $y = \ln u$ . Then

$$\frac{d}{dx}(\ln u) = \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} u' = \frac{u'}{u}.$$

In effect, to find the derivative of  $\ln u$  with respect to  $x$  we first differentiate  $\ln u$  as if  $u$  were an independent variable, obtaining  $1/u$ . To take account that  $u$  in fact depends on  $x$  we multiply the result by  $u'$ , so in this context  $u$  plays the role of a dependent variable. A similar interpretation applies to the other statements.

**Example 1.5:** Compute derivatives of the following expressions:

a)  $e^{2x}$

b)  $e^{-x}$

c)  $(3\ln(2x+1)-2)^5$

*Solution:*

a) Here  $u = 2x$  so we find using Rule 1.6b) that  $\frac{d}{dx}(e^{2x}) = e^{2x} \frac{d}{dx}(2x) = 2e^{2x}$  (See also Example 1.3b)

b)  $\frac{d}{dx}(e^{-x}) = e^{-x} \frac{d}{dx}(-x) = -e^{-x}$  (See Example 1.3c)

c) Since the expression is a 5<sup>th</sup> power we first we use Rule 1.6:

$$\frac{d}{dx}((3\ln(2x+1)-2)^5) = 5(3\ln(2x+1)-2)^4 \frac{d}{dx}(3\ln(2x+1)-2)$$

Then we must compute the derivative of the expression  $u$  that is “inside” the power.

$$\frac{d}{dx}(3\ln(2x+1)-2) = 3 \frac{(2x+1)'}{2x+1} = \frac{6}{2x+1}.$$

Hence, our final answer:

$$\frac{d}{dx}((3\ln(2x+1)-2)^5) = \frac{30(3\ln(2x+1)-2)^4}{2x+1}. \blacksquare$$

## 1.2 Antidifferentiation or Indefinite Integration

Given a function  $f(x)$ , can we find a function  $F(x)$  with the property that  $F'(x) = f(x)$ ? Such a function  $F(x)$  is called an *antiderivative* of  $f(x)$ . In general, finding an explicit formula for an antiderivative of  $f(x)$  is much more difficult than differentiating the same function. In fact, an explicit antiderivative in terms of elementary functions may not exist. In this section we will consider some techniques for finding antiderivatives.

**Example 1.6:** Find all antiderivatives of the function  $f(x) = x^2$ .

*Solution:*

We can easily guess an answer using what we know about differentiation. If we differentiate  $x^3$  we get  $3x^2$ , which is not quite what we want. However, dividing our initial guess by 3 will do the trick, as the unwanted 3 will be canceled. Namely,

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2.$$

Having found one antiderivative we easily construct infinitely many in the form  $\frac{1}{3}x^3 + C$ , where  $C$  is an arbitrary constant. In fact, according to a basic theorem in calculus, all antiderivatives can be obtained from any one in this way. ■

It is convenient to have a notation for the antiderivatives of a function, whether or not we can find an explicit formula. Because antidifferentiation has a fundamental connection with the evaluation of definite integrals, the integral sign, without limits of integration is used to represent the antiderivatives of a function. In this notation, the computation above would be indicated by writing

$$\int x^2 dx = \frac{1}{3}x^3 + C.$$

The symbol  $\int$  is an integral sign and  $\int x^2 dx$  is called an indefinite integral. Thus indefinite integration is simply another term for antidifferentiation. The expression  $dx$  that is part of the integral notation helps keep track of the variable of integration, as we will consider below when we discuss changing that variable. It also arises because of the connection with definite integrals. Although not absolutely essential, it is best not to omit it.

Every rule of differentiation when viewed in reverse gives rise to a rule of antidifferentiation. For instance, corresponding to the basic formulas (1.1), (1.2) and (1.3) we have

$(1.5) \int x^k dx = \frac{x^{k+1}}{k+1} + C, \text{ if } k \neq -1$	$(1.6) \int e^x dx = e^x + C$	$(1.7) \int \frac{dx}{x} = \ln x  + C$
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As with any purported indefinite integration, the answers can be checked by differentiation. Here, the right sides differentiate to give the expression following the integral sign. In (1.5) observe the restriction on  $k$ , since clearly the right side is not even defined when  $k = -1$ . The case  $k = -1$  is covered by (1.7). Note that to be technically correct, when  $x$  is negative the antiderivative of the function  $f(x) = 1/x$  is  $\ln|x|$ , not  $\ln x$ .

Rule 1.2 and Rule 1.3 give rise to corresponding principles of integration.

**Rule 1.7:**  $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx.$  ■

This statement means that an arbitrary antiderivative of the sum  $f(x) + g(x)$  can be obtained by adding antiderivatives of the summands  $f(x)$  and  $g(x)$ . From Rule 1.2, if  $F'(x) = f(x)$  then



$(cF(x))' = cF'(x) = cf'(x)$  so  $cF(x)$  is an antiderivative of  $cf'(x)$ . Writing this statement in the integral notation yields

**Rule 1.8:**  $\int c f(x)dx = c \int f(x)dx$ . ■

Operationally, this rule is usually invoked by saying that a multiplicative constant can be passed from under the integral sign.

**Example 1.7:** Compute each of the following antiderivatives:

a)  $\int (2\sqrt{x} + x^2 - 3)dx$                       b)  $\int \frac{x^2 + 2}{x} dx$

*Solution:*

a) Using Rule 1.7 we have

$$\int (2\sqrt{x} + x^2 - 3)dx = \int 2\sqrt{x}dx + \int x^2 dx - \int 3dx. \quad (1.8)$$

From Rule 1.8 and (1.5) we have  $\int 2\sqrt{x}dx = 2\int \sqrt{x}dx = 2\frac{x^{3/2}}{3/2} + C = \frac{4}{3}x^{3/2} + C$ . Similarly the

last two integrals in (1.8) evaluate as  $\int x^2 dx = \frac{1}{3}x^3 + C^*$  and  $\int 3dx = 3x + \tilde{C}$ . Adding these we can combine the three arbitrary constants into a single constant  $D$ . Thus our final answer is

$$\int (2\sqrt{x} + x^2 - 3)dx = \frac{4}{3}x^{3/2} + \frac{1}{3}x^3 - 3x + D.$$

b) We split the fractional integrand into a sum of terms with denominator  $x$  and then integrate each term as we did in a).

$$\int \frac{x^2 + 2}{x} dx = \int \left( x + \frac{2}{x} \right) dx = \frac{x^2}{2} + 2\ln|x| + C. \blacksquare$$

The Product Rule used in reverse gives rise to an integration technique called integration by parts, which we will not pursue. Reversing the Quotient Rule is too restrictive to be of practical use. The Chain Rule leads to a fundamental technique of integration, known as change of variables. We will describe how this works only in the context of the special forms described in Rule 1.6. For instance, Rule 1.6b) implies that

$$\frac{d}{dx} e^{x^2} = 2xe^{x^2}. \quad (1.9)$$

Therefore, the antiderivative of the right side of (1.9) is the function  $e^{x^2}$ , or  $\int 2xe^{x^2} dx = e^{x^2} + C$ .

The factor  $2x$  in the integrand “disappears” in the answer for the same reason that the same factor appears in the differentiation (1.9). Both are consequences of the Chain Rule. We can then reinterpret Rule 1.6 in the language of integration.

**Rule 1.9:** Suppose  $u$  is a function of  $x$ , then

a)  $\int u^k u' dx = \frac{u^{k+1}}{k+1} + C$ , provided  $k \neq -1$ .

b)  $\int e^u u' dx = e^u + C$ .

c)  $\int \frac{u'}{u} dx = \ln|u| + C$ . ■

Applying these rules requires that the integrand have a special form. In any specific integration the difficulty lies in matching the integrand to one of these templates, perhaps after making some preliminary manipulations. To do this with assurance and efficiency requires some practice. The basic strategy is to identify the part of the integrand that plays the role of  $u$  and to verify that the factor  $u'$  appears in the appropriate place. For types a) and b) this will often be clear because expression  $u$  appears in a distinctive part of the integrand. For type c) the selection of  $u$  may be more difficult (see Example 1.8c).

**Example 1.8:** Evaluate each of the following indefinite integrals:

a)  $\int 3x^2 \sqrt{x^3 + 1} dx$

b)  $\int \frac{2x}{x^2 - 1} dx$

c)  $\int \frac{dx}{x \ln x}$

*Solution:*

a) We have  $\int 3x^2 \sqrt{x^3 + 1} dx = \int 3x^2 (x^3 + 1)^{1/2} dx$ . This seems to fit the pattern of Rule 1.9a) with  $u = x^3 + 1$  and  $u' = 3x^2$ . With this selection for  $u$  we have

$$\int \overbrace{3x^2}^{u'} \underbrace{\sqrt{x^3 + 1}}_{u^{1/2}} dx = \int u^{1/2} u' dx = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (x^3 + 1)^{3/2} + C.$$

b) The key observation is to see that  $(x^2 - 1)' = 2x$ , so the numerator is exactly the derivative of the denominator. Hence we can apply Rule 1.9c) with  $u = x^2 - 1$ . This gives

$$\int \frac{2x}{x^2 - 1} dx = \int \frac{u'}{u} dx = \ln|u| + C = \ln|x^2 - 1| + C.$$

c) This is more difficult. You must first notice that  $(\ln x)' = 1/x$ , so with  $u = \ln x$  the integrand may be written

$$\frac{1}{x \ln x} = \frac{1}{x} \frac{1}{\ln x} = u' \frac{1}{u} = \frac{u'}{u}.$$

Therefore,  $\int \frac{dx}{x \ln x} = \int \frac{u'}{u} dx = \ln |u| + C = \ln |\ln x| + C$ . ■

The examples we just solved appear to require a special structure, which is indeed correct. However, we can deviate from that structure a bit and still carry out the integration.

**Example 1.9:** Evaluate  $\int x^2 \sqrt{x^3 + 1} dx$ .

*Solution:*

Proceeding as in Example 1.8a), we let  $u = x^3 + 1$ . Since  $u' = 3x^2$  the integrand is missing the factor 3 from the derivative. However, we can insert the missing factor and, so as not to change the value of the integrand, compensate by multiplying the integrand by the reciprocal  $1/3$ . Rule 1.8 allows us to pass the extra constant factor of  $1/3$  outside the integral. Thus we have,

$$\int x^2 \sqrt{x^3 + 1} dx = \int \left(\frac{1}{3}\right) (3x^2) \sqrt{x^3 + 1} dx = \frac{1}{3} \int 3x^2 \sqrt{x^3 + 1} dx.$$

The last integral was evaluated in Example 1.8a). Using the antiderivative obtained in that result we have

$$\int x^2 \sqrt{x^3 + 1} dx = \left(\frac{1}{3}\right) \left(\frac{2}{3}\right) (x^3 + 1)^{3/2} + C = \frac{2}{9} (x^3 + 1)^{3/2} + C$$
. ■

The evaluation technique described in the previous examples can be put into a more standardized form by making use of the differential term  $dx$ , which until now appears as an annoying encumbrance. The technique for doing this is called the *Method of Substitution*. Just as in the Chain Rule of differentiation, the new substitution variable  $u$  will function both as an independent and dependent variable.

**Example 1.10:** Use the Method of Substitution to evaluate  $\int x^2 \sqrt{x^3 + 1} dx$ .

*Solution:*

As above we let  $u = x^3 + 1$ . We next compute  $du$ , the differential of  $u$ , defined as  $du = u' dx$ .

$$u = x^3 + 1$$

$$du = 3x^2 dx$$

The idea of the method is to express the entire original integrand (including the differential term  $dx$ ) in terms of  $u$  and  $du$ . Clearly  $x^2\sqrt{x^3+1} dx = u^{1/2} du / 3$ , since  $x^2 dx = du / 3$ . Thus, using Rule 1.8 we have

$$\int x^2 \sqrt{x^3 + 1} dx = \int u^{1/2} du / 3 = \frac{1}{3} \int u^{1/2} du .$$

So far  $u$  has been thought of as a dependent variable. Now in the last integral we treat  $u$  as an independent variable, using (1.5) to do the evaluation. This gives

$$\int x^2 \sqrt{x^3 + 1} dx = \frac{1}{3} \int u^{1/2} du = \left(\frac{1}{3}\right) \left(\frac{2}{3}\right) u^{3/2} + C = \frac{2}{9} (x^3 + 1)^{3/2} + C . \blacksquare$$


To summarize, the Method of Substitution employs four steps:

- i) Selection of the new variable  $u$  and the equation that relates it to the original integration variable.
- ii) Transformation of the entire original integrand, including the differential term, to an expression involving the new variable  $u$  and its differential  $du$ .
- iii) Evaluation of the new integral, with  $u$  as the integration variable.
- iv) Back-substitution for  $u$  in terms of  $x$  to express the solution of the original problem.

**Example 1.11:** Try using the Method of Substitution to evaluate  $\int \sqrt{x^2 + 1} dx$ .

*Solution:*

If we let  $u = x^2 + 1$  then  $du = 2x dx$ . Expressing  $dx$  in the latter equation in terms of  $du$ , we have

  $dx = \frac{du}{2x}$ . The integral would then become  $\int u^{1/2} \frac{du}{2x}$ . **Unfortunately, the  $x$  in the denominator is not a constant and so we cannot remove it from under the integral sign.** Using the defining equation  $u = x^2 + 1$ , the extraneous  $x$  term could be eliminated. Indeed solving the latter equation for  $x$  gives  $x = \sqrt{u-1}$  which can be substituted for the  $x$  in the denominator. Thus we would arrive at the correct statement that with  $u = x^2 + 1$ ,  $\int \sqrt{x^2 + 1} dx = \int \frac{u^{1/2}}{2\sqrt{u-1}} du$ . Unfortunately, the latter integral does not appear any easier to evaluate than the original and we must conclude that

the method has failed. It turns out that a different substitution, involving trigonometric functions, succeeds in this case. However, we will not pursue the details. ■

Rule 1.9 is often expressed in differential form, which makes it easier to remember. In fact the statements appear identical to (1.5), (1.6), and (1.7), except the quantity  $u$  is in reality a dependent variable, notationally disguised to resemble an independent one.

**Rule 1.10:** If  $u$  is a function of  $x$  then

$$(1.10) \int u^k du = \frac{u^{k+1}}{k+1} + C, \text{ if } k \neq -1 \quad (1.11) \int e^u du = e^u + C \quad (1.12) \int \frac{du}{u} = \ln|u| + C \blacksquare$$

Of course since  $du = u' dx$  these formulas merely repeat Rule 1.9 with a different notation.

### 1.3 Summary

The following table summarizes the differentiation and integration rules discussed in this review. The table is arranged so that parallel rules for the two processes appear adjacent to each other.

Rule #	Differentiation	Integration
Rule 1.1 & Rule 1.7	$(u+v)' = u' + v'$	$\int (u+v) dx = \int u dx + \int v dx$
Rule 1.2 & Rule 1.8	$(cu)' = cu', c \text{ constant}$	$\int cu dx = c \int u dx$
Rule 1.3	$(uv)' = u'v + uv'$	Integration by parts (not discussed)
Rule 1.4	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$	Not used
Rule 1.6 & Rule 1.10	$\frac{d}{dx} u^k = ku^{k-1}u'$	$\int u^k du = \frac{u^{k+1}}{k+1} + C, k \neq -1,$ where $du = u' dx$
Rule 1.6 & Rule 1.10	$\frac{d}{dx} e^u = e^u u'$	$\int e^u du = e^u + C$
Rule 1.6 & Rule 1.10	$\frac{d}{dx} \ln u = \frac{u'}{u}$	$\int \frac{du}{u} = \ln u  + C$

### 1.4 Exercises

In exercises 1 to 18 compute the derivative of the given expression. Try to simplify your answer, if it seems appropriate.

1.  $x^3 - 5x^2 + \frac{3}{x} + \frac{2}{\sqrt{x}}$

2.  $x^{-2} - 3e^x + x^2$

3.  $e^x \ln x$

4.  $\sqrt{x} \ln x$

5.  $(\ln x)^2$

6.  $\frac{2x^2 + x + 1}{x^2}$

7.  $\frac{e^x}{1+x^2}$

8.  $\frac{\sqrt{x}}{2+\ln x}$

9.  $(1+3x-x^3)^5$

10.  $(1+3e^x)^4$

11.  $\frac{1}{(2x+1)^3}$

12.  $\frac{1}{1+2e^{-t}}$

13.  $\sqrt{\ln x}$

14.  $\ln(\ln x)$

15.  $\ln(e^{x^2} + 1)$

16.  $e^{1/x^2}$

17.  $e^{kx}$ ,  $k$  a constant

18.  $e^{e^{-t}}$

In exercises 19 to 37 compute the indefinite integral.

19.  $\int (x^2 - \frac{2}{x^2} + 2e^x) dx$

20.  $\int (x^3 + 1)^2 dx$

21.  $\int \left( \sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx$

22.  $\int e^{2x} dx$

23.  $\int e^{kx} dx$ ,  $k$  a constant

24.  $\int (e^{2x} + e^{-2x}) dx$

25.  $\int (e^x + e^{-x})^2 dx$

26.  $\int \frac{2x+1}{(x^2+x)^5} dx$

27.  $\int \frac{dx}{(2x+1)^3}$

28.  $\int \left( \frac{1}{2(x-1)} + \frac{2}{x} \right) dx$

29.  $\int \frac{x+1}{x+2} dx$  (\*Hint below)

30.  $\int \frac{\ln x}{x} dx$

31.  $\int x e^{-x^2} dx$

32.  $\int \frac{\sqrt{x}}{x^{3/2} + 1} dx$

33.  $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

34.  $\int \frac{e^{2x}}{e^{2x} + 1} dx$

35.  $\int \frac{x^2}{\sqrt{1+x^3}} dx$

36.  $\int \frac{(\sqrt{x}+1)^4}{\sqrt{x}} dx$

37.  $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$

\*(Hint: Using long division write the integrand in the form  $A + \frac{B}{x+2}$ , for some constants  $A$  and  $B$ .)