

Math 20300

Calculus III

Lesson 21

Double Integrals Over Rectangles

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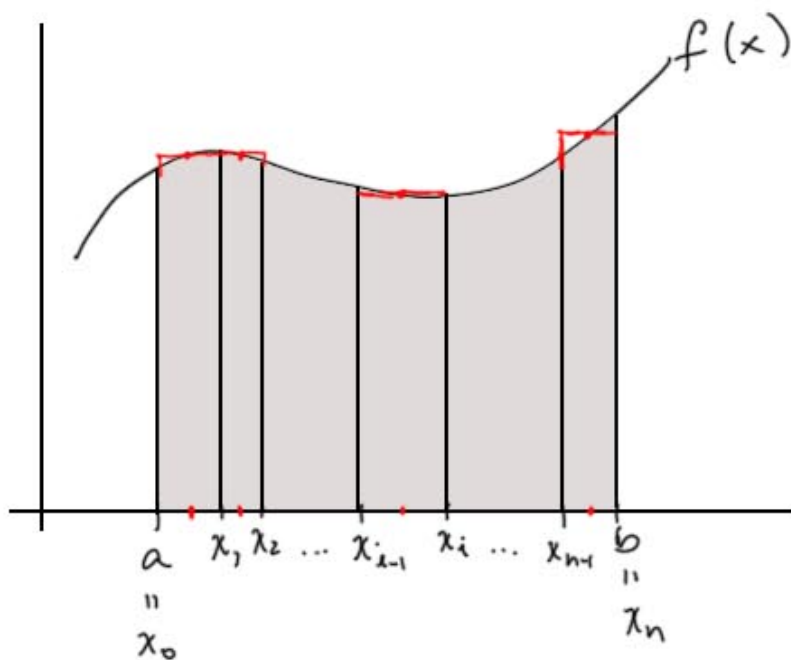
Double Integrals over Rectangles

Recall: For $f(x) \geq 0$, the area under the graph of $f(x)$ over the interval $[a, b]$ is given by

$$\int_a^b f(x) dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

where $[x_{i-1}, x_i]$ is a partition of $[a, b]$,

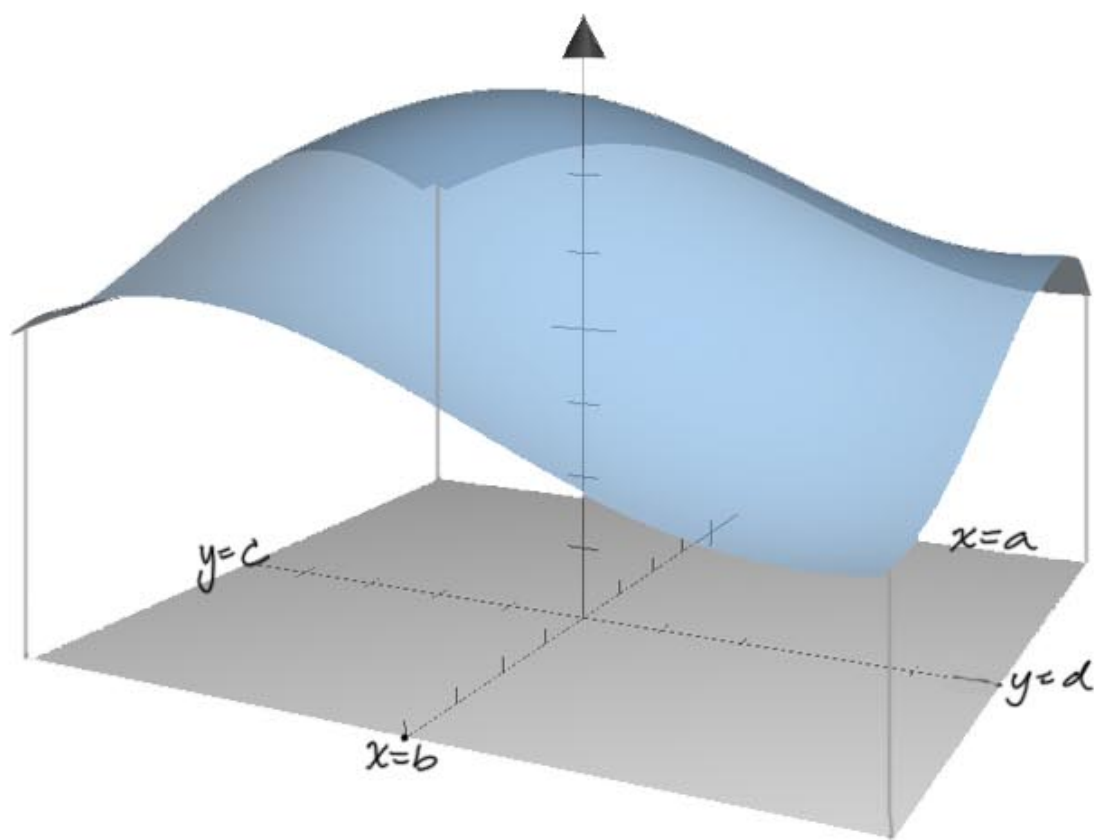
$x_i^* \in [x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$.



Ex, midpoint rule

And we compute $\int_a^b f(x)dx$ by The
Fundamental Theorem of Calculus.

Now consider $f(x,y) \geq 0$ and the volume
under the surface over the rectangular
region $\{(x,y) : a \leq x \leq b, c \leq y \leq d\}$



where $[x_{i-1}, x_i]$ is a partition of $[a, b]$

$[y_{j-1}, y_j]$ is a partition of $[c, d]$

$$(x_{ij}^*, y_{ij}^*) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

and $\Delta A_{ij} = \Delta x_i \Delta y_j$

$$\Delta x_i = x_i - x_{i-1}$$

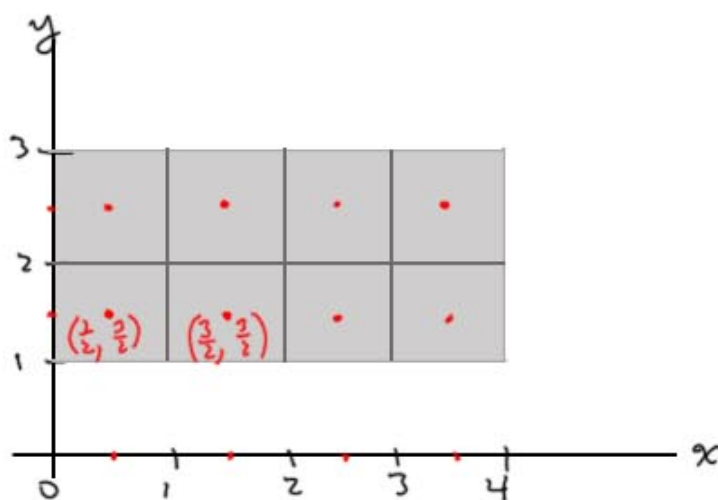
$$\Delta y_j = y_j - y_{j-1}$$

Ex. Estimate The volume of The solid enclosed

by $x=0, x=4, y=1, y=3$, The x - y plane and

$z = 4 + 2x + xy$ with a Riemann sum using

8 pieces.



Midpoint Rule:

$$\frac{dA}{\Delta x \Delta y} = 1 \cdot 1$$

$$\iint_R f(x,y) dA \approx f\left(\frac{1}{2}, \frac{3}{2}\right)(1) + f\left(\frac{3}{2}, \frac{3}{2}\right)(1) + f\left(\frac{5}{2}, \frac{3}{2}\right)(1) + f\left(\frac{7}{2}, \frac{3}{2}\right)(1) + f\left(\frac{1}{2}, \frac{5}{2}\right)(1) + f\left(\frac{3}{2}, \frac{5}{2}\right)(1) + f\left(\frac{5}{2}, \frac{5}{2}\right)(1) + f\left(\frac{7}{2}, \frac{5}{2}\right)(1)$$

$$f(x,y) = 4 + 2x + xy$$

$$f\left(\frac{1}{2}, \frac{3}{2}\right) = 4 + 2\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{3}{2}\right) = 5\frac{3}{4} = \frac{23}{4}$$

$$f\left(\frac{3}{2}, \frac{3}{2}\right) = 4 + 2\left(\frac{3}{2}\right) + \frac{3}{2}\left(\frac{3}{2}\right) = 7 + \frac{9}{4} = \frac{37}{4}$$

$$f\left(\frac{5}{2}, \frac{3}{2}\right) = 4 + 2\left(\frac{5}{2}\right) + \frac{5}{2}\left(\frac{3}{2}\right) = 9 + \frac{15}{4} = \frac{51}{4}$$

$$f\left(\frac{7}{2}, \frac{3}{2}\right) = 4 + 2\left(\frac{7}{2}\right) + \frac{7}{2}\left(\frac{3}{2}\right) = 11 + \frac{21}{4} = \frac{65}{4}$$

$$f\left(\frac{1}{2}, \frac{5}{2}\right) = 4 + 2\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{5}{2}\right) = 5 + \frac{5}{4} = \frac{25}{4}$$

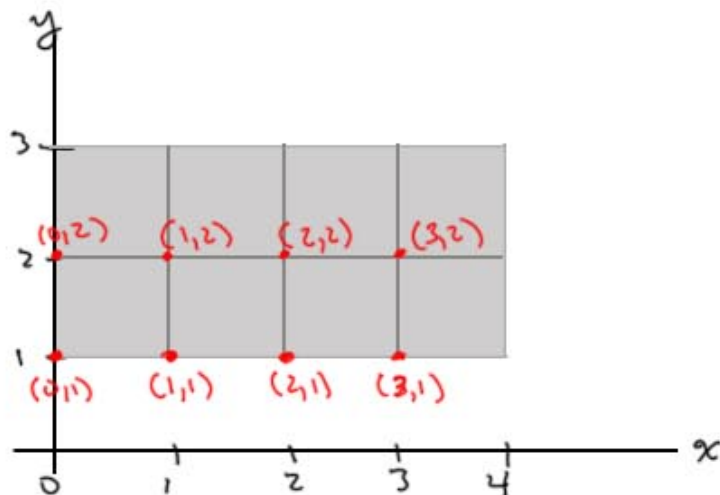
$$f\left(\frac{3}{2}, \frac{5}{2}\right) = 4 + 2\left(\frac{3}{2}\right) + \frac{3}{2}\left(\frac{5}{2}\right) = 7 + \frac{15}{4} = \frac{43}{4}$$

$$f\left(\frac{5}{2}, \frac{5}{2}\right) = 4 + 2\left(\frac{5}{2}\right) + \frac{5}{2}\left(\frac{5}{2}\right) = 9 + \frac{25}{4} = \frac{61}{4}$$

$$f\left(\frac{7}{2}, \frac{5}{2}\right) = 4 + 2\left(\frac{7}{2}\right) + \frac{7}{2}\left(\frac{5}{2}\right) = 11 + \frac{35}{4} = \frac{79}{4}$$

$$\text{So } \iint_R f(x,y) dA \approx \frac{384}{4} = 96 \quad \text{using the midpoint rule.}$$

Using The "lower left corner" rule:



$$\iint_R f(x,y) dA \approx f(0,1)(1) + f(1,1)(1) + f(2,1)(1) + f(3,1)(1) \\ + f(0,2)(1) + f(1,2)(1) + f(2,2)(1) + f(3,2)(1)$$

$$f(x,y) = 4 + 2x + xy$$

$$= 4 + 7 + 10 + 13 +$$

$$+ 4 + 8 + 12 + 16 = 74$$

Can also get overall overestimate and underestimate

by using one box with height = function max

and one box with height = function min

$$f(x,y) = 4 + 2x + xy$$

$$f_x(x,y) = 2+y = 0$$

$$y = -2$$

$$f_y(x,y) = x = 0$$

$$x = 0$$

not in our rectangle R

function values on the boundaries of R :

$$x = 0 \quad 1 \leq y \leq 3$$

$$f(0,y) = 4 + 2(0) + (0)y = 4$$

constant

$$x = 4 \quad 1 \leq y \leq 3$$

$$f(4,y) = 4 + 2(4) + 4y =$$
$$12 + 4y$$

$$\text{min at } (4,1) = 16$$

$$\text{max at } (4,3) = 24$$

$$y = 1 \quad 0 \leq x \leq 4$$

$$f(x,1) = 4 + 2x + x$$
$$= 4 + 3x$$

$$\text{min at } (0,1) = 4$$

$$\text{max at } (4,1) = 16$$

$$y = 3 \quad 0 \leq x \leq 4$$

$$f(x,3) = 4 + 2x + 3x$$
$$= 4 + 5x$$

$$\text{min at } (0,3) = 4$$

$$\text{max at } (4,3) = 24$$

So, maximum function value is 24.

$$\text{one box: } 4(2)(24) = 192 \quad \text{overestimate for volume}$$

minimum function value is 4

$$4(2)(4) = 32 \quad \text{underestimate for volume}$$

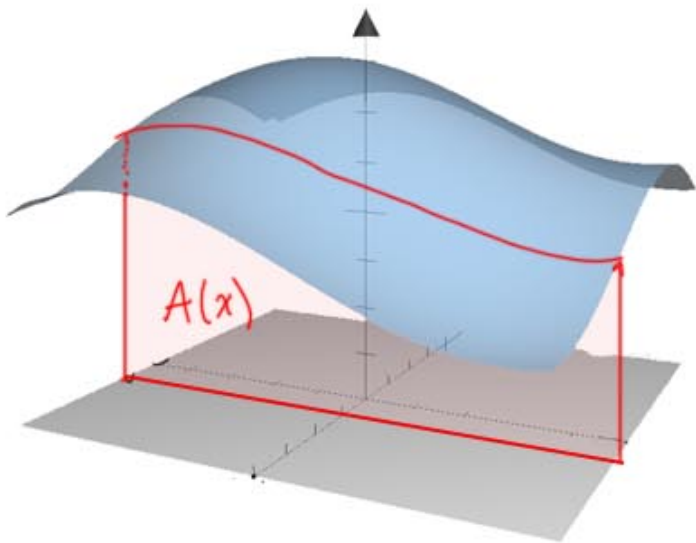
Note that The double integral above can be used for any (continuous) $f(x,y)$, and volumes under the xy plane then have a negative sign (because there, $f(x_{ij}^*, y_{ij}^*) < 0$).

How to compute a double integral without limits: iterated integrals.

Iterated Integrals:

Consider $A(x) = \int_c^d f(x,y) dy =$ area of slice for fixed x

↑
integrating with respect to y ,
 x is fixed



Then integrate $\int_a^b A(x) dx = \int_a^b \int_c^d f(x,y) dy dx$

Iterated integral

This iterated integral also gives

the volume:

Fubini's Theorem:

If f is continuous on the rectangle

$R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, then:

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Note: Switching the order of integration is only this easy for rectangles!!!

Ex. Find the volume of the solid enclosed by $x=0$, $x=4$, $y=1$, $y=3$, the x - y plane and $z = 4 + 2x + xy$

can compute $\int_0^4 \int_1^3 (4 + 2x + xy) dy dx$

$$\text{or } \int_1^3 \int_0^4 (4 + 2x + xy) dx dy .$$

$$\int_0^4 \int_1^3 (4 + 2x + xy) \underline{dy} dx = \int_0^4 \left[4y + 2xy + \frac{xy^2}{2} \right]_1^3 dx$$

*y is the variable
x is constant*

← y=3
↑ y=1

$$= \int_0^4 \left[\left(4(3) + 2x(3) + \frac{x(3)^2}{2} \right) - \left(4(1) + 2x(1) + \frac{x(1)^2}{2} \right) \right] dx$$

$$= \int_0^4 \left[\left(12 + 6x + \frac{9}{2}x \right) - \left(4 + 2x + \frac{1}{2}x \right) \right] dx$$

$$= \int_0^4 [8 + 8x] dx = 8x + \frac{8x^2}{2} \Big|_0^4 = 8x + 4x^2 \Big|_0^4$$

$$= (8(4) + 4(4)^2) - (8(0) + 4(0)^2) = 96.$$

Other order:

$$\int_1^3 \int_0^4 (4 + 2x + xy) dx dy =$$

x is the variable
y is constant

$$\int_1^3 \left[4x + x^2 + \frac{x^2}{2} y \right]_0^4 dy =$$

← x=4
← x=0

$$= \int_1^3 \left[(4(4) + 4^2 + \frac{4^2}{2} y) - (4(0) + 0^2 + \frac{0^2}{2} y) \right] dy$$

$$= \int_1^3 (32 + 8y) dy = 32y + \frac{8y^2}{2} \Big|_1^3 = 32y + 4y^2 \Big|_1^3$$

$$= (32(3) + 4(3)^2) - (32(1) + 4(1)^2)$$

$$= 96 + 36 - 36 = 96. \text{ (same answer)}$$

Ex. Evaluate $\iint_R y e^{xy} dA$ where

$$R = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 4\}$$

$$\int_0^2 \int_0^4 y e^{xy} dy dx \quad \text{or} \quad \int_0^4 \int_0^2 y e^{xy} dx dy$$

needs integration by parts *easier*

$$\rightarrow = \int_0^4 y \frac{1}{y} e^{xy} \Big|_0^2 dy = \int_0^4 e^{xy} \Big|_0^2 dy$$

x=2 *x=0*

$$= \int_0^4 (e^{2y} - e^{0y}) dy = \int_0^4 (e^{2y} - 1) dy$$

$$= \frac{1}{2} e^{2y} - y \Big|_0^4 =$$

$$= \left(\frac{1}{2} e^{2(4)} - 4 \right) - \left(\frac{1}{2} e^{2(0)} - 0 \right)$$

$$= \frac{1}{2} e^8 - 4 - \frac{1}{2} = \frac{1}{2} e^8 - \frac{9}{2} = \frac{e^8 - 9}{2} .$$

Properties of double integrals (as a result of properties of sums) :

$$1) \iint_R [f(x,y) + g(x,y)] dA = \iint_R f(x,y) dA + \iint_R g(x,y) dA$$

$$2) \iint_R c f(x,y) dA = c \iint_R f(x,y) dA \quad c \text{ constant}$$

3) if $f(x,y) \geq g(x,y) \quad \forall (x,y) \in R$, then

$$\iint_R f(x,y) dA \geq \iint_R g(x,y) dA$$