

Math Review for Algebra and Precalculus

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Table of Contents

Part I: Algebra Notes for Math 195

| | |
|---|----|
| Introduction..... | 3 |
| 1. Basic algebra laws; order of operations..... | 4 |
| 2. How algebra works..... | 12 |
| 3. Simplifying polynomial expressions..... | 29 |
| 4. Functions..... | 41 |
| 5. When to use parentheses..... | 55 |
| 6. Working with fractions..... | 61 |
| 7. Adding fractions..... | 76 |

Math Review for Precalculus and Calculus

Part I: Algebra

Introduction

Algebra is the language of calculus, and calculus is needed for science and engineering. When you attack a real-world problem, you want to represent the problem using algebra expressions. When you read technical books, you want to be comfortable deciphering and working with these expressions. Computers can't do either of these tasks for you.

Algebra used in undergraduate mathematics involves three main activities: rewriting expressions, solving equations, and solving inequalities. You need to perform these somewhat mechanical activities quickly and accurately. It's very difficult to achieve this goal unless you understand how algebra works.

Algebra is a symbolic language that allows communication between people who don't know each others' spoken language. The grammar of the language involves three main components: expressions, identities, and equations.

An *expression* involves numbers, variables, parentheses, and algebra operations. Basic types of expressions are integers, variables, monomials, polynomials, and so forth. We'll deal mostly with expressions in one variable, such as the polynomial $x^3 - x + 4$.

An *identity* between two expressions, written with an equals sign, is a statement that each expression can be obtained by rewriting the other. A simple example is $x + 2 = 1 + x + 1$. With rare exceptions, substituting numbers for variables turns an identity into a true statement about numbers. For example, setting x to 4 yields $4 + 2 = 1 + 4 + 1$. An important part of algebra is using identities to rewrite expressions.

An *equation* is also a statement that two expressions are equal. In most equations, however, equality holds only for specific values of the variable. For example, the statement $x^2 + 2 = 3$ is true only when x is 1 or -1 . We say that the solutions of the equation $x^2 + 2 = 3$ are $x = 1$ and $x = -1$. Please remember;; we rewrite expressions but we solve equations.

In this preliminary edition, section headings such as AN1 are used for Algebra Notes, Section 1.

CHAPTER 1: BASIC ALGEBRA LAWS; ORDER OF OPERATIONS.

1.1 Algebra operations and notation

Let's begin with two tricky examples.

Example 1.1.1: Rewrite $2x^{-1}$ as an expression with no negative powers.

Right: $2x^{-1} = 2\left(\frac{1}{x}\right) = \frac{2}{x}$ **Wrong:** $2x^{-1} = \frac{1}{2x}$

Example 1.1.2: Simplify the expression -5^2

Right: $-5^2 = -(5^2) = -25$ **Wrong:** $-5^2 = (-5)^2 = 25$

To understand what's going on, we need to review in some detail the five *algebra operations*: *addition*, *subtraction*, *multiplication*, *division*, and *exponentiation*. Each of these is called a *binary* operation because it is used to combine two expressions. The table below lists notation and terminology for these operations. The last entry shows a special operation called *negation*, which operates on one expression and is an abbreviation for multiplication by -1 .

| Operation | Write | Say | Describe the answer |
|----------------|---|--|---|
| Addition | $4 + 3$ | 4 plus 3 | The <i>sum</i> of 4 and 3 |
| Subtraction | $4 - 3$ | 4 subtract 3 or 4 minus 3 | The <i>difference</i> of 4 and 3 |
| Multiplication | $4 \cdot 3$ Traditional notation 4×3 Traditional notation $4 * 3$ Calculator notation $4(3 + x)$ Implied times sign $4xy$ Implied times sign | 4 times 3 | The <i>product</i> of 4 and 3 |
| Division | $4 \div 3$ Seldom used $4/3$ Calculator notation $\frac{4}{3}$ Traditional notation | 4 divided by 3 four thirds 4 over 3 (slang) | The <i>quotient</i> of 4 by 3 |
| Exponentiation | 4^3 Traditional notation 4^3 Calculator notation | 4 (raised) to the 3rd (power) | The 3 rd <i>power</i> of 4 4 is the <i>base</i> . 3 is the <i>exponent</i> . |
| Negation | -3 $-(-5) = 5$ | Negative 3 or Minus 3 Negative of minus 5 | The negation (additive opposite) of 3 is -3 . |

It's a bit annoying that the minus sign ' $-$ ' is used for three different purposes: naming a negative number, subtraction, and negation. Specifically:

- A minus sign **before a number** (as in -3) is part of the **number name**.
- A minus sign **between** two expressions (as in $x - y$) means **subtract**.
- A minus sign that is the leftmost symbol in an expression (as in $-x$), or one that follows a left parenthesis (as in $(-x + 2)$), means negation: **multiply by -1** .

To reduce the number of meanings from three to two, some books reserve the long dash for negation or subtraction but write negative numbers with a short dash (e.g., as -3).

The fewer symbols an expression contains, the easier it is to understand. The multiplication sign is left out as often as possible: between a digit and a parenthesis, and between two parentheses, as shown in the examples below.

| Multiplication sign omitted | Multiplication sign included |
|-----------------------------|------------------------------|
| $3(4)$ | $3 \times (4) = 12$ |
| $(3)4$ | $(3) \times 4 = 12$ |
| $(3)(4)$ | $(3) \times (4) = 12$ |
| $-3(4)$ | $-3 \times (4) = 12$ |

1.2 Evaluating Numerical Expressions

Algebra begins with arithmetic and the properties of whole numbers: 0, 1, 2, 3, , , ,

The most basic property is the **identity law**, which states that adding zero to any number or multiplying it by 1 leave the number unchanged.

One important property of addition of whole numbers is that **addition is commutative**: two numbers can be added in either order. Some examples of this property:

$$12345 + 678910 = 678910 + 12345$$

$$123456789 + 987654321 = 987654321 + 123456789$$

To actually state the law, we use letters a and b to stand for any two whole numbers.

Commutative law for addition: $a + b = b + a$.

Another important property of addition of whole numbers is that **addition is associative**: when you add three numbers, the two possible methods of grouping yield the same answer. Some examples:

$$(2 + 3) + 4 = 4 + (3 + 2)$$

$$(123 + 456) + 789 = 123 + (456 + 789)$$

To state the associative law, we let a , b , and c stand for any three whole numbers.

Associative law for addition: $(a + b) + c = a + (b + c)$

The associative law allows us to write expressions such as $2 + 3 + 4$ and $123 + 456 + 789$ without using parentheses.

Together, the associative and commutative laws for addition allow us to write sums without parentheses *and* to reorder the numbers in the sum. Thus
 $a + b + c = c + a + b = b + c + a$ etc. For example, using $a = 2; b = 3; c = 4$, we have
 $2 + 3 + 4 = 4 + 2 + 3 = 3 + 4 + 2$.

Similar laws apply to multiplication:

Commutative law for multiplication: $a \times b = b \times a$

Associative law for multiplication: $(a \times b) \times c = a \times (b \times c)$

Together, the associative and commutative laws for multiplication allow us to write products without parentheses *and* to reorder the numbers in the product. Thus
 $a \times b \times c = c \times a \times b = b \times c \times a$ etc. For example, $2 \times 3 \times 4 = 4 \times 2 \times 3 = 3 \times 4 \times 2$.

Addition and multiplication are connected by the two versions of the distributive law, which states that when a sum is multiplied by a number, the multiplier can be distributed among the numbers that make up the sum.

Distributive law: $a \times (b + c) = (a \times b) + (a \times c)$ and $(b + c) \times a = (b \times a) + (c \times a)$

The basic rules, or laws, of arithmetic and algebra are listed in the following table, in which a, b, c represent any whole numbers, addition is indicated by +, and multiplication is indicated by \times .

Basic algebra laws for whole numbers

| Name of Law | Statement of law | Example with $a = 2; b = 3; c = 4$; | Check the example by working out both sides |
|------------------------------|--|--|--|
| Identity law for + | $a + 0 = a$ | $2 + 0 = 2$ | $2 = 2$ |
| Commutative law for + | $a + b = b + a$ | $2 + 3 = 3 + 2$ | $5 = 5$ |
| Associative law for + | $(a + b) + c = a + (b + c)$ | $(2 + 3) + 4 = 2 + (3 + 4)$ | $(5) + 4 = ? 2 + (7)$ $9 = 9$ |
| Identity law for \times | $a \times 1 = a$ | $2 \times 1 = 2$ | $2 = 2$ |
| Commutative law for \times | $a \times b = b \times a$ | $2 \times 3 = 3 \times 2$ | $6 = 6$ |
| Associative law for \times | $(a \times b) \times c = a \times (b \times c)$ | $(2 \times 3) \times 4 = 2 \times (3 \times 4)$ | $(6) \times 4 = ? 2 \times (12)$ $24 = 24$ |
| Distributive law | $a \times (b + c) = (a \times b) + (a \times c)$ $(b + c) \times a = (b \times a) + (c \times a)$ | $2 \times (3 + 4) = (2 \times 3) + (2 \times 4)$ | $2 \times (7) = ? (6) + (8)$ $14 = 14$ |

In the above discussion, we said we were working with whole numbers. As you already know, there are more general classes of numbers.

Integers (signed whole numbers) are ..., -3, -2, -1, 0, 1, 2, 3, ...

Rational numbers are fractions of integers: $\frac{1}{7}, \frac{3}{4}, \frac{-3}{4}, \frac{123}{1}$

Decimal numbers are rational numbers with denominators 1, 10, 100, 1000, ...:

Examples are $3.14 = \frac{314}{100}$, $0.0314 = \frac{314}{10000}$, and $314 = \frac{314}{1}$

Real numbers include rational numbers and other numbers such as $e, \pi, \sqrt{2}$ that cannot be expressed as fractions of integers.

Other laws and definitions that apply to the categories of numbers listed above are:

- $-a = -1 \times a$
- $a + (-a) = 0$: The sum of a number and its negative is zero.
- **Subtraction** is defined by $a - b = a + (-b)$.
- $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ where a, b, c, d are integers. (Here b and d are not zero.)
- $\frac{b}{a}$ is called the **reciprocal** of $\frac{a}{b}$ if $b \neq 0$.

The product of a non-zero rational number and its reciprocal is 1.

- **Division** is defined by $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ if b, c, d are not zero.

To divide by a non-zero fraction, multiply by its reciprocal.

Exercise 1.2.1: Using the table above as a model, please fill in the following tables.

Basic algebra laws for integers

| Name of Law | Statement of law | Example with $a = -2; b = 3; c = -4$; | Check the example by working out both sides |
|-----------------------|--|---|--|
| Identity law for + | $a + 0 = a$ | | |
| Commutative law for + | $a + b = b + a$ | | |
| Associative law for + | $(a + b) + c = a + (b + c)$ | | |
| Identity law for × | $a \times 1 = a$ | | |
| Commutative law for × | $a \times b = b \times a$ | | |
| Associative law for × | $(a \times b) \times c = a \times (b \times c)$ | | |
| Distributive law | $a \times (b + c) = (a \times b) + (a \times c)$ $(b + c) \times a = (b \times a) + (c \times a)$ | | |

Basic algebra laws for rational numbers

| Name of Law | Statement of law | Example with $a = \frac{2}{3}; b = \frac{1}{2}; c = \frac{2}{5}$ | Check the example by working out both sides |
|-----------------------|--|---|--|
| Identity law for + | $a + 0 = a$ | | |
| Commutative law for + | $a + b = b + a$ | | |
| Associative law for + | $(a + b) + c = a + (b + c)$ | | |
| Identity law for × | $a \times 1 = a$ | | |
| Commutative law for × | $a \times b = b \times a$ | | |
| Associative law for × | $(a \times b) \times c = a \times (b \times c)$ | | |
| Distributive law | $a \times (b + c) = (a \times b) + (a \times c)$ $(b + c) \times a = (b \times a) + (c \times a)$ | | |

Basic algebra laws for decimal numbers

| Name of Law | Statement of law | Example with $a = 1.2; b = 2.2; c = 1.5$ | Check the example by working out both sides |
|-----------------------|--|---|--|
| Identity law for + | $a + 0 = a$ | | |
| Commutative law for + | $a + b = b + a$ | | |
| Associative law for + | $(a + b) + c = a + (b + c)$ | | |
| Identity law for × | $a \times 1 = a$ | | |
| Commutative law for × | $a \times b = b \times a$ | | |
| Associative law for × | $(a \times b) \times c = a \times (b \times c)$ | | |
| Distributive law | $a \times (b + c) = (a \times b) + (a \times c)$ $(b + c) \times a = (b \times a) + (c \times a)$ | | |

1.3 PEMDAS and Order of Operations

We just saw that $2 + 3 + 4$ can be written without parentheses, because the associative law tells us that $(2 + 3) + 4 = 2 + (3 + 4)$. Similarly, there are many ways to insert parentheses in $2 + 3 + 4 + 5 + 6$, but the associative law guarantees that the answer will be the same (in this case, 20) no matter where the parentheses are inserted.

Very often, however, the choice of parentheses matters. For example,
 $(2 + 3) \times 4 = 5 \times 4 = 20$ but $2 + (3 \times 4) = 2 + 12 = 14$.

When you construct longer expressions, the situation gets worse. For example, what does $4 + 8 \times 4 - 2$ mean?

$$\text{Some possibilities: } \begin{cases} (4 + 8) \times (4 - 2) = 12 \times 2 = 24 \\ [(4 + 8) \times 4] - 2 = [12 \times 4] - 2 = 48 - 2 = 46 \\ 4 + [8 \times (4 - 2)] = 4 + [8 \times 2] = 4 + 16 = 20 \end{cases}$$

To resolve the ambiguity in such expressions, math people have specified **order of operations rules** that **tell you what operation to do first** in an expression that doesn't have parentheses. Another way to say this is that **order of operations rules show where to insert parentheses in arithmetic expressions**.

You probably know the acronym PEMDAS (Parentheses, Exponents, Multiplication and Division, Addition and Subtraction) for order of operations in numerical or more general expressions. PEMDAS is usually described as a step-by-step procedure for removing parentheses from a numerical expression. However, it applies to expressions with variables as well.

One way to explain PEMDAS is as follows (the word "evaluate" means "find the value of"):

To evaluate a numerical expression that doesn't involve parentheses:

Step 1: Evaluate all **Exponents** (powers), working from right to left.

Note: In the following discussion, we will refer to exponents as powers; do not confuse the word powers with the P in PEMDAS, which stands for **P**arentheses.

Step 2: Evaluate all **Multiplications** and **Divisions**, working from left to right.

Step 3: Evaluate all **Additions** and **Subtractions**, working from left to right.

To evaluate a numerical expression that does involve parentheses:

Step I: Find an innermost set of **Parentheses**: a pair of parentheses that don't have other parentheses between them. Evaluate the expression between them using Steps 1, 2, 3 above.

Step II: If any parentheses remain, go to Step I. Otherwise you are done.

You don't need to begin with an innermost set of parentheses: you can simplify parenthesized expressions in any order you like. For example, to simplify the expression $1 + (7(4 - 2))$, you could either

- begin with the inner set of parentheses: $1 + (7(4 - 2)) = 1 + 7((2)) = 1 + 14 = 15$ or
- begin with the outer set of parentheses, obtaining the same answer:
 $1 + (28 - 14) = 1 + 14 = 15$.

Furthermore, you can work on several sets of parentheses simultaneously. This is done in the right column of the following example: the left column works on one set at a time and follows the PEMDAS rules strictly.

| | |
|--------------------------------|--------------------------------|
| $(3 + 2(4))^2 + (8/2/2 - 3)^3$ | |
| $= (3 + 8)^2 + (8/2/2 - 3)^3$ | |
| $= (11)^2 + (8/2/2 - 3)^3$ | $(3 + 2(4))^2 + (8/2/2 - 3)^3$ |
| $= 121 + (8/2/2 - 3)^3$ | $= (3 + 8)^2 + (4/2 - 3)^3$ |
| $= 121 + (4/2 - 3)^3$ | $= 121 + (-1)^3$ |
| $= 121 + (2 - 3)^3$ | $= 121 + (-1)$ |
| $= 121 + (-1)^3$ | $= 120$ |
| $= 121 + (-1)$ | |
| $= 120$ | |

Reading expressions with minus signs:

It's worth mentioning a few tricky examples involving minus signs. The word "minus" and the word "negative" mean pretty much the same thing, although the phrase "minus number" is not used. **Don't use "minus" to mean subtraction.**

| Symbols | Words |
|----------------------------|--|
| $7 - 3 = 4$ | 7 subtract 3 equals 4 |
| $3 - 7 = -4$ | 3 subtract 7 equals minus 4 |
| $-3 - 7 = -3 + (-7) = -10$ | Minus 3 subtract 7 equals minus 10 |
| $-3 - (-7) = -3 + 7 = 4$ | Minus 3 subtract minus 7 equals 4 |
| $4(3 - 7) = 4(-4) = -16$ | 4 times the quantity: 3 subtract 7: equals -16 |
| $4(3) - 7 = 12 - 7 = 5$ | 4 times 3 subtract 7 equals 5 |

Here are some simple examples with numerical expressions. In each example, we begin by inserting parentheses around the operation that comes first as specified by the order of operations rules.

| | |
|---|--|
| 1) $4^2 + 1 = (4^2) + 1 = (16) + 1 = 17$ | Powers before addition |
| 2) $4 + 2 \times 3 = 4 + (2 \times 3) = 4 + (6) = 10$ | Multiplication before addition |
| 3) $4 \times 2^3 = 4 \times (2^3) = 4 \times (8) = 32$ | Powers before multiplication |
| 4) $2^{2^3} = 2^{(2^3)} = 2^8 = 256$ | Do powers from right to left |
| 5) $4 - 2 + 1 = (4 - 2) + 1 = (2) + 1 = 3$ | Add/subtract from left to right |
| 6) $4 \times 2 \div 2 = (4 \times 2) \div 2 = (8) \div 2 = 4$ | Multiply/divide from left to right. |
| The following examples omit parentheses implied by order of operations. | |
| 7) $12 - 2 + 5 = 10 + 5 = 15$ | Add/subtract from left to right |
| 8) $3 \times 8 \div 2 = 24 \div 2 = 12$ | Multiply/divide from left to right |
| 9) $36 / 3 \times 8 = 12 \times 8 = 96$ | Multiply/divide from left to right |
| 10) $3^{2^3} = 3^8 = 3^4 3^4 = 6561$ | Do powers from right to left (top to bottom) |
| 11) $4 + 3 \times 5 = 4 + 15 = 19$ | Multiply before you add or subtract |
| 12) $4 - 3 \times 5 = 4 - 15 = -11$ | Multiply before you add or subtract |
| 13) $4 \times 3^2 = 4 \times 9 = 36$ | Do powers before anything else. |
| 14) $3 + 4^2 = 3 + 16 = 19$ | Do powers before anything else. |
| 15) $4 - 3^2 = 4 - 9 = -5$ | Do powers before anything else. |
| 16) $16 / 2^2 = 16 / 4 = 4$ | Do powers before anything else. |

Exercise 1.3.1: Use PEMDAS to simplify each of the following numerical expressions. Start by inserting parentheses that are implied by the order of operations.

| | |
|--|--|
| a) $2 + 3 \times 5 = 2 + (3 \times 5) = 2 + 15 = 17$ | m) $10 - 8 \times 2^2 =$ |
| b) $2 + 3 \times 5 + 2 =$ | n) $10 - 2(3 - 5) =$ |
| c) $2 - 3 \times 5 =$ | o) $(5 - 2)(2 + 3)^2 =$ |
| d) $17 - 5 - 4 + 3 =$ | p) $10 - (5 - 2)(2 + 3)^2 =$ |
| e) $16 / 8 \times 2 =$ | q) $10 - 3(5 - 2)^2 =$ |
| f) $-3^2 =$ | r) $10 - (3 + 4)(5 - 2)^2 =$ |
| g) $10 - 8^2 =$ | s) $10 - 3 + 4(5 - 2)^2 =$ |
| h) $6 / 4 / 4 =$ | t) $6 - 5(4 - 3) =$ |
| i) $2 \times 2 - 3 \times 5 =$ | u) $6^2 - 35(1 + 1) =$ |
| j) $15 / 3 \times 5 =$ | v) $-5 \times 3^2 =$ |
| k) $-2 + 3 \times 5 =$ | w) $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 - 9 =$ |
| l) $5 \times 2^2 =$ | x) $2^{2^3} =$ |

CHAPTER 2: HOW ALGEBRA WORKS.

2.1 Implied multiplication signs in expressions involving variables

The multiplication sign is left out as often as possible whenever it appears between two variables, between two parentheses, or between any two of the symbols: digit, variable, parenthesis. One exception: the expression $x3$, which might reasonably be taken to mean x times 3, is not used; write $(x)3$ or, preferably, $3(x)$.

| Expression with multiplication signs omitted | Signs included |
|---|--------------------------|
| $3x$ (digit followed by variable) | $3 \times x$ |
| xy (variable followed by variable) | $x \times y$ |
| $x(2 + x)$ (variable followed by left parenthesis) | $x \times (2 + x)$ |
| $(x + 2)x$ (variable following right parenthesis) | $(x + 2) \times x$ |
| $3(x + y)$ (digit followed by left parenthesis) | $3 \times (x + y)$ |
| $(x + y)3$ (digit following left parenthesis) | $(x + y) \times 3$ |
| $(x)(x + 3)$ (left parenthesis following right parenthesis) | $(x + 2) \times (x + 3)$ |
| $x^2 y^3$ (exponent followed by any base) | $x^2 \times y^3$ |

2.2 Using the distributive law to remove parentheses

Earlier, we wrote the distributive law as $a(b + c) = (a \times b) + (a \times c)$

Omitting the multiplication signs, we have $a(b + c) = (ab) + (ac)$

Finally we can rewrite this as $a(b + c) = ab + ac$

since multiplication precedes addition in the order of operations.

The distributive law comes up in many situations but is tricky to work with when minus signs appear. The following are the patterns that commonly occur.

$$-a(b) = (-a)(b) = (-a)b = a(-b) = -ab$$

$$-a(-b) = (-a)(-b) = ab$$

Caution: $-a - b$ is read “minus a subtract b ” and is not related to the products on the previous line. Note that $-a - b = (-a) - b = -a + (-b) = (-a) + (-b) = -1(a + b)$.

Removing parentheses following a minus sign:

The following identity illustrates a very important rewriting operation: removing parentheses that follow a minus sign.

$$a - (b - c + d) = a - b + c - d$$

This operation should be automatic: think of multiplying the minus sign by each term inside the parentheses. Here are the details, which you should understand once, then put aside in favor of the automatic procedure.

$$\begin{aligned}
 & a - (b - c + d) && \text{This is a subtraction problem.} \\
 & = a + (-(b - c + d)) && \text{Apply the definition of subtraction:} \\
 & && u - v = u + (-v) \text{ with } \begin{cases} u = a \\ v = b - c + d \end{cases} \\
 & = a + (-1(b - c + d)) && \text{Apply the definition of negation: } -v = -1(v) = -1 \times v \text{ with} \\
 & && v = b - c + d \\
 & = a + (-b + c - d) && \text{Use the distributive law to multiply by -1:} \\
 & && -1(b - c + d) \\
 & && = -1 \times b + (-1 \times (-c)) + (-1 \times d) \\
 & && = -b + c + (-d) \\
 & && = -b + c - d \\
 & = a - b + c - d && \text{Remove parentheses.}
 \end{aligned}$$

Here is a related example of the form $a - 2(b - c + d) = a - 2b + 2c - 2d$

$$\begin{aligned}
 & a - 2(b - c + d) && \text{This is a subtraction problem.} \\
 & = a + (-(2b - 2c + 2d)) && \text{Apply the definition of subtraction.} \\
 & = a + (-1(2b - 2c + 2d)) && \text{Apply the definition of negation.} \\
 & = a + (-2b + 2c - 2d) && \text{Apply the distributive law.} \\
 & = a - 2b + 2c - 2d && \text{Remove parentheses.}
 \end{aligned}$$

Exercise 2.2.1: Fill in the table below.

| Version of distributive law | Example with $a = 7; b = -4; c = 2; d = 3$ | Check the example by working out both sides |
|--|---|--|
| $a(b + c) = ab + ac$ | $7(-4 + 3) = ? 7(-4) + 7(3)$ | $7(-1) = ? -28 + 21$ $-7 = -7$ |
| a) $a(b - c) = ab - ac$ | | |
| b) $(b + c)a = ?$ | | |
| c) $(b - c)a = ?$ | | |
| d) $-a(b + c) = ?$ | | |
| e) $-(b + c) = -b - c$ | | |
| f) $-1(b + c) = -b - c$ | | |
| g) $-a(b - c) = ?$ | | |
| h) $(b + c)(-a) = -ba - ca$ but | | |
| i) $(b + c) - a = b + c - a$ Why? | | |

| | | |
|---|--|--|
| j) $a(b + c + d) = ab + ac + ad$ | | |
| k) $a(b - c + d) = ?$ | | |
| l) $a(b - c - d) = ?$ | | |
| m) $-a(b - c + d) = ?$ | | |
| n) $-(b - c + d) = -b + c - d$ | | |
| o) $-1(b - c + d) = -b + c - d$ | | |

The preceding was a review of arithmetic. Now we move on to algebra.

2.3 We do algebra by rewriting expressions

An *expression* involves numbers, variables, parentheses, and algebra operations. Some basic types of expressions (separated by semicolons) are

- *integers*: -2 ; 0 ; 345 ; -9 ; 87654
- *variables*: x ; y ; z ; a ; b ; c
- *monomials* in x : 0 ; -4 ; x ; x^5 ; $2x^2$; $3x^{100}$; $-34x^{23}$
- *polynomials* in x : 0 ; 7 ; $5x$; $7 + 5x$; $5x - 6 - 3x^2$; $x^{100} - 17x + 1$
- *rational polynomials* in x : $\frac{3}{4}$; $\frac{x}{x^2}$; $\frac{x^3 + 1}{3x^2 + 17x - 3}$

Algebra and first year calculus focus on polynomials and rational polynomials with one or two variables (usually x and y), as well as on equations and inequalities involving such expressions.

The hardest part of algebra is rewriting expressions correctly. A typical algebra problem is a sequence of rewriting operations, which can be written using two possible formats:

- Horizontal:
Original expression = Expression 2 = Expression 3 ... = Final Expression
- Vertical:
Original expression
= Expression 2
= Expression 3
...
= Final Expression

It's much easier to avoid mistakes when you use vertical notation, because you are copying each expression from the one directly above it. You should think of this format as a basic paragraph of mathematics, in which each expression is a sentence.

In most cases the final expression is obtained by *simplifying* the original expression to make it easier to understand. Simplifying a numerical expression means rewriting it as a single number. However, there are two ways to simplify an expression with variables, by rewriting it as a *simplified sum* or as a *simplified product*. For example, once you review how to multiply using FOIL, you could use vertical format to rewrite $(x^2 + x)(x^3 + x^2)$

- as a **simplified product**, with no repeated factors:

$$(x^2 + x)(x^3 + x^2)$$

$$= (x)(x+1)(x^2)(x+1)$$

$$= x^3(x+1)^2$$

- or as a **simplified sum**, with no repeated terms:

$$(x^2 + x)(x^3 + x^2)$$

$$= x^2x^3 + x^2x^2 + xx^3 + xx^2$$

$$= x^5 + x^4 + x^4 + x^3$$

$$= x^5 + 2x^4 + x^3$$

2.4 Substituting in expressions

Simplifying expressions uses a very basic rewriting process called *substitution*. In a basketball game, substituting player A for player B means replacing player B by player A. Similarly, to substitute expression A for expression B, replace expression B by expression A, with one technical requirement:

Definition of substitution: To substitute expression S for a variable a in expression E , erase every letter a in E and replace it by the expression S enclosed in parentheses.

Please engrave the following statement in the innermost core of your mathematical subconscious:

When you substitute an expression for a variable, put parentheses around the expression.

Here is an informal exercise: try to find a textbook that correctly describes substitution.

Example 2.4.1: Substitute $x + 7$ for a in the expression $12 - 4a$

Correct Method: Erase every ' a ' (there's just one) in the expression and replace it by ' $(x + 7)$ '.

Correct Answer: $12 - 4(x + 7)$

Incorrect Method: Replace every ‘a’ by ‘x+7’

Incorrect Answer: $12 - 4x + 7$

Two important observations:

- Parentheses make a difference, since $12 - 4(x + 7) = 12 - 4x - 28 = -4x - 16$ whereas $12 - 4x + 7 = 19 - 4x = -4x + 19$
- The answer $12 - 4(x + 7)$ is *not simplified*. It’s correct, because the question said ‘substitute’ rather than ‘substitute and simplify.’

Substitution and simplification are different processes:

Substitution is a purely mechanical “erase and replace” operation. *Don’t think, just erase and replace!*

Simplification involves rewriting the expression by applying algebra rules. Thinking about algebra when you do substitution very often leads to errors.

The table below lists common errors that occur rather frequently. To reveal each error, we simplify the result of each correct substitution. Question marks appear in the last column when omitting parentheses produces a result with consecutive minus signs, notation that is usually considered illegal.

| Substitute | Correct | Incorrect |
|----------------------------|----------------------------|-----------------------------|
| $x + 2$ for a in $b - a$ | $b - (x + 2) = b - x - 2$ | $b - x + 2$ |
| $x + 2$ for a in a^2 | $(x + 2)^2 = x^2 + 4x + 4$ | $x + 2^2 = x + 4$ |
| $x + 2$ for a in ab | $(x + 2)b = xb + 2b$ | $x + 2b$ |
| $x + 2$ for a in $3a$ | $3(x + 2) = 3x + 6$ | $3x + 2$ |
| $-3x$ for a in $b - a$ | $b - (-3x) = b + 3x$ | $b - -3x = ?$ |
| $-3x$ for a in a^2 | $(-3x)^2 = 9x^2$ | $-3x^2$ |
| $-3x$ for a in $3a$ | $3(-3x) = -9x$ | $3 - 3x$ |
| -2 for a in $b - a$ | $b - (-2) = b + 2$ | $b - -2 = ?$ |
| -2 for a in a^2 | $(-2)^2 = 4$ | $-2^2 = -1 \times 2^2 = -4$ |
| -2 for a in $3a$ | $3(-2) = -6$ | $3 - 2 = 1$ |

The following table gives examples of substitution in expressions, one per row:

| Substitute this | For this | In this | Correct Answer | Incorrect |
|-----------------|----------|-------------------------|-------------------------------------|-----------------------------|
| -3 | x | x^2 | $(-3)^2$ | -3^2 |
| $a + b$ | x | x^2 | $(a + b)^2$ | $a + b^2$ $a^2 + b^2$ |
| -3 | x | $7 - x^2$ | $7 - (-3)^2$ | $7 + 3^2$ $(7 - (-3))^2$ |
| $x + h$ | x | $7 - x^2$ | $7 - (x + h)^2$ | $7 - x^2 + h^2$ |
| $x + h$ | x | $f(3x)$ | $f(3(x + h))$ | $f(3x + h)$ |
| $x + h$ | a | $f(3x)$ | $f(3x)$ | $f(3(x + h))$ |
| $x + h$ | a | $f(ax)$ | $f((x + h)x)$ | $f(x + hx)$ |
| $x + h$ | x | $(x + h)^2 - (x - h)^2$ | $((x + h) + h)^2 - ((x + h) - h)^2$ | |

Exercise 2.4.1: Substitute as indicated but do not simplify in each of the following expressions.

| |
|--|
| a) $x + 2$ for x in $ax + b$ |
| b) $x + 2a$ for x in $ax - x^2$ |
| c) -3 for x in $3x - 2x^2$ |
| d) $x + h$ for x in $3 - x - x^2$ |
| e) $x - h$ for a in $x^2 + bx + c$ |
| f) $x + h$ for x in $ab + cx$ |
| g) $x + h$ for x in $ab + c$ |
| h) b for a in $ax + b$ |
| i) $(x + 2)^2$ for a and $3x + 2$ for b in $ab - a^2$ |
| j) $(x + 2)^2$ for a and $3x + 2$ for b in $ac - c^2$ |
| k) $a + b$ for a and $a - b$ for b in $a^2 - b^2$ |
| l) $x + 3$ for a and $x - 2$ for b in $\frac{a - b}{c}$ |

Exercise 2.4.2: Even though we haven't discussed simplification methods in detail, see if you can rewrite each answer in Exercise 2.4.2 as a simplified sum. Here is an example using the answer to 2.4.2 i), $ab - a^2 = (x + 2)^2(3x + 2) - ((x + 2)^2)^2$:

$$\begin{aligned} & \left((x+2)^2(3x+2) - ((x+2)^2)^2 \right) \\ &= (x+2)^2(3x+2) - (x+2)^2(x+2)^2 \\ &= (x+2)^2(3x+2 - (x+2)^2) \\ &= (x^2 + 4x + 4)(3x+2 - (x^2 + 4x + 4)) \\ \text{Answer to 2.4.2 i): } &= (x^2 + 4x + 4)(3x+2 - x^2 - 4x - 4) \\ &= (x^2 + 4x + 4)(-x^2 - x - 2) \\ &= -1(x^2 + 4x + 4)(x^2 + x + 2) \\ &= -1(x^4 + 5x^3 + 10x^2 + 12x + 8) \\ &= -x^4 - 5x^3 - 10x^2 - 12x - 8 \end{aligned}$$

The laws of algebra are listed below. Run through them quickly, then continue to the next section and refer back to this list as necessary.

Addition laws

Identity: $a + 0 = a$

Commutative: $a + b = b + a$

Associative: $a + (b + c) = (a + b) + c$ Both are written as $a + b + c$.

Inverse: For each expression a there is an expression $-a$ with $a + (-a) = 0$.

$-a$, called the *negation* (or *negative*) of a , is equal to the product $-1(a)$.

The commutative and associative laws allow us to compute a sum of three or more expressions in any order. Example: $(a + b) + ((c + d) + e) = (((a + e) + b) + d) + c$.

All such sums can be written simply as $a + b + c + d + e$.

Definition of subtraction: The **difference of a and b** is written $a - b$ and is defined by $a - b = a + (-b)$.

Multiplication laws

Identity: $1(a) = a(1) = a$

Commutative: $ab = ba$.

Associative: $a(bc) = (ab)c$ Both are written as abc .

Inverse: Let a be any nonzero expression. Then there is an expression $\frac{1}{a}$ with the

property $a\left(\frac{1}{a}\right) = 1$. In calculator notation, we write $a(1/a) = 1$.

The commutative and associative laws allow us to compute a product of three or more expressions in any order. For example, $(ab)((cd)e) = (((ae)d)c)b$.

All such products can be written simply as $abcde$.

Definition of division:

- If b is nonzero then the **quotient of a by b** , written $\frac{a}{b}$, is defined to be $a\left(\frac{1}{b}\right)$.

In calculator notation the quotient is $a / b = a(1 / b)$.

- If $b = 0$, the quotient is undefined.

Distributive law: the connection between products and sums

- $a(b + c) = ab + ac$

This law is the basis for multiplication rules such as FOIL, which says that

$$(a + b)(c + d) = ac + ad + bc + bd.$$

Power laws (rules for exponents):

$$\text{a) } a^1 = a \quad \text{b) } a^{b+c} = a^b a^c; \quad \text{c) } (a^b)^c = a^{bc}$$

In calculator notation:

$$\text{a) } a^1 = E \quad \text{b) } a^{(b+c)} = (a^b)(a^c) \quad \text{c) } (a^b)^c = a^{(bc)}$$

The above laws are called axioms or definitions. All other algebra rules can be derived as theorems. In particular, you might want to prove the following:

Theorem 1: Exponential notation

$$\text{a) } a^0 = 1 \quad \text{b) } a^{-1} = \frac{1}{a} \quad \text{c) } a^2 = aa; \quad a^3 = aaa, \text{ etc.}$$

$$\text{d) } a^{-2} = (a^{-1})^2 = \left(\frac{1}{a}\right)^2 = \frac{1}{a^2} \quad \text{e) } a^{-b} = \frac{1}{a^b}$$

Theorem 2: Subtraction and addition are inverse operations:

$$\text{a) } (a + b) - b = a \quad \text{b) } (a - b) + b = a$$

Theorem 3: Division and multiplication are inverse operations.

$$\text{a) } \frac{ab}{b} = a \quad \text{b) } \frac{a}{b}(b) = a$$

In calculator notation:

a) $ab/b = a$

b) $(a/b)b = a$

Theorem 4: Fraction rules:

a) $\frac{ab}{ac} = \frac{b}{c}$

b) $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

c) $\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$

In calculator notation: **a)** $(ab)/(ac) = b/c$ **b)** $(a/b) + (c/d) = (ad + bc)/(bd)$

c) $(a/b)(c/d) = (ac)/(bd)$

2.5 Substituting in Identities

In Chapter 1, the discussion of algebra rules such as the distributive law $a(b + c) = ab + ac$ seemed to assume that the variables used were stand-ins for numbers. The big picture of algebra is that the variables in those laws can stand for arbitrary expressions which themselves involve variables. The main idea is as follows.

The laws of algebra, listed in the last section, are basic identities that apply to all expressions. When you substitute expressions for variables in algebra laws, you get new identities. Just remember to put parentheses around the substituted expressions!

Algebra involves using what students usually call formulas, or rules: we call both of these identities, because they are statements that two expressions are identical. Given any expression E , the statement $E = E$ is an obvious identity.

Instead of saying that $F = E$ is an identity, we will simply write $F = E$.

When you do algebra, you substitute for variables in algebra rules to get new identities.

Fundamental Substitution Principle: When you substitute an expression for a variable in an identity, the result is an identity.

In practice, this principle is applied as follows: Suppose x is a variable, F is an expression you are working with containing variable x , and E is an expression that you know is equal to x —that is, $x = E$. If you substitute expression E for variable x in expression F to get a new expression F^* , then $F^* = F$.

Example 2.5.1:

Substitute $3x$ for a , x^2 for b , and 10 for c
in the distributive law: $a(b + c) = ab + ac$.

You get a new identity: $3x(x^2 + 10) = (3x)(x^2) + (3x)(10)$

Now substitute $x + 2$ for x in this new identity to get a third identity:

$$3(x + 2)((x + 2)^2 + 10) = (3(x + 2))((x + 2)^2) + (3(x + 2))(10)$$

Again we caution that many, many algebra errors occur because students forget the parentheses when they substitute. Remember:

When you substitute an expression for a variable, erase and replace the variable by the expression *written inside parentheses*.

Exercise 2.5.1: Substitute as indicated but do not simplify

| |
|--|
| a) $x + 3$ for a and $x - 2$ for b in $\frac{a}{b} - \frac{b}{a} = 1$ |
| b) $x + a$ for x in $x - x^2 = x(1 - x)$ |
| c) -3 for x in $3 - x - x^2 = 3 - x(1 + x)$ |
| d) $2 - h$ for a , $3 - h$ for b , and $4 - h$ for c in $a - b - c = -b - c + a$ |
| e) $x - h$ for a in $-a(x - a)(x + a) = a^3 - ax^2$ |
| f) $x + h$ for x in $4 - x^2 = (2 + x)(2 - x)$ |
| g) $x + h$ for x in $-x + y = y - x$ |
| h) b for a in $ax + b = \frac{ax^2 + bx}{x}$ |
| i) $(x + 2)^2$ for a and $3x + 2$ for b in $ab - a^2 = a(b - a)$ |
| j) $(x + 2)^2$ for a and $3x + 2$ for b in $ac - c^2 = c(a - c)$ |
| k) $a + b$ for a and $a - b$ for b in $a^2 - b^2 = (a + b)(a - b)$ |
| l) $x + 3$ for a and $x - 2$ for b in $\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}$ |
| m) $a + b$ for a and $a - b$ for b in $\frac{b - a - 2}{a - b + 2} = -1$ |
| n) $x + h$ for x in $f(x) = x(f)$ |
| o) $x + h$ for x in $f(x) = 3 - x - x^2$ |
| p) $x + h$ for x in $f(x) = 3 - z - z^2$ |

Examples n), o), and p) may seem puzzling. Example n) is an identity if f is a variable. In examples o) and p), f is the name of a function and the identity shown is a special kind of identity called a function definition, which will be discussed in the following section.

Exercise 2.5.2: Find an algebra law with the following property: Omitting parentheses when you substitute 4 for a and 5 for b yields a false statement.

2.6 Order of operations for general expressions

It's tricky to work with expressions that involve both variables and parentheses. In special cases such as $a + b + c + d + e$ and $abcde$, you can perform the operations in any

order. In most cases, the order of operations does matter and the safest procedure is to insert parentheses that indicate the order of operations.

Example 2.6.1: Write $7 - (x + 1)(x + 7)$ as a simplified sum.

Notice that the expression doesn't have parentheses showing whether the first operation is $7 - (x + 1)$ or $(x + 1)(x + 7)$. However, according to the order of operations rules, the multiplication is done first.

Now comes the tricky part: Since $(x + 1)(x + 7) = x^2 + 8x + 7$, you might begin by writing $7 - (x + 1)(x + 7) = 7 - x^2 + 8x + 7$. That's not correct! The order of operations rule tells you where to insert parentheses: the ambiguous expression $7 - (x + 1)(x + 7)$ really means $7 - [(x + 1)(x + 7)]$. Here we used square brackets to show the new parentheses that were inserted to indicate the order of operations.

Therefore, you should proceed as follows:

| | |
|--|--------------------------|
| The original expression is | $7 - (x + 1)(x + 2)$ |
| Insert parentheses around the product, since multiplication comes before subtraction. | $= 7 - [(x + 1)(x + 7)]$ |
| Find the product inside the parentheses. | $= 7 - [x^2 + 8x + 7]$ |
| Use the distributive law. | $= 7 - x^2 - 8x - 7$ |
| Collect like terms. | $= -x^2 - 8$ |

To save a step, figure out which algebra operation needs to be done first and then place parentheses around the result of the operation, as follows:

| | |
|--|------------------------|
| The original expression is | $7 - (x + 1)(x + 2)$ |
| Insert parentheses around the product. | $= 7 - [x^2 + 8x + 7]$ |
| Use the distributive law. | $= 7 - x^2 - 8x - 7$ |
| Collect like terms. | $= -x^2 - 8$ |

Always keep in mind the following: algebra works by rewriting expressions. You should

- **Know and use PEMDAS to follow the order of operations.**
- **Put parentheses around the result of each algebra operation.**

In some cases, the inserted parentheses might be superfluous, but it's better to be safe than sorry!

Keep in mind the order of operations as you read through the examples below. For clarity, they all show the inserted parentheses as square brackets.

In 1a), parentheses are inserted both before and after the algebra operation is carried out. You can safely omit the “before” step as in 1b). You may omit the “after” step as well, as in 1c), but be careful! Other problems are written out in full detail.

$$\begin{array}{lll} \mathbf{1a)} \left\{ \begin{array}{l} 3 - 4(x + 2) \\ = 3 - [4(x + 2)] \\ = 3 - [4x + 8] \\ = 3 - 4x - 8 \\ = -4x - 5 \end{array} \right. & \mathbf{1b)} \left\{ \begin{array}{l} 3 - 4(x + 2) \\ = 3 - [4x + 8] \\ = 3 - 4x - 8 \\ = -4x - 5 \end{array} \right. & \mathbf{1c)} \left\{ \begin{array}{l} 3 - 4(x + 2) \\ = 3 - 4x - 8 \\ = -4x - 5 \end{array} \right. \end{array}$$

$$\begin{array}{lll} \mathbf{2)} \left\{ \begin{array}{l} x^2 - x(x + 2) \\ = x^2 - [x(x + 2)] \\ = x^2 - [x^2 + 2x] \\ = x^2 - x^2 - 2x \\ = 0 - 2x \\ = -2x \end{array} \right. & \mathbf{3)} \left\{ \begin{array}{l} x^2 - (x + 2)(x - 3) \\ = x^2 - [(x + 2)(x - 3)] \\ = x^2 - [x^2 + 2x - 3x - 6] \\ = x^2 - [x^2 - x - 6] \\ = x^2 - x^2 + x + 6 \\ = x + 6 \end{array} \right. & \mathbf{4)} \left\{ \begin{array}{l} x(x + 2)^2 \\ = x[(x + 2)^2] \\ = x[x^2 + 4x + 4] \\ = x^3 + 4x^2 + 4x \end{array} \right. \end{array}$$

$$\begin{array}{ll} \mathbf{5a)} \left\{ \begin{array}{l} x^3 / (x + 1)^2 \\ = x^3 / [(x + 1)^2] \\ = x^3 / (x^2 + 2x + 1) \end{array} \right. & \mathbf{5b)} \left\{ \begin{array}{l} x^3 \div (x + 1)^2 \\ = x^3 \div [(x + 1)^2] \\ = x^3 \div (x^2 + 2x + 1) \end{array} \right. \end{array}$$

$$\begin{array}{ll} \mathbf{6)} \left\{ \begin{array}{l} 2x^2 - x(x - 2)^2 \\ = 2x^2 - x[(x - 2)^2] \\ = 2x^2 - x[x^2 - 4x + 4] \\ = 2x^2 - [x^3 - 4x^2 + 4x] \\ = 2x^2 - x^3 + 4x^2 - 4x \\ = -x^3 + 2x^2 + 4x^2 - 4x \\ = -x^3 + 6x^2 - 4x \end{array} \right. & \mathbf{7)} \left\{ \begin{array}{l} x^2 - (x + 2)(x + 3) - (1 - x)^2 \\ = x^2 - [(x + 2)(x + 3)] - [(1 - x)^2] \\ = x^2 - [x^2 + 5x + 6] - [1 - 2x + x^2] \\ = x^2 - x^2 - 5x - 6 - 1 + 2x - x^2 \\ = x^2 - x^2 - x^2 - 5x + 2x - 6 - 1 \\ = -x^2 - 3x - 7 \end{array} \right. \end{array}$$

$$\begin{array}{ll} \mathbf{8)} \left\{ \begin{array}{l} 7x^3 - (x^2 - 2x + 5) \\ = 7x^3 + [(-1)(x^2 - 2x + 5)] \\ = 7x^3 + [-x^2 + 2x + 5] \\ = 7x^3 - x^2 + 2x + 5 \end{array} \right. & \mathbf{9)} \left\{ \begin{array}{l} 7x^3 + 4(x^2 - 2x + 5) \\ = 7x^3 + 4x^2 - 8x + 20 \end{array} \right. \end{array}$$

$$\begin{array}{ll}
 \text{10)} \left\{ \begin{array}{l} 7x^3 - 3x^4(x^2 - 2x) \\ = 7x^3 - [3x^4(x^2 - 2x)] \\ = 7x^3 - [3x^6 - 6x^5] \\ = 7x^3 - 3x^6 + 6x^5 \\ = -3x^6 + 6x^5 + 7x^3 \end{array} \right. & \text{11)} \left\{ \begin{array}{l} 1 - 4x - 2x(x^2 - 4x - 3) \\ = 1 - 4x - [2x(x^2 - 4x - 3)] \\ = 1 - 4x - [2x^3 - 8x^2 - 6x] \\ = 1 - 4x - 2x^3 + 8x^2 + 6x \\ = -2x^3 + 8x^2 + 6x - 4x + 1 \\ = -2x^3 + 8x^2 + 2x + 1 \end{array} \right.
 \end{array}$$

Exercise 2.6.1: Rewrite each of the following as a simplified sum.

| |
|-----------------------------------|
| a) $2 + 2(2 + 2)$ |
| b) $x + 3(x + 2)$ |
| c) $x - 3(x + 2)$ |
| d) $7 + 3(x + 2)$ |
| e) $7 - 3(x + 2)$ |
| f) $7 - 3(4 + 2)$ |
| g) $x^2 - 3(x + 1)^2$ |
| h) $x^2 - 3(x + 1)^2$ |
| i) $x^2 - (x + 2)(x + 1)$ |
| j) $x^2 - 3(x + 2)(x + 1)$ |
| k) $x^2 + 3(x + 2)(x + 1)$ |
| l) $x^2 - 3(x + 2)(x + 1)$ |
| m) $x^2 - (x + 1)^2$ |
| n) $x^2 - (1 - x)^2$ |

Exercise 2.6.2: Rewrite each expression in one variable as a standard form polynomial, with no repeated powers and powers decreasing from right to left. Rewrite each expression involving two variables as a simplified sum. Remember to put parentheses around the result of each algebra operation.

| |
|--|
| a) $A] - 3 - (x + 2)^2$ |
| b) $B] (x + 2)^2 + (x - 3)^2 =$ |
| c) $C] x - (x + 1)(x + 2) =$ |
| d) $x^2 - 7x - (x + 2)^2 =$ |
| e) $(x + 2)(x + 1) - 2(x + 3)(x + 4) =$ |
| f) $(x + 2)(x + 1)^2$ |
| g) $(a + b)^2 - (a - b)^2 =$ |

| |
|--|
| h) $x - 3 - 2(x - 3)^2 =$ |
| i) $-5(a - b) - 3(a - b)^2$ |
| j) $7 - x^2 - 7(1 - x^2) - (1 - x)^2$ |
| k) $(x + y)(x + 2y) - (2x + y)^2$ |
| l) $x^2(y + 1)^2 - y^2(x + 1)^2$ |
| m) $-3x - 4x - 5x - 6x - 7x^2 - 3x^2$ |
| n) $(y + 1 - 2y)^2$ |

2.7 Classification of Expressions

This section is a kind of technical reference to terminology found in mathematics texts. It lists the kinds of expressions that can be substituted for variables in the basic identities discussed in the previous section. Some of this material has been covered before: it's always good to review.

An integer is a whole number, possibly preceded by a sign: $0; \pm 1; \pm 2; \pm 3, \dots$

A rational number is a quotient of integers: $\frac{3}{4}; -\frac{7}{8}; \frac{453}{5}; \frac{8}{1}$

A variable is a letter, usually taken from the English or Greek alphabet. The most frequently used variables are x, y, z, s and t .

A positive integer power of x is one of the following: $x^0, x^1, x^2, x^3, \dots$. These symbols are abbreviations for repeated multiplication, as follows.

$$x^0 = 1;$$

$$x^1 = x;$$

$$x^2 = xx, \text{ which means } x \text{ times } x;$$

$$x^3 = xxx$$

.....

Negative integer powers are alternate notation for reciprocals: x^{-3} is a way of writing $\frac{1}{x^3}$ that avoids using fractions. Another point of view: $\frac{1}{x^3}$ is a way of writing x^{-3} that avoids using negative exponents.

A monomial in x is an integer, a power of x , or an integer times a positive integer power of x .

The **degree** of a monomial in x is the power to which x is raised. It is undefined if the monomial is zero, or is zero if the monomial is an integer. For example

-3 , 1 , x , $7x$, x^{24} , $-444x^{25}$ are monomials with degrees
 0 , 0 , 1 , 1 , 24 , and 25 respectively.

The degree of the zero monomial is undefined.

The **coefficient** of a monomial is defined separately for the three cases just listed:

The coefficient of an integer is the integer.

The coefficient of a power is 1 .

The coefficient of an integer times a power of x is the integer.

Here are nine monomials: 0 , 3 , -3 , x^0 , $3x^0$, x , x^3 , $-12x^3$, $3456x^{2345}$

Their coefficients are: 0 , 3 , -3 , 1 , 3 , 1 , 1 , -12 , 3456 .

A **polynomial** is a monomial or a sum (or difference) of monomials.

Examples are $4 + 4x + 3x^2$; $-2x^{34}$; $x^3 + -17x + 2$.

The last example is always written $x^3 - 17x + 2$.

A **standard form polynomial** is one in which all the monomials have different powers and the powers of the monomials decrease as you read from left to right. If a polynomial is in standard form, then

- the **leading term** is the monomial with the highest power of x .
- the **leading coefficient** is the coefficient of the leading term.

Every polynomial can be rewritten in standard form: reorder and then collect terms by using the distributive law..

$$\begin{aligned} & x^2 + 3x^3 - 7x^2 + 2x^3 \\ \text{Example: } &= 3x^3 + 2x^3 + 1x^2 - 7x^2 \\ &= (3 + 2)x^3 + (1 - 7)x^2 \\ &= 5x^3 - 6x^2 \end{aligned}$$

The **degree of a polynomial** is the highest power of the monomials in the polynomial obtained by rewriting the original polynomial in standard form.

Examples:

0 has leading coefficient 0 .

-32 has leading coefficient -32

$7x - 32$ has leading coefficient 7.

$x^{128} + 67x^{127} - 33$ has leading coefficient 1.

$x^3 + x - 17$ has degree 3.

$x^2 + x^3$ has degree 3.

$x^3 + 1 + 2x^3 + x^2 - 3x$ in standard form is $x^2 + 1$ and so has degree 2.

A monic polynomial is a standard form polynomial with leading coefficient 1.

Make sure you know the above definitions well. The remaining ones below are more technical: refer back to them when necessary.

A polynomial expression is built out of polynomials by the operations of adding, subtracting, multiplying and raising to a positive integer power.

Every polynomial expression can be rewritten as a polynomial by “multiplying out” to remove parentheses. For example:

$$\begin{aligned} & ((x+1)(x+2) - 3x)^2 \\ &= (x^2 + 3x + 2 - 3x)^2 \\ &= (x^2 + 2)^2 \\ &= x^4 + 4x^2 + 4 \end{aligned}$$

A rational polynomial is a fraction of polynomials, possibly preceded by a minus sign. The fraction line means that the top (numerator) is divided by the bottom (denominator).

A standard form rational polynomial is one whose numerator and denominator are in standard form and have positive leading coefficients.

Example: Rewrite $\frac{3 - x^2}{2x + x^7 + 18}$ in standard form as follows:

$$\frac{3 - x^2}{2x + x^7 + 18} = \frac{-(x^2 - 3)}{x^7 + 2x + 18} = -\frac{x^2 - 3}{x^7 + 2x + 18}$$

A polynomial is a special kind of rational function, since $p(x) = \frac{p(x)}{1}$.

A rational polynomial expression is a fraction of polynomial expressions. Example:

$$\frac{(4 - (x+2)^2)^3 + x + 2}{((4 - (x+2)^2)^3 + x + 2)^2}$$

rational polynomial by “multiplying out” to remove parentheses.

A (general) rational expression is built out of polynomial expressions by adding, subtracting, or multiplying two rational polynomial expressions, or by raising a rational polynomial expression to a positive *or negative* integer power.

Here is a modest example

$$\frac{\frac{(4-x^2)^3 + x + 2}{\left((4-(x-2)^2)^3 + x + 2\right)^2} + \frac{(4-(x-1)^2)^3 + x + 2}{\left((4-(x+1)^2)^3 + x + 2\right)^2}}{\frac{(4-(x+2)^2)^3 + x + 2}{\left((4-(x+3)^2)^3 + x + 2\right)^2}} + \frac{(4-(x+2)^2)^3 + x + 2}{\left((4-(x+3)^2)^3 + x + 2\right)^2}$$

Every rational expression can be rewritten as a rational polynomial by (possibly a great deal of) multiplying out and working with fractions to combine everything into a standard form rational polynomial. You might try to do that simplification for the above example. You will quickly be discouraged unless you have access to a computer algebra system.

In the above example, exponents are restricted to be integers. If they are allowed to be rational numbers, then the expressions are called **algebraic expressions**. You know, of course, that rational powers can be written with radical signs: the basic example is that \sqrt{x} can be rewritten as $x^{1/2}$.

CHAPTER 3: SIMPLIFYING POLYNOMIAL EXPRESSIONS

You are familiar with polynomial expressions in a single variable x . Such expressions involve numbers, the variable x , and whole number exponents. Division signs and fractions are not permitted. Some examples of polynomial expressions, in a list separated by semicolons, are:

$$\begin{aligned} &0; \quad 2; \quad -15 + 3; \quad 2^3 \\ &x; \quad x^2; \quad 7x; \quad {}^3 27x; \quad 3x^3; \quad x^3 x^7 \\ &(x+2)^2; \quad (x+2)^2 + 3x^7 x^8 \\ &7x^3 + 5x^2 - 7 \end{aligned}$$

In this section, we show how to use the laws of algebra to rewrite any polynomial expression as a polynomial in a standard way.

3.1 Monomials and Polynomials

A **monomial** in x is a number, a whole number power of x , or a non-zero number times a positive integer power of x .

Examples: 17 ; x ; x^3 ; $-17x^3$

We will automatically rewrite the monomial kx^0 as k , since $x^0 = 1$

A **polynomial** in x is a sum of simplified monomials in x .

Examples: 0 ; 3 ; $7 - 3$; $7 - 3x$; $2x - 7x$; $3x + x^2 - 7x - 7$;

A **simplified polynomial** in x is a sum of simplified monomials in x in which no two monomials have the same power.

Examples: 0 ; 3 ; $7 - 3x$; $3x + x^2 - 7x^4$

A **standard form polynomial** in x (also called a polynomial in standard form) is a simplified polynomial in x in which the powers of x decrease as you read from left to right.

Examples: 0 ; $3x + 1$; $-3x$; $x^2 - 12x + 9$; $-17x^5 + x^3 + 2$

Almost everything we have studied so far: the associative, commutative, and distributive laws, as well as the order of operations, will be used in the next section.

3.2 Rewriting polynomial expressions in standard form

1. To multiply powers of x , add exponents

Example: $x^3x^2 = x^3 \times x^2 = xxx \times xx = xxxxx = x^{3+2} = x^5$

2. Use the commutative and associative laws for multiplication to rewrite a product of numbers and powers of x as a simplified monomial.

Example: $3(4x^3)(-7x^{10})(2x^5) = 3(4)(-7)(2)x^3x^{10}x^5 = 12(-14)x^{3+10+5} = -168x^{18}$

3. Use the distributive law to combine like terms

Example: $3x^2 - 7x^2 + 9x^2 = (3 - 7 + 9)x^2 = (-4 + 9)x^2 = 5x^2$

We have described a polynomial as a sum of monomials. Remember that a combination of additions and subtractions is still a sum. That's because subtraction is defined by $a - b = a + (-b)$. Here is an example, shown with an intermediate level of detail.

| | |
|---|-----------------------------------|
| Original expression with additions and subtractions | $-7x - 9 + x^3 + 6x^2$ |
| Subtractions rewritten as additions | $= (-7x) + (-9) + (x^3) + (6x^2)$ |
| Reorder the additions (commutative law) | $= (x^3) + (6x^2) + (-7x) + (-9)$ |
| Rewrite some additions as subtractions | $= x^3 + 6x^2 - 7x - 9$ |

4. Use the commutative and associative laws for addition to rewrite a polynomial in standard form by parenthesizing and reordering terms, and then use the distributive law to combine like terms.

When you rearrange the terms of the polynomial, remember that the sign in front of each monomial is part of the term, and must be carried along when the term is moved.

Example: All details are shown. Try to solve the problem by leaving out as many intermediate steps as possible!

$$\begin{aligned}
 & 2x^2 - 7 - 5x^2 - 2x^3 + 17 - 9 + 4x^2 - 4x^3 \\
 &= -2x^3 - 4x^3 - 5x^2 + 2x^2 + 4x^2 - 7 + 17 - 9 \\
 &= -2x^3 + (-4x^3) + (-5x^2) + 2x^2 + 4x^2 + (-7) + (-9) + 17 \\
 &= [-2 + (-4)]x^3 + [-5 + 2 + 4]x^2 + [(-7) + (-9) + 17] \\
 &= -6x^3 + 1x^2 + [-16 + 17] \\
 &= -6x^3 + x^2 + 1
 \end{aligned}$$

More examples are shown below. Example 3) shows details of reordering, and omits the parentheses used in the above example. Example 4) omits details of reordering, and that's how you should eventually be able to write out such problems.

$$\begin{array}{lcl}
 \mathbf{1)} \left\{ \begin{array}{l} 3x + 2x \\ = (3 + 2)x \\ = 5x \end{array} \right. & \mathbf{2)} \left\{ \begin{array}{l} 4x - 7x + 2x \\ = (4 - 7 + 2)x \\ = (-1)x \\ = -x \end{array} \right. & \mathbf{3)} \left\{ \begin{array}{l} -9 - 6x^2 + x^3 - 7x^2 \\ = -9 + -6x^2 + x^3 + -7x^2 \\ = x^3 + -6x^2 + -7x^2 + -9 \\ = x^3 + -13x^2 + -9 \\ = x^3 - 13x^2 - 7x - 9 \end{array} \right.
 \end{array}$$

$$\begin{array}{lcl}
 \mathbf{4)} \left\{ \begin{array}{l} 4x^2 - 7x + 4x^3 - 8x^2 + 9x \\ = 4x^3 + 4x^2 - 8x^2 - 7x + 9x \\ = 4x^3 - 4x^2 + 2x \end{array} \right. & \mathbf{5)} \left\{ \begin{array}{l} 3ab + 6ba + 8ab \\ = 3ab + 6ab + 8ab \\ = (3 + 6 + 8)ab \\ = 17ab \end{array} \right. &
 \end{array}$$

The following examples, preparation for the next section illustrate how the distributive law is used to remove parentheses when a monomial is multiplied by a polynomial.

$$\begin{array}{lcl}
 \mathbf{1)} \left\{ \begin{array}{l} -(x^2 - 2x + 5) \\ = -1(x^2 - 2x + 5) \\ = -x^2 + 2x - 5 \end{array} \right. & \mathbf{2)} \left\{ \begin{array}{l} 4(x^2 - 2x + 5) \\ = 4(x^2) + 4(-2x) + 4(5) \\ = 4x^2 - 8x + 20 \end{array} \right. & \\
 \mathbf{3)} \left\{ \begin{array}{l} -3(x^2 - 2x) \\ = -3x^2 - 3(-2x) \\ = -3x^2 + (-3)(-2x) \\ = -3x^2 + 6x \end{array} \right. & \mathbf{4)} \left\{ \begin{array}{l} 2x(x^2 - 4x - 3) \\ = (2x)(x^2) + (2x)(-4x) + (2x)(-3) \\ = 2x^3 - 8x^2 - 6x \end{array} \right. &
 \end{array}$$

3.3 Rewriting a product of sums as a sum of products

Each of the basic laws of algebra described in the section on numerical expressions applies as well to expressions involving variables. All of the algebra laws that you know are the result of using substitution in the basic algebra laws. As an example, the following material shows how to use the distributive law to derive the FOIL law

$$(a + b)(c + d) = ac + ad + bc + bd.$$

Start with the distributive law

$$a(b + c) = ab + ac$$

Substitute $x + y$ for a to obtain

$$(x + y)(b + c) = (x + y)b + (x + y)c$$

On the right side, apply the distributive law

$$(x + y)(b + c) = xb + yb + xc + yc$$

Use commutative law for $+$ to reorder the right side

$$(x + y)(b + c) = xb + xc + yb + yc$$

The result is the FOIL law. It may look more familiar if you substitute a for x , b for y , c for b , d for c to obtain the identity

$$(a + b)(c + d) = ac + ad + bc + bd$$

In turn, you can substitute any expressions whatsoever for the variables a , b , c , d to obtain new identities that are consequences of the FOIL law.

Any identity tells you two methods for rewriting expressions. In this case:

- You can rewrite $(a + b)(c + d)$ as $ac + ad + bc + bd$, or
- You can rewrite $ac + ad + bc + bd$ as $(a + b)(c + d)$

The first rewriting operation, called **“multiplying out,”** rewrites a product of sums as a sum of products.

The second rewriting operation, called **“factoring,”** rewrites a sum of products as a product of sums.

The basic idea of multiplying out is that to multiply two sums of variables, you multiply each variable in the first sum by each variable in the second sum and add the resulting products:

$$\begin{aligned}(a + b)(x + y) &= ax + ay + bx + by \\(a + b)(x + y + z) &= ax + ay + az + bx + by + bz \\(a + b + c)(x + y + z) &= ax + ay + az + bx + by + bz + cx + cy + cz \\(a + b)^2 &= a^2 + 2ab + b^2\end{aligned}$$

The last line is an important special case of the FOIL law, since

$$(a + b)(a + b) = aa + ab + ba + bb = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

The same ideas work if some additions are replaced by subtractions. Recall that

$$(a)(-b) = (-a)(b) = -a(b) = a(-b) = -ab \text{ and } (-a)(-b) = ab. \text{ Then}$$

$$\begin{aligned}(a - b)(x + y) &= ax + ay - bx - by \\(a - b)(x - y) &= ax - ay - bx + by \\(a + b)(x - y + z) &= ax - ay + az + bx - by + bz \\(a - b)^2 &= a^2 - 2ab + b^2\end{aligned}$$

We repeat an earlier warning: $-a - b$ is read “minus a subtract b ” and has nothing to do with the products $(a)(-b) = (-a)(b) = -a(b) = a(-b) = -ab$

The following identity (difference of squares) is of particular importance:

$$(a + b)(a - b) = a^2 - b^2 \text{ and is usually presented as a method for factoring the right side.}$$

Examples include

$$(x^2 - 4) = (x + 2)(x - 2)$$

$$(x^2 - 7) = (x + \sqrt{7})(x - \sqrt{7})$$

$$(4x^2 - 9y^4) = (2x + 3y^2)(2x - 3y^2)$$

Example 3.3.1: Use a similar rule to rewrite $(x^2 - 2x)(x^3 - x + 7)$ as a standard form polynomial.

Solution:

$(x^2 - 2x)(x^3 - x + 7)$ clearly follows the pattern of the following identity, in which we will substitute x^2 for a ; $2x$ for b ; x^3 for c ; $-x$ for d ; and 7 for e :

$$\begin{aligned} (a - b)(c - d + e) &= ac - ad + ae - bc + bd - be \\ (x^2 - 2x)(x^3 - x + 7) &= x^2(x^3) - x^2(x) + x^2(7) - 2x(x^3) - 2x(x) - 2x(7) \\ &= x^5 - x^3 + 7x^2 - 2x^4 + 2x^2 - 14x \\ &= x^5 - 2x^4 - x^3 - 2x^2 + 7x^2 + 2x^2 - 14x \\ &= x^5 - 2x^4 - x^3 + 9x^2 - 14x \end{aligned}$$

Exercise 3.3.1: Convert each product of polynomials to a simplified sum: first multiply out and then reorder and collect terms. The last two expressions were generated by substitution in the distributive law and should yield the same standard form polynomial.

| |
|--------------------------------------|
| a] $(a + b)(a - b) =$ |
| b] $(a^2 + b^2)(a^2 - b^2) =$ |
| c] $(a^2 - b^2)^2 =$ |
| d] $3x^2(x^2 - 2) =$ |
| e] $3x^2(x^2 - x - 2) =$ |
| f] $-3x^2(x^2 - x - 2) =$ |
| g] $(x^2 - 2)(x - 5) =$ |
| h] $(3x + 2)(x^3 - 5) =$ |
| i] $(3x - 2)(x - 5) =$ |
| j] $(x + 1)(x - 1) =$ |
| k] $(3x - 2)^2 =$ |
| l] $(3x + 2)(x - 5) =$ |
| m] $x^2(3x + 2)(x - 5) =$ |
| n] $x^3(x + 1)^2 =$ |
| o] $(x + 2)(x + 5)(x + 1) =$ |

| |
|--|
| p] $2x^2(x+2)(x+5) =$ |
| q] $(x+2)(x+2)^2$ |
| r] $(x-2)^3 =$ |
| s] $3(x+2)((x+2)^2+10)$ |
| t] $(3(x+2))((x+2)^2) + (3(x+2))(10)$ |

3.4 Factoring: rewriting a sum as a product

In the previous section, we used the distributive law to rewrite products as sums, a process called “multiplying out.” Now we reverse direction and use the distributive law to rewrite a sum as a product. When you factor (verb) an expression, you rewrite it as a product of simpler expressions, which are called factors (noun) of the original expression.

The distributive law $a(b+c) = ab+ac$ says that the product of a and $b+c$ is identical to the sum of ab and ac . In other words, you can factor $ab+ac$ as follows

$$ab+ac = a(b+c)$$

Substituting x for a , $2x^2$ for b , and x^3 for c , gives

$$(x)(2x^2) + (x)(x^3) = (x)((2x^2) + (x^3))$$

Multiply on the left side and remove extra parentheses: $2x^3 + x^4 = (x)(2x^2 + x^3)$

However, we could factor the right side further by pulling out x more times:

$$2x^3 + x^4 = (x)(2x^2 + x^3) = (x)(x)(2x + x^2) = (x)(x)(x)(2 + x) = x^3(2 + x)$$

We want to automate the process of factoring sums such as $x^3 + x^4$. To do this, look for the greatest common factor of x^3 and x^4 , namely the largest expression (in this case, the highest power of x) which is a factor of both.

In the expression $x^3 + x^4$, we see that $x^3 = x^3 \times 1$ and $x^4 = x^3 \times x$. We “pull out” x^3 from $x^3 + x^4$ and obtain $2x^3 + x^4 = x^3(2 + x)$. Note that this is what you get from the distributive law $ab + ac = a(b + c)$ when you substitute x^3 for a , 2 for b , and x for c .

Technical note: in the following definitions, all expressions are assumed to be rational polynomial expressions with integer coefficients.

Definitions involving factoring: (all expressions are polynomial expressions)

- Expression a is a **factor** of expression E if there is a third expression b such that $E = ab$. In this case, b is also a factor of E .
- Expression E is **prime** if its only factors are 1, -1 , E , and $-E$. Examples are 3, 19, $x+2$, and x^2+1 .

- Expression a is the **greatest common factor** (GCF) of expressions E and F if there are factors e of E and f of F such that
 - $E = ae$, $F = af$, and
 - e and f don't have a common factor.
 In such a case we can factor $E + F$ as $a(e + f)$, but $e + f$ can't be factored any more, at least by using this method.

Example 3.4.1: Factor $2x^4 + 4x^3 + 2x^2$ by pulling out the GCF of the terms.

Solution: $2x^4 + 4x^3 + 2x^2 = 2x^2(x^2 + 2x + 1)$

The terms of the $x^2 + 2x + 1$ have no common factor, but you probably realize that we can factor by the FOIL method. Here's the language that would be used.

Example 3.4.2: Factor $2x^4 + 4x^3 + 2x^2$ completely.

Solution: $2x^4 + 4x^3 + 2x^2 = 2x^2(x^2 + 2x + 1) = 2x^2(x + 1)^2$

Please master the examples in the following table of factoring problems of increasing difficulty. It is followed by a table of related exercises.

Example 3.4.3

Rewrite each sum of terms as a product by factoring out the GCF of its terms.

| Factor the expression: | Solution (sometimes uses substitution, shown in the column to the right) | Comments |
|------------------------|---|---|
| $4a + 8b + 6c$ | $4a + 8b + 6c$ $= 2(2a + 4b + 3c)$ Finished: 2, 4, 3 have no common factor. | 2 goes into the coefficients 4,8,6. |
| $20a + 40b + 30c$ | $30a + 45b + 60c$ $= 5(6a + 9b + 12c)$ $= 5 \times 3(2a + 3b + 4c)$ $= 15(2a + 3b + 4c)$ | 5 goes into 30, 45 and 60. Then 3 goes into 6,9,12 |
| $x^3 + x^4 + x^7$ | $x^3 + x^4 + x^7$ $= x^3(1 + x + x^4)$ | Pull out the lowest x -power, which is the GCF |
| $12x^3 + 9x^4 + 18x^7$ | $12x^3 + 9x^4 + 18x^7$ $= 3(4x^3 + 3x^4 + 6x^7)$ $= 3x^3(1 + x + x^4)$ | Pull out the GCF of the coefficients 12, 9, 18 and also pull out lowest x -power. |

| | | |
|--|--|--|
| $a^3b^5 + a^2b^7 + a^4b^4$ | $a^3b^5 + a^2b^7 + a^4b^4$ $= a^2(ab^5 + b^7 + a^2b^4)$ $= a^2b^4(ab + b^3 + a^2)$ | Pull out lowest a -power. Then pull out lowest b -power. |
| $4(x+1)^3(x+2)^5$ $+ 5(x+1)^4(x+2)^4$ | $4(x+1)^3(x+2)^5 + 5(x+1)^4(x+2)^4$ $= 4a^3b^5 + 5a^4b^4$ $= a^3b^4(4b + 5a)$ $= (x+1)^3(x+2)^4[4(x+2) + 5(x+1)]$ $= (x+1)^3(x+2)^4[4x+8+5x+5]$ $= (x+1)^3(x+2)^4[9x+13]$ | Substitution used was $\begin{cases} a \leftrightarrow x+1 \\ b \leftrightarrow x+2 \end{cases}$ Problem arises in Calc 1 as derivative of $(x+1)^4(x+2)^5$ |

Exercise 3.4.1: Factor each of the following sums by pulling out the GCF of the terms, or write DNF (does not factor using GCF method). Then rewrite each factor as a simplified sum.

| |
|---|
| a) $9a + 8b + 6c$ |
| b) $42a + 35b + 49c$ |
| c) $30a + 42b + 66c$ |
| d) $ax^3 + ax^4 + ax^{12}$ |
| e) $30x^3 + 60x^4 + 45x^{17}$ |
| f) $a^3b^5 + a^2b^7 + a^4b^4$ |
| g) $12a^3b^5 + 4a^2b^7 + 24a^4b^4$ |
| h) $12a^3b^5 + 4a^2b^7 + 24a^4c^7$ |
| i) $16a^3c^5 + 24a^2b^7 + 24a^4c^7$ |
| j) $4x^3(x+2)^5 + 5x^4(x+2)^4$ |
| k) $4(x+1)^3(x+2)^5 + 5(x+1)^4(x+2)^4$ |
| l) $14x^4(x+1)^3(x+2)^5 + 15x^2(x+1)$ |
| m) $24x^3(x+2)^5 + 45x^4(x+1)^4(x+2)^4 + 6x^4$ |
| n) $24x^3(x+2)^5 + 45x^4(x+1)^4(x+2)^4 + 6(x+1)(x+2)$ |
| o) $10x^3(x+2)^5 - 45x^4(x+1)(x+2)^4 + 15(x+1)(x+2)^4$ |

Factoring using FOIL is an important technique that is discussed in Chapter 9. The identities in the following exercise are important examples of the FOIL method.

Exercise 3.4.2: Rewrite each expression as a product of sums by using one of the following identities

$$a^2 + 2ab + b^2 = (a + b)^2;$$

$$a^2 - 2ab + b^2 = (a - b)^2;$$

$$a^2 - b^2 = (a + b)(a - b).$$

| |
|----------------------------------|
| a) $x^2 - 16 =$ |
| b) $16 - x^2 =$ |
| c) $9x^2 - 16 =$ |
| d) $25 - 16x^2 =$ |
| e) $x^2 - y^2 =$ |
| f) $x^2 - y^4 =$ |
| g) $x^4 - y^4 =$ |
| h) $x^4 - 9y^4 =$ |
| i) $x^2 + 4x + 4 =$ |
| j) $x^2 + 10x + 25 =$ |
| k) $x^2 + 2x + 1 =$ |
| l) $x^2 + 2xy + y^2 =$ |
| m) $x^2 + 12xy + 36y^2 =$ |
| n) $x^2 - 12xy + 36y^2 =$ |
| o) $x^2 - 6x + 9 =$ |
| p) $x^2 - 18x + 81 =$ |

3.5 Rewriting expressions with radicals

A square root can be written in two ways, \sqrt{a} and $a^{\frac{1}{2}}$. The defining property of these expressions is that $a^{\frac{1}{2}} a^{\frac{1}{2}} = a$, which could also be written in the form $\sqrt{a} \sqrt{a} = a$. Be aware that there is a subtlety when we deal with numbers rather than expressions: the usual convention is that $\sqrt{4} = 2$ but not -2 , even though $(-2)(-2) = 4$.

In the following identities involving notation for exponents, the letters m and n can represent any expression. In almost all the examples we'll work with, m and n are integers or rational numbers.

$$\text{Laws of exponents} \left\{ \begin{array}{l} a^m a^n = a^{m+n} \\ a^{-m} = \frac{1}{a^m} \\ (a^m)^n = a^{mn} \end{array} \right. \quad \text{Radical notation:} \left\{ \begin{array}{l} \sqrt[m]{a} = a^{\frac{1}{m}} \\ \sqrt{a} = a^{\frac{1}{2}} \end{array} \right.$$

$$\text{Examples:} \left\{ \begin{array}{l} a^3 b^2 a^4 b^5 = a^3 a^4 b^2 b^5 = a^7 b^7 \\ (3a^4 b^6)(-4a^2 b^3) = -12a^{4+2} b^{6+3} = -12a^6 b^9 \\ a^4 a^{-6} = a^{4+(-6)} = a^{-2} = \frac{1}{a^2} \\ a\sqrt{a} = \sqrt{a}\sqrt{a}\sqrt{a} = (\sqrt{a})^3 = (a^{1/2})^3 = a^{3/2} \\ \sqrt{a}(\sqrt[3]{a}) = a^{1/2} a^{1/3} = a^{\frac{1}{2}+\frac{1}{3}} = a^{5/6} = a^{5\left(\frac{1}{6}\right)} = (a^5)^{\frac{1}{6}} = \sqrt[6]{a^5} \end{array} \right.$$

For the time being, use the following basic facts :

$$\sqrt{x}\sqrt{x} = x \quad x\sqrt{x} = x^1 x^{\frac{1}{2}} = x^{\frac{3}{2}} \quad x^2 \sqrt{x} = x^2 x^{\frac{1}{2}} = x^{2+\frac{1}{2}} = x^{\frac{5}{2}}$$

Exercise 3.5.1: rewrite the following products as simplified sums in which powers of x decrease as you read from left to right. The first one is done for you

| |
|--|
| a) $(\sqrt{x} + x)^2 = (\sqrt{x})^2 + 2x\sqrt{x} + x^2 = x + 2x\sqrt{x} + x^2 = x^2 + 2x\sqrt{x} + x$ |
| b) $(\sqrt{x} + 2)^2 =$ |
| c) $x - (\sqrt{x} + 1)^2 =$ |
| d) $(\sqrt{x} + 1)^2 - 2(\sqrt{x} + 2)^2$ |
| e) $(\sqrt{x} + 2)(\sqrt{x} + 1) - 2\sqrt{x}(\sqrt{x} + 3)$ |
| f) $(x + \sqrt{x})(x + 1)^2$ |
| g) $(2x\sqrt{x} - \sqrt{x})^2$ |
| h) $(x + \sqrt{x})(x - \sqrt{x})$ |

The following examples show how a sum of terms involving radicals can be rewritten as a product by factoring out the lowest power of x . Do not assume that one form is “simpler” than the other. Please focus on the instructions in the left column: they specify how the answer should be written. Be aware that the examples

| | |
|---|--|
| <p>Pull out lowest x-power in $\sqrt{x} + x^2 + x^3$</p> <p>Your answer should contain radical signs, not fractional powers</p> | $\begin{aligned}\sqrt{x} + x^2 + x^3 \\ &= x^{1/2} + x^2 + x^3 \\ &= x^{1/2}(1 + x^{2-1/2} + x^{3-1/2}) \\ &= x^{1/2}(1 + x^{3/2} + x^{5/2}) \\ &= \sqrt{x}(1 + x\sqrt{x} + x^2\sqrt{x})\end{aligned}$ |
|---|--|

| | |
|---|--|
| <p>Pull out lowest x-power in $x^{1/2} + x^2 + x^3$ Write the answer with fractional powers but no radical signs.</p> | $\begin{aligned}&= x^{1/2} + x^2 + x^3 \\ &= x^{1/2}(1 + x^{2-1/2} + x^{3-1/2}) \\ &= x^{1/2}(1 + x^{3/2} + x^{5/2})\end{aligned}$ |
|---|--|

| | |
|--|--|
| <p>Pull out lowest x-power and rewrite $14x^{-2} + 7x^2 + 21x^3$ using only positive powers.</p> | $\begin{aligned}14x^{-2} + 7x^2 + 21x^3 \\ &= 7x^{-2}(2 + x^{2-(-2)}3x^{3-(-2)}) \\ &= 7x^{-2}(2 + x^4 + 3x^5) \\ &= 7\left(\frac{1}{x^2}\right)(2 + x^4 + 3x^5) \\ &= \frac{7(2 + x^4 + 3x^5)}{x^2}\end{aligned}$ |
|--|--|

| | |
|---|---|
| <p>Rewrite $\frac{1}{\sqrt{x}} + 3x^2 + x\sqrt{x}$ as a single fraction without using fractional powers.</p> | <p>Easy way: to rewrite any non-fraction as a fraction with denominator \sqrt{x}, use the identity $e = \frac{e\sqrt{x}}{\sqrt{x}}$. Then</p> $\begin{aligned}\frac{1}{\sqrt{x}} + 3x^2 + x\sqrt{x} \\ &= \frac{1}{\sqrt{x}} + \frac{3x^2\sqrt{x}}{\sqrt{x}} + \frac{x\sqrt{x}\sqrt{x}}{\sqrt{x}} = \frac{1 + 3x^2\sqrt{x} + x^2}{\sqrt{x}}\end{aligned}$ |
|---|---|

| | |
|--|--|
| <p>Rewrite $\frac{1}{\sqrt{x}} + 3x^2 + x\sqrt{x}$ as a single fraction without using fractional powers..</p> | $\begin{aligned} & \frac{1}{\sqrt{x}} + 3x^2 + x\sqrt{x} \\ &= x^{-\frac{1}{2}} + 3x^2 + x^{\frac{3}{2}} \\ \text{Hard way: } &= x^{-\frac{1}{2}} \left(1 + 3x^{2 - (-\frac{1}{2})} + x^{\frac{3}{2} - (-\frac{1}{2})} \right) \\ &= \frac{\left(1 + 3x^{\frac{5}{2}} + x^2 \right)}{x^{\frac{1}{2}}} = \frac{1 + 3x^2\sqrt{x} + x^2}{\sqrt{x}} \end{aligned}$ |
|--|--|

Exercise 3.5.2: rewrite each expression as indicated.

| | | |
|---|---|--|
| a) $\frac{1}{\sqrt{x}} + 1 + \sqrt{x}$ | As one fraction, without fractional powers of x | |
| b) $\frac{1}{\sqrt{x}} + 1 + \sqrt{x}$ | Without using fractions (other than 1/2) | |
| c) $x + \sqrt{x} + x^3$ | By factoring out \sqrt{x} , with no fractional powers in the answer | |
| d) $3x^{-2} + 6x + 7x^3$ | By factor out the lowest power of x | |
| e) $3x^{-2} + 6x + 7x^3$ | Without negative powers | |
| f) $3x^{-2} + 6\sqrt{x} + 12x^3$ | Without negative powers | |

CHAPTER 4: FUNCTIONS

4.1 Real life functions

Very often, in real-life situations, one measurement depends on another. Suppose we want to use 24 feet of fence to surround a rectangular field. Thus the perimeter of the rectangle will be 24 feet. What are possible dimensions (length and width) for the field?

Since the perimeter is twice the width plus twice the length, it follows that $2L + 2W = 24$, where L and W are the length and width of the field. It follows that $L + W = 12$ and therefore $L = 12 - W$. Of course, it's important to note that both the length and the width are between 0 and 12. In particular, we require $0 < W < 12$, and the following table shows a few choices of such W .

Possible length and width of a rectangle with perimeter 24

| Width W | Length $L = 12 - W$ | Check: $2L + 2W = 24$ |
|-----------|---------------------|--------------------------|
| 0.1 | 11.9 | $2(0.1) + 2(11.9) = 24$ |
| 2 | 10 | $2(2) + 2(10) = 24$ |
| 3 | 9 | $2(3) + 2(9) = 24$ |
| 4 | 8 | $2(4) + 2(8) = 24$ |
| 5 | 7 | $2(5) + 2(7) = 24$ |
| 6 | 6 | $2(6) + 2(6) = 24$ |
| 7.32 | 4.68 | $2(7.32) + 2(4.68) = 24$ |
| 11.9 | 0.1 | $2(11.9) + 2(0.1) = 24$ |

The table shows that if we know W , then we know L . We say that the rectangle's length depends upon its width, or that its *length is a function of its width*. Here's a summary of what we did.

Example 4.1.1: A rectangle has width W feet and perimeter 24 feet. What is the length of the rectangle?

Solution: Let L be the length. Solve the equation $2L + 2W = 24$ to obtain $L = 12 - W$.

Answer: The length of the rectangle is $12 - W$.

Example 4.1.2: I have a board 12 feet long. I cut off a damaged piece that is D feet long. How long is the remaining good piece?

Here we will use a very basic principle: The whole is the sum of its parts. In this case: When you cut a board into pieces, the length of the board is obtained by adding the lengths of the pieces

Solution: Call the length of the good piece G . Then $G + D = 12$. Solve this equation to get $G = 12 - D$.

Answer: The length of the left over piece is $12 - D$ feet.

4.2 Function definitions: substituting for the argument variable

Many textbooks show a picture of a “function machine,” a gadget that swallows a number called the “input” and spits out a number called the “output.” That’s a reasonable description of a real-life function. In this section, we generalize that description by allowing not only numbers, but also general expressions with variables, as inputs and outputs of functions.

The equations $G = 12 - D$ and $L = 12 - W$ from the previous section express the same idea. A new measurement of interest to us (G or L) depends on a given original measurement (D or W). In both cases the relationship can be expressed by calling the original measurement x .

| | |
|-------------------------------------|--|
| Then we can write | New measurement = $12 - \text{original measurement}$. |
| Call the original measurement x : | New measurement = $12 - x$ |
| Abbreviate this as | New (x) = $12 - x$ |

On the last line, the right side is an ordinary expression involving variable x . However, the left side is not an ordinary expression. It looks odd because it has an English word “New” in it. Write “ N ” instead of “New.” Now we have:

$$N(x) = 12 - x$$

The left side looks like N times x . But N is not a variable, and there is no multiplication involved. The entire statement $N(x) = 12 - x$ is a special kind of identity called a *function definition*. It says that a new measurement, abbreviated by $N(x)$, depends on an original measurement, abbreviated by x . Here’s what you need to know:

- The letter N is the *name of a function*.
- $N(x)$ **has nothing whatsoever to do with N times x** .
- x is the *argument variable*.
- Both sides of the function definition are referred to as *the value of N at x* .
- The right side of the function definition can be any expression whatsoever: it may or may not include an instance of the variable x .
- The *function definition* $N(x) = 12 - x$ is a special kind of identity
- The Fundamental Substitution Principle remains true: If E is any expression, then substituting E for x in the identity $N(x) = 12 - x$ produces a new identity $N((E)) = 12 - (E)$, which can be rewritten as $N(E) = 12 - (E)$

We have been working with many different kinds of expressions. Each and every expression turns into a single number when numbers are substituted for variables. However:

There's no such thing as an expression that produces more than one output for a given input.

In particular, things such as $\pm x$ and $3 \pm \sqrt{x}$ are not expressions and may not appear on the right side of a function definition.

Example 4.2.1: Substitute for x in the function definition $f(x) = 12 - x - x^2$ to find

a) $f(x + h)$ **b)** $f(gx)$ **c)** $f(g(x))$

Solutions:

| | |
|--|---|
| <p>a) $f(x) = 12 - x - x^2$ $f(x + h) = 12 - (x + h) - (x + h)^2$</p> | <p>b) $f(x) = 12 - x - x^2$ $f(gx) = 12 - (gx) - (gx)^2$</p> |
| <p>c) $f(x) = 12 - x - x^2$ $f(g(x)) = 12 - (g(x)) - (g(x))^2$</p> | |

The last example is interesting, because the substituted expression $g(x)$ could be either a function value or simply the product g times x . It doesn't matter: in either case, the substitution principle guarantees that $f(g(x)) = 12 - (g(x)) - (g(x))^2$ is an identity.

A function is a kind of algebra law that's valid only for the duration of a particular mathematical discussion, in contrast to algebra laws such as the commutative law $a + b = b + a$, which is true always.

The beauty of a function definition is that substituting any expression whatsoever for the argument variable x yields a new algebra law. A function definition is a template for an infinite number of algebra laws describing the values of the function at various input expressions.

For example, substituting $a + b$ for x in the function definition $f(x) = 7 - x - x^2$ yields $f((a + b)) = 7 - (a + b) - (a + b)^2$.

The double parentheses to the left of the equals sign are unnecessary: we write $f(a + b) = 7 - (a + b) - (a + b)^2$.

However, **the parentheses on the right of the equals sign are absolutely essential**, since $7 - (a + b) - (a + b)^2$ and $7 - a + b - a + b^2$ are very different expressions!

Any other expression, such as $\frac{a - 2^{3-b}}{x + 16}$ can be substituted for x . Then

$$f(x) = 7 - x - x^2 \text{ becomes } f\left(\frac{a - 2^{3-b}}{x + 16}\right) = 7 - \left(\frac{a - 2^{3-b}}{x + 16}\right) - \left(\frac{a - 2^{3-b}}{x + 16}\right)^2.$$

$$g(x) = (3 - x)(4 - x) \text{ becomes } g\left(\frac{a - 2^{3-b}}{x + 16}\right) = \left(3 - \left(\frac{a - 2^{3-b}}{x + 16}\right)\right)\left(4 - \left(\frac{a - 2^{3-b}}{x + 16}\right)\right)$$

The process just demonstrated is called argument substitution. It's easy (and fun) to do it with a word processor: just paste the substituted expression into the function definition, replace each occurrence of the argument variable, and insert parentheses!

Study carefully the table below. It contains some surprises. In particular, notice the last column, in which every function value that appears as part of a larger expression is enclosed in square brackets. Reasons will be given in the next section.

| Given f defined below: | Then $f(-2) =$ | and $f(a) =$ | and $f(a + h) =$ | and finally $\frac{f(a + h) - f(a)}{h} =$ |
|--------------------------|----------------|--------------|------------------|---|
| $f(x) = 1 - x^2$ | $1 - (-2)^2$ | $1 - (a)^2$ | $1 - (a + h)^2$ | $\frac{[1 - (a + h)^2] - [1 - a^2]}{h}$ |
| $f(q) = 1 - q^2$ | $1 - (-2)^2$ | $1 - (a)^2$ | $1 - (a + h)^2$ | $\frac{[1 - (a + h)^2] - [1 - a^2]}{h}$ |
| $f(z) = \sin z$ | $\sin(-2)$ | $\sin(a)$ | $\sin(a + h)$ | $\frac{[\sin(a + h)] - [\sin(a)]}{h}$ |
| $f(r) = \sin z$ | $\sin z$ | $\sin z$ | $\sin z$ | $\frac{[\sin z] - [\sin z]}{h}$ |
| $f(x) = 11 + a$ | $11 + a$ | $11 + a$ | $11 + a$ | $\frac{[11 + a] - [11 + a]}{h}$ |

4.3 Substituting expressions for function values

The last section emphasized that when you substitute an expression for the argument variable in a function definition, the substituted expression must be enclosed in parentheses. For example: given the function definition $f(x) = 1 - x^2$, then

$$f(x + h) = 1 - (x + h)^2$$

When a function value is used in an expression, you also need to use parentheses whenever you substitute for the function value. That's because the symbol $f(x)$ behaves like a variable, as shown in the following example. Again define $f(x) = 1 - x^2$.

Example 4.3.1: Substitute $1 - x^2$ for u in the expression $u^2 - u$.

Answer: $u^2 - u = (1 - x^2)^2 - (1 - x^2)$

Example 4.3.2: Define $f(x) = 1 - x^2$ and substitute for $f(x)$
in the expression $f(x)^2 - f(x)$.

Solution: Treat $f(x)$ exactly like the variable u in the previous example.

Answer: $f(x)^2 - f(x) = (1 - x^2)^2 - (1 - x^2)$

Example 4.3.3: Define $f(x) = 1 - x^2$ and substitute in the expression $\frac{f(x+h) - f(x)}{h}$

Answer: $f(x+h) = 1 - (x+h)^2$ and so $\frac{f(x+h) - f(x)}{h} = \frac{[1 - (x+h)^2] - [1 - x^2]}{h}$

In this example, the first set of inserted parentheses (here written for clarity as brackets) is not needed, but omitting the second set is a serious error, as seen from the following

Exercise 4.3.1 Simplify both the given answer to 4.3.3 and the incorrect answer obtained by omitting brackets.

4.4 Function evaluation examples

The following examples show how to an expression for the argument of a function. They also show how to simplify the result. The simplification process depends on following the order of operations carefully.

Given $f(x) = 1 - x^2$, find each function value and rewrite it as a simplified sum.

$$\begin{array}{llll}
 \mathbf{1)} \left\{ \begin{array}{l} f(-3) \\ = 1 - (-3)^2 \\ = 1 - 9 \\ = -8 \end{array} \right. &
 \mathbf{2)} \left\{ \begin{array}{l} f(z) \\ = 1 - (z)^2 \\ = 1 - z^2 \end{array} \right. &
 \mathbf{3)} \left\{ \begin{array}{l} f(x+h) \\ = 1 - (x+h)^2 \\ = 1 - [(x+h)^2] \\ = 1 - [x^2 + 2hx + h^2] \\ = 1 - x^2 - 2hx - h^2 \end{array} \right. &
 \mathbf{4)} \left\{ \begin{array}{l} f(3-x) \\ = 1 - (3-x)^2 \\ = 1 - [(3-x)^2] \\ = 1 - [9 - 6x + x^2] \\ = 1 - 9 + 6x - x^2 \\ = -x^2 + 6x - 9 + 1 \\ = -x^2 + 6x - 8 \end{array} \right.
 \end{array}$$

In the following examples, inserted parentheses will be written as braces { }.

5) Given $f(x) = 1 - x^2$ **simplify** $f(x+h) - f(x)$

$$\left\{ \begin{aligned} & f(x+h) - f(x) \\ &= \{f(x+h)\} - \{f(x)\} \\ &= \{1 - (x+h)^2\} - \{1 - x^2\} \\ &= \{1 - [(x+h)^2]\} - \{1 - x^2\} \\ &= \{1 - [x^2 + 2hx + h^2]\} - \{1 - x^2\} \\ &= 1 - x^2 - 2hx - h^2 - 1 + x^2 \\ &= -x^2 + x^2 - 2hx - h^2 + 1 - 1 \end{aligned} \right.$$

$= -2hx - h^2$ This answer is a simplified sum.

$= -h(2x + h)$ This answer is a simplified product.

6) Given $f(x) = 1 - x^2$ **simplify** $\frac{1}{h}(f(x+h) - f(x-h))$

Step 1: Simplify $f(x+h) - f(x-h)$:

Step 2: Multiply by $\frac{1}{h}$ **and simplify:**

$$\left\{ \begin{aligned} & f(x+h) - f(x-h) \\ &= \{f(x+h)\} - \{f(x-h)\} \\ &= \{1 - (x+h)^2\} - \{1 - (x-h)^2\} \\ &= \{1 - [(x+h)^2]\} - \{1 - [(x-h)^2]\} \\ &= \{1 - [x^2 + 2hx + h^2]\} - \{1 - [x^2 - 2hx + h^2]\} \\ &= \{1 - x^2 - 2hx - h^2\} - \{1 - x^2 + 2hx - h^2\} \\ &= 1 - x^2 - 2hx - h^2 - 1 + x^2 - 2hx + h^2 \\ &= -x^2 + x^2 - 2hx - 2hx - h^2 + h^2 + 1 - 1 \\ &= -4hx \end{aligned} \right. \quad \left\{ \begin{aligned} & \frac{1}{h}(f(x+h) - f(x-h)) \\ &= \frac{1}{h}(-4hx) \\ &= \frac{-4hx}{h} \\ &= -4x \end{aligned} \right.$$

7) Given $f(x) = 7 - x - x^2$, **simplify** $\frac{f(x+h) - f(x)}{h}$

$$\begin{aligned} & \frac{[f(x+h)] - [f(x)]}{h} \\ &= \frac{[7 - (x+h) - (x+h)^2] - [7 - x - x^2]}{h} \\ &= \frac{[7 - x - h - (x^2 + 2hx + h^2)] - 7 + x + x^2}{h} \\ &= \frac{7 - x - h - x^2 - 2hx - h^2 - 7 + x + x^2}{h} \end{aligned}$$

$$\begin{aligned}
 &= \frac{-h - 2hx - h^2}{h} \\
 &= \frac{h(-1 - 2x - h)}{h} \\
 &= -1 - 2x - h
 \end{aligned}$$

In each exercise below, find and try to simplify each of the specified function values.

Exercise 4.4.1 $f(x) = x - x^2$

a) $f(-3)$ (b) $f(3/2)$; (c) $f(h)$; (d) $f(x+h)$; (e) $f(x+h) - f(x-h)$

Exercise 4.4.2 $f(x) = \frac{1}{x+a}$

a) $f(-3)$ (b) $f(3/2)$; (c) $f(h)$; (d) $f(x+h)$; (e) $f(x+h) - f(x-h)$

Exercise 4.4.3 $f(x) = -1 - 3x$

a) $f(-3)$ (b) $f(3/2)$; (c) $f(h)$; (d) $f(x+h)$; (e) $f(x+h) - f(x-h)$

Exercise 4.4.4 $f(x) = \frac{1}{b+a}$

a) $f(-3)$ (b) $f(3/2)$; (c) $f(h)$; (d) $f(x+h)$; (e) $f(x+h) - f(x-h)$

4.5 Composition of functions

Example 4.5.1: A real life example with a number game.

I give you a number. You have one chance to double the number and one chance to add ten to the number. Which procedure gives the bigger answer: adding 10, then doubling, or doubling, then adding 10?

Solution:

Call the starting number x . In this problem, there are two functions.

The doubling function is $d(x) = 2x$. The adding 10 function is $a(x) = x + 10$

Start with x . Doubling first gives $d(x) = 2x$.

Adding 10 to the result gives $a(d(x)) = d(x) + 10 = 2x + 10$

This strategy starts with x , yields $2x + 10$, and can be thought of as a function:

$$f(x) = a(d(x)) = 2x + 10$$

Start with x . Adding 10 first gives $a(x) = x + 10$

Doubling the result gives $d(a(x)) = d(x + 10) = 2(x + 10) = 2x + 20$

This strategy starts with x , yields $2x + 20$, and can be thought of as a function:

$$g(x) = d(a(x)) = 2x + 20$$

The result $g(x)$ of adding first, then doubling, is always 10 more than the result $f(x)$ of doubling first, then adding.

The functions $g(x) = d(a(x))$ and $f(x) = a(d(x))$ are calculated by putting together, or composing, the functions a and d . We've just seen that the order of composition matters. In other words, composition of functions is not commutative.

We just assigned a somewhat random name g to the composite function formed by first applying a , then applying d . The official name of this function is $d \circ a$. It is usually enclosed in parentheses when we write the function definition: $(d \circ a)(x) = d(a(x))$.

Similarly, composing in the reverse order is defined by $(a \circ d)(x) = a(d(x))$

Take note: When a composite function is named using the composition operator \circ , the rightmost function is evaluated first. For example: to find $(f \circ g \circ h)(x)$, start with $h(x)$, next find $g(h(x))$, and lastly find $f(g(h(x)))$

Example 4.5.2: Buying lumber. I am buying wood that costs 3 dollars per foot. A 12 foot long board has a damaged part whose length is x feet. I am allowed to cut off the damaged part and pay only for the good part that I take. The charge for cutting the wood is 10 dollars. How much do I pay?

In this problem there are three important measurements and two functional relationships.

- The length of the good part equals 12 minus the length of the damaged part.
- The cost of the good part is 3 times the length of the good part.

For example, if the length of the good part is 8 feet, it costs $3 \times 8 = 24$ dollars. The charge for cutting the wood is an additional 10 dollars and therefore I pay $(3 \times 8) + 10 = 34$ dollars.

In general, if the length of the good part is G , then I pay $(3 \times G) + 10 = 3G + 10$ dollars.

There are two ways to proceed, each related to a way of phrasing the problem.

Method 1. Let x be the length of the damaged piece. Find a formula for C , the amount I must pay.

Solution 1: Name the quantities as follows. Length is measured in feet.

Let D be the length of the damaged part.

Let G be the length of the good part. Then $G = 12 - D$ as before.

Let C be the cost of the good part. Then $C = 3G + 10$

Substitute $G = 12 - x$ in the identity $C = 3G + 10$ to obtain $C = 3(12 - x) + 10$.

Answer 1: If the length of the damaged piece is x , then I pay $C = 3(12 - x) + 10$ dollars.

$$\text{Summary 1: } \begin{cases} G = 12 - x \\ C = 3G + 10 \\ C = 3(12 - x) + 10 \end{cases}$$

This method was straightforward. The following one seems a bit more complicated. We use it for reasons that will become clear in Calculus I.

Method 2: Let x be the length of the damaged piece. Find the definition of a function of x that gives the cost of the good piece.

Solution 2: The length of the good part is given by the function definition $g(x) = 12 - x$. Call the length of the good part u : then the cost of the good part is given by the function definition $c(u) = 3u + 10$.

Combine these function definitions: the cost of the good part is $c(g(x)) = c(12 - x) = 3(12 - x) + 10$

Answer 2: The cost of the good piece is $(c \circ g)(x) = 3(12 - x) + 10$ dollars.

$$\text{Summary 2: } \begin{cases} g(x) = 12 - x \\ c(u) = 3u + 10 \\ (c \circ g)(x) = c(g(x)) = c(12 - x) = 3(12 - x) + 10 \end{cases}$$

The method of combining functions shown above is called composition: the word “compose” means “put together.” The idea is simple: you have definitions for functions f

and g , for example $\begin{cases} f(x) = 2 - x - x^2 \\ g(x) = 5 - 3x \end{cases}$

It might have been nicer to write the definition of g using a letter other than x , but the letter used in the function definition is arbitrary and there’s nothing wrong with using the same letter in two different function definitions.

To define composite functions, we substitute as in the following example.

Example 4.5.3: Given $\begin{cases} f(x) = 2 - x - x^2 \\ g(x) = 5 - 3x \end{cases}$, find $f(g(x))$ and $g(f(x))$.

Solution:

$$\begin{array}{ll}
 (g \circ f)(x) = g(f(x)) & (f \circ g)(x) = f(g(x)) \\
 = g(2 - x - x^2) & = f(5 - 3x) \\
 = 5 - 3(2 - x - x^2) & = 2 - (5 - 3x) - (5 - 3x)^2 \\
 \text{This completes the substitution. In each case continue by simplifying.} & \\
 5 - 3(2 - x - x^2) & = 2 - (5 - 3x) - [25 - 30x + 9x^2] \\
 = 5 - [6 - 3x - 3x^2] & = 2 - 5 + 3x - 25 + 30x - 9x^2 \\
 = 5 - 6 + 3x + 3x^2 & = 2 - 5 - 25 + 3x + 30x - 9x^2 \\
 = -1 + 3x + 3x^2 & = -28 + 33x - 9x^2 \\
 = (g \circ f)(x) & = (f \circ g)(x)
 \end{array}$$

As before, we see that the order of composition matters.

A function may be composed with itself. For the function $g(x) = 5 - 3x$ we have
 $(g \circ g)(x) = g(g(x)) = g(5 - 3x) = 5 - (5 - 3x) = 5 - 5 + 3x = 3x$

Exercise 4.5.1 Given $f(x) = 2x - 1$ and $g(x) = 1 - 2x$ find and simplify
 a) $g(f(-1))$ (b) $f(f(-1))$; (c) $f(f(a))$; (d) $f(g(x+h))$; (e) $g(g(2x))$

Exercise 4.5.2 Given $f(x) = x^2$ and $g(x) = \frac{1}{x}$ find and simplify
 a) $g(f(-1))$ (b) $f(f(-1))$; (c) $f(f(a))$; (d) $f(g(x+h))$; (e) $g(g(2x))$

4.6 Piecewise defined functions

A very generous bank offers the following deal:

Deposit your money with us at the beginning of any year. At the end of the year we will deposit interest in your account as follows:

- if you have less than 80 dollars in your account, we'll add 80 dollars to your account..
- if you have 80 dollars or more, we'll double your account.

Example 4.6.1:

a) Assume your account starts out with x dollars. Give a function definition for the amount in your account after one year.

b) If your account starts out with 20 dollars, how much is your account after one year? after two years? after three years?

Part a) asks how to write the symbolic definition of a piecewise defined function.

Part b) involves finding the value of the function f and its composites $f \circ f$ and $f \circ f \circ f$ for the specific input $x = 20$.

Solution a) If you start the year with x dollars, let $f(x)$ be the amount after one year.

According to the bank's rules:

If $x < 80$, then $f(x) = x + 80$. If $x \geq 80$, then $f(x) = 2x$

These two sentences answer part a). They are usually combined and written in math books as

Answer a) If your account at the beginning of a year has x dollars, then at the end of that year, your account is given by the function $f(x) = \begin{cases} x + 80 & \text{if } x < 80 \\ 2x & \text{if } x \geq 80 \end{cases}$

We say that the function just defined is a *piecewise defined* function because the expression that will be substituted for the input variable x depends on which piece of the number line the number x lies on.

Solution b) If you start with x dollars then the amount in your account (in dollars) is $f(x)$ after one year ; $f(f(x))$ after two years; $f(f(f(x)))$ after three years, and so forth.

Start with $x = 20$. Note that $20 < 80$.

When you substitute 20 for x in the function definition $f(x) = \begin{cases} x + 80 & \text{if } x < 80 \\ 2x & \text{if } x \geq 80 \end{cases}$

and omit unnecessary parentheses, the result is $f(20) = \begin{cases} (20 + 80) & \text{if } 20 < 80 \\ 2(20) & \text{if } 20 \geq 80 \end{cases}$

Since $20 < 80$, it follows that the top line of the definition should be used. Therefore $f(20) = 20 + 80 = 100$ is the amount after one year.

The amount after two years is $f(f(20)) = f(100) = \begin{cases} 100 + 80 & \text{if } 100 < 80 \\ 2(100) & \text{if } 100 \geq 80 \end{cases}$

In this case the second line applies and so the amount after two years is $f(f(20)) = 2(100) = 200$. From this point on, the account doubles each year.

Answer b) The account has $f(20) = 100$ dollars after one year;

$f(f(20)) = f(100) = 200$ dollars after two years; and

$f(f(f(20))) = f(f(100)) = f(200) = 400$ dollars after three years.

A piecewise defined function is always a real-life function in the sense that the input variable is assumed to be a real number. Certainly the right side of the function definition above is not an expression of the sort we have been discussing. However, it satisfies the crucial property of a function: there's exactly one output for a given input number x .

You should watch out for possible mistakes in piecewise function definitions.

If you change the inequality sign in the top line of the function from $<$ to \leq , then you would write $f(x) = \begin{cases} x + 80 & \text{if } x \leq 80 \\ 2x & \text{if } x \geq 80 \end{cases}$ This is still a legal function definition, since the top line and the bottom line both give 160 when you substitute 80 for x . The function so defined is exactly the same as the one above.

However, $g(x) = \begin{cases} x + 80 & \text{if } x \leq 80 \\ 3x & \text{if } x \geq 80 \end{cases}$ is not a legal function definition, since when 80 is substituted for x , the top line becomes 160 but the bottom line gives 240. Whatever this thing is, it's not a function definition, since the single input 80 produces two different outputs.

Exercise 4.6.1 Given $\begin{cases} f(x) = 1 - x^2 \\ g(x) = 2 - x \end{cases}$ find each of the following:

- | | | | |
|--------------|---------------|----------------|--|
| a) $f(g(x))$ | e) $f(g(-3))$ | h) $f(g(1-u))$ | m) $(f \circ g)(w+1)$ |
| b) $g(f(x))$ | f) $g(f(-3))$ | i) $g(f(1-u))$ | n) $(f \circ g \circ g)(w+1)$ |
| c) $g(g(x))$ | g) $g(g(-3))$ | j) $g(g(1-u))$ | o) $(g \circ g \circ g)(x)$ |
| d) $f(f(x))$ | h) $f(f(-3))$ | k) $f(f(1-u))$ | p) $(g \circ g \circ g \circ g)(x^2 + 17)$ |

Exercise 4.6.2 Given $f(x) = \begin{cases} -x & \text{if } x < 0 \\ 12 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$, find each of the following:

- | | |
|-----------------|-----------------------------|
| a) $f(f(0))$ | e) $f(f(-3))f(3)$ |
| b) $f(f(-2))$ | f) $f(f(-3))f(f(3))$ |
| c) $f(f(2))$ | g) $(f \circ f)(0)$ |
| d) $f(f(f(0)))$ | h) $(f \circ f \circ f)(0)$ |

4.7 Warnings about function notation

- If f is a function, the symbol f is neither a number nor a variable. The **symbol** $f(x)$ **has nothing to do with f times x** . Computer science and logic books avoid this problem by using square brackets exclusively to enclose function arguments, but we will follow the custom of math books and use round parentheses.
- If f were a variable, we could use the distributive law to write $f(x + y) = f(x) + f(y)$. This tempting statement is responsible for many errors in students' algebra work in pre-calculus and calculus. The statement is usually but not always false: the

exceptions are functions defined by laws of the form $f(x) = kx$. Here are some examples where it's true:

- Let $f(x) = 5x$. Then $5(x + y) = 5x + 5y$.
- Let $f(x) = abcx$. Then $abc(x + y) = abc(x) + abc(y)$
- Let $f(x) = \frac{1}{5}x = \frac{x}{5}$. Then $\frac{1}{5}(x + y) = \frac{1}{5}x + \frac{1}{5}y$ This can be rewritten

$$\frac{x + y}{5} = \frac{x}{5} + \frac{y}{5}.$$
- Let $f(x) = \frac{1}{z}x = \frac{x}{z}$. Then $\frac{1}{z}(x + y) = \frac{1}{z}x + \frac{1}{z}y$. This can be rewritten

$$\frac{x + y}{z} = \frac{x}{z} + \frac{y}{z}.$$

The last two lines are examples of a principle you already know:

A fraction can be rewritten as a sum of fractions provided its numerator is a sum.

The corresponding statement for denominators is false: see the third line of the following table, which contains mistakes frequently found on students' exams.

Every error in the table is based on the false belief that a function name can be treated as a variable. That in turn leads to the misconception that $f(x + y) = f(x) + f(y)$, which in turn is responsible for most of the errors shown in the table below.

You will make your math teachers very happy if you avoid the errors in the table below.

Example 4.7.1: Some common function errors to avoid

| Function | Error! Error! NO! NO! | Correct statement |
|--|---|---|
| $f(x) = -x$ | $-(x + y) = -x + y$ | $-(x + y) = -x - y$ |
| $f(x) = x $ | $ x + y = x + y $ | $ x + y \leq x + y $ |
| $f(x) = \sqrt{x}$ | $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$ | $\sqrt{x + y}$ does not simplify. |
| $f(x) = x^2$ | $(x + y)^2 = x^2 + y^2$ | $(x + y)^2 = x^2 + 2xy + y^2$ |
| $f(x) = \frac{k}{x}$ | $\frac{k}{x + y} = \frac{k}{x} + \frac{k}{y}$ | $\frac{k}{x + y}$ does not simplify. |
| $f(x) = \frac{k}{x^2}$ | $\frac{k}{x^2 + y^2} = \frac{k}{x^2} + \frac{k}{y^2}$ | $\frac{k}{x^2 + y^2}$ does not simplify. |
| $f(x) = \cos(x)$ | $\cos(x + y) = \cos(x) + \cos(y)$ | $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ |
| $f(x) = \sin(x)$ | $\sin(x + y) = \sin(x) + \sin(y)$ | $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ |
| $f(x) = \log(x)$ | $\log(x + y) = \log x + \log y$ | $\log(x + y)$ does not simplify. |
| $f(x) = e^x$ | $e^{x+y} = e^x + e^y$ | $e^{x+y} = e^x e^y$ |
| $f(x) = kx$ | $k(x + y) = kx + y$ | $k(x + y) = kx + ky$ |
| $f(x) = \frac{x}{k} = \left(\frac{1}{k}\right)x$ | (Hard to mess this one up) | $\frac{x + y}{k} = \frac{x}{k} + \frac{y}{k}$ |

Exercise 4.7.1 Fill in the following table. First substitute in the function definition, then simplify the result

| Function | Substitute to find the function value. Then simplify your answer or write DNS (does not simplify) |
|---------------------------------------|--|
| a) $f(x) = -x$ | $f(x - y) =$ |
| b) $f(x) = \sqrt{x}$ | $f(2^2 + 3^2) =$ |
| c) $f(x) = \sqrt{x}$ | $f(x^2 + y^2) =$ |
| d) $f(x) = x^2$ | $f(x - y) =$ |
| e) $f(x) = \frac{5}{x}$ | $f(2 + 3) =$ |
| f) $f(x) = \frac{5}{x}$ | $f(\frac{1}{2} + \frac{1}{3}) =$ |
| g) $f(x) = \frac{5}{x}$ | $f(u + v) =$ |
| h) $f(x) = \frac{25}{x^2}$ | $f(u + v) =$ |
| i) $f(x) = \frac{1}{x}$ | $f(x + h) - f(x - h) =$ |
| j) $f(x) = \frac{1}{x}$ | $\frac{f(x + h) - f(x)}{h} =$ |
| k) $f(x) = 1 - x - x^2$ | $\frac{f(x + h) - f(x)}{h} =$ |
| l) $f(x) = a^2 + 1$ | $f(x + h) - f(x) =$ |
| m) $f(x) = \frac{\sin(2x)}{4}$ | $f(2z) =$ |
| n) $f(x) = \frac{1}{x} - \frac{1}{u}$ | $f(x + h) - f(x) =$ |

CHAPTER 5: WHEN TO USE PARENTHESES

As you may suspect, every step in an algebra problem can be thought of as a substitution operation. Many students get into trouble because they don't use parentheses correctly when they substitute. The following summary of previous sections shows when parentheses are needed in substitutions that are done in precalculus.

When you substitute, put parentheses

- **Around the result of an algebra operation.**
- **Around an expression that is substituted for**
 - **a variable in a larger expression;**
 - **a variable in an algebra law;**
 - **a function argument;**
 - **a function value.**

Parentheses are sometimes optional, of course. However, it's very difficult to give complete laws for when parentheses are needed. When in doubt, use them!

This section presumes that the reader knows the basic FOIL identity, discussed in more detail in Section 6, for multiplying expressions: $(a + b)(c + d) = ac + ad + bc + bd$.

5.1 Put parentheses around the result of each algebra operation.

Example: Rewrite $x^2 - (x + 1)(x + 2)$ as a simplified sum.

Solution: Do the multiplication first. Put parentheses around the product. We'll write these inserted parentheses as brackets [...] for clarity.

$$\begin{array}{ll}
 x^2 - (x + 1)(x + 2) & \text{There are two operations, subtraction and multiplication.} \\
 = x^2 - [x^2 + 3x + 2] & \text{Multiply using FOIL and put the product inside parentheses.} \\
 = x^2 - x^2 + 3x - 2 & \text{Remove parentheses (see below)} \\
 = 3x - 2 & \text{Collect terms.}
 \end{array}$$

Inserting the parentheses was absolutely crucial. Otherwise, you would have written

$$\begin{array}{ll}
 x^2 - (x + 1)(x + 2) & \text{There are two operations, subtraction and multiplication.} \\
 = x^2 - x^2 - 3x + 2 & \text{FOIL is used correctly, but omitting the brackets is a **fatal error**.} \\
 = -3x + 2 & \text{Full credit off!}
 \end{array}$$

Even though the order of operations is correct the result is wrong!

In the correct version, we performed a very tricky operation

Removing parentheses following a minus sign

It's important to understand why we wrote

$x^2 - (x^2 - 3x + 2) = x - x^2 + 3x - 2$. This operation should be automatic: think of multiplying the minus sign by each term inside the parentheses. After a while, you should be able to remove the parentheses without showing intermediate details. For now, here are the details for this example: note that

$$x^2 - (x^2 - 3x + 2) = x - x^2 + 3x - 2 \text{ is an example of the identity}$$

$$a - (b - c + d) = a - b + c - d$$

| | |
|------------------------------|---|
| $x^2 - (x^2 - 3x + 2)$ | This is a subtraction problem. |
| $= x^2 + -(x^2 - 3x + 2)$ | Definition of subtraction: $a - b = a + (-b)$ |
| $= x^2 + (-1(x^2 - 3x + 2))$ | Negation: $-b = -1(b) = -1 \times b$ |
| $= x^2 + (-x^2 + 3x - 2)$ | Do the multiplication. |
| $= x^2 - x^2 + 3x - 2$ | Remove parentheses. |
| $= 3x - 2$ | Combine terms. |

Here is a related example of the form $a - 2(b - c + d) = a - 2b + 2c - 2d$

| | |
|--------------------------------------|---|
| $x^2 - 2(x^2 - 3x + 2)$ | This is a subtraction problem. |
| $= x^2 + (-2 \times (x^2 - 3x + 2))$ | Definition of subtraction: $a - b = a + (-b)$ |
| $= x^2 + (-2x^2 + 6x - 4)$ | Do the multiplication. |
| $= x^2 - 2x^2 + 6x - 4$ | Remove parentheses. |
| $= -x^2 + 6x - 4$ | Combine terms. |

Here's another example, in which you follow the laws for order of operations by evaluating a power before doing a subtraction

| | | | |
|-----------------|---|-----------------------------|--|
| Correct: | $\begin{cases} 1 - (x + 2)^2 \\ = 1 - [(x + 2)^2] \\ = 1 - [x^2 + 4x + 4] \\ = 1 - x^2 - 4x - 4 \\ = -x^2 - 4x - 3 \end{cases}$ | Correct and shorter: | $\begin{cases} 1 - (x + 2)^2 \\ = 1 - [x^2 + 4x + 4] \\ = 1 - x^2 - 4x - 4 \\ = -x^2 - 4x - 3 \end{cases}$ |
|-----------------|---|-----------------------------|--|

5.2 Put parentheses around any expression that you substitute for a variable in another expression.

When you do this, the Fundamental Substitution Principle guarantees that the new expression is equivalent (identical) to the original.

For example, if $x = a + b$ then $7 - x - x^2 = 7 - (a + b) - (a + b)^2$.

5.3 Put parentheses around any expression that you substitute for a variable in an identity or algebra law.

An identity is a statement that two expressions are equal. However, it says much more: substituting any expression for every instance of the variable produces another identity.

For example, the commutative law says that $AB = BA$. Substituting any expressions for A and B will produce a true statement, *provided you enclose those expression in parentheses*. Here are three examples.

If you omit parentheses in either of the first two examples below, you are making a serious error. In those examples, we use language such as “Let $B = 4$ ” as a shorter way of saying: “substitute 4 for B .”

- Let $A = -3$ and $B = 4$. Then $(-3)(4) = (4)(-3)$.
- Let $A = x + y$ and $B = u + v$. Then $(x + y)(u + v) = (u + v)(x + y)$
- Substitute A^2 for A and A^5 for B . Then $(A^2)(A^5) = (A^5)(A^2)$

In the last example we didn't say “Let $A^2 = A$.” Why?

Perhaps the most important algebra law is the **cancellation law**, which can be written

$$\frac{ax}{bx} = \frac{a}{b}$$

Warning #1: Many students misuse the cancellation law. **Don't cancel in a situation where the cancellation law doesn't apply.** For example, the attempted cancellation

$\frac{x + yz}{xyz} = \frac{1 + yz}{yz}$ is wrong, because x isn't a factor of the numerator.

Warning #2: When you simplify expressions, you may be using an algebra law or identity without realizing it. For example, suppose you want to perform the easy fraction

subtraction $\frac{2x + 2}{x} - \frac{2 + x}{x}$.

If you realize that the relevant identity is $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$, you will substitute

$2x + 2$ for a , $2 + x$ for b , and x for c , using parentheses and correctly obtaining

$$\frac{(2x+2)}{x} - \frac{(2+x)}{x} = \frac{(2x+2)-(2+x)}{x} = \frac{2x+2-2-x}{x} = \frac{x}{x} = 1$$

whereas if you forget to put parentheses around the $2 + x$ when you substitute it for b , your answer will be 3!

5.4 Put parentheses around any expression that is substituted for a function argument.

Suppose the function definition is $f(x) = 1 - x^2$

- 1) Substitute -3 for x : $f(-3) = 1 - (-3)^2$ is the value of f at -3 .
- 2) Substitute z for x : $f(z) = 1 - (z)^2$ is the value of f at z .
- 3) Substitute $x + h$ for x : $f(x + h) = 1 - (x + h)^2$ is the value of f at $x + h$.
- 4) Substitute $3 - x$ for x : $f(3 - x) = 1 - (3 - x)^2$ is the value of f at $3 - x$.

5.5 Put parentheses around an expression that is substituted for a function value.

This is completely analogous to putting parentheses around an expression that's substituted for a variable. For example, suppose $f(x) = 1 - x^2$ and we want to simplify the expression $7 - f(-3)$. You need two sets of parentheses: one when you substitute -3 for x to get $f(-3) = 1 - (-3)^2$, and another to rewrite $7 - f(-3)$ as $7 - [1 - (-3)^2]$. One way to proceed is shown below. If you're comfortable omitting the second line, do so.

| Correct | Incorrect |
|----------------------|--------------------|
| $7 - f(-3)$ | $7 - f(-3)$ |
| $= 7 - [f(-3)]$ | $= 7 - f(-3)$ |
| $= 7 - [1 - (-3)^2]$ | $= 7 - 1 - (-3)^2$ |
| $= 7 - [1 - 9]$ | $= 7 - 1 - 9$ |
| $= 7 - [-8]$ | $= 6 - 9$ |
| $= 15$ | $= -3$ |

5.6 One more time:

1. Decide what do next by using the order of operations rules. Put the result of each algebra operation in parentheses.

2. Put parentheses around an expression that is substituted for any of the following:

- a variable in another expression;
- a variable in an algebra law or identity;
- a function argument;
- a function value.

Exercise 5.6.1: Perform the indicated substitution in the given expression, identity, or function. Use $f(x) = 1 - x^2$; $g(x) = -9 - x$

| |
|--|
| a) u for w in $x^2 - y^2$ |
| b) $u + 1$ for w in $x^2 - w^2$ |
| c) $u + 1$ for w and u for x in $x^2 - w^2$ |
| d) $u + 1$ for a and $v + 1$ for b in $a^2 - b^2$ |
| e) u for w in $x^2 - w^2$ |
| f) $u + 1$ for a and $v + 1$ for b in $ab = ba$ |
| g) $u + 1$ for a and $v + 1$ for b and $w + 1$ for c in $a(b + c) = ab + ac$ |
| h) $u + 1$ for a and $v + 1$ for b and $w + 1$ for c in $a(b - c) = ab - ac$ |
| i) $u + 1$ for a and $v + 1$ and for b and $w + 1$ for c in $(a^2 - ab)(c - a)$ |
| j) $x + h$ for x in $f(x) = x - 3x^2$ |
| k) $x + h$ for x in $f(x) = 1 - x - x^2$ |
| l) Find $f(x + h) - f(x)$ |
| m) Find $f(x + h) - f(x - h)$ |
| n) Find $g(x + h) - g(x)$ |
| o) Find $g(x + h) - g(x - h)$ |
| p) Find $f(x) - 3xg(x)$ |
| q) Find $g(x) - g(-x)$ |
| r) Find $f(x) - f(-x)$ |

Exercise 5.6.2 Rewrite each result of the last exercise without parentheses as a simplified sum.

CHAPTER 6: WORKING WITH FRACTIONS

Your initial goal in this chapter is to rewrite each of the following examples as a completely reduced fraction by canceling common factors of numerator and denominator. Two examples are done for you. If you are comfortable doing these problems, skip to section 6.4.

| |
|--|
| $\frac{2^8 \cdot 3^{10} \cdot 5^{12} \cdot 7^4}{2^4 \cdot 3^{12} \cdot 5^6 \cdot 11} = \frac{2^4 \cdot 5^6 \cdot 7^4}{3^2 \cdot 11}$ |
| $\frac{2^8 \cdot 3^{10} \cdot 5^{12} \cdot 7^4}{2^{10} \cdot 3^2 \cdot 5^6 \cdot 7 \cdot 11} =$ |
| $\frac{2^8 \cdot 3^{10} \cdot 5^{12} \cdot 7^4}{2^{10} \cdot 3^{32} \cdot 5^{64} \cdot 7^{45} \cdot 11^3} =$ |
| $\frac{abcde}{a^2b^2c^2d^2e^2} =$ |
| $\frac{e^4d^4a^4b^4cde}{a^2b^2c^2d^2e^2} =$ |
| $\frac{(2x+1)^4}{(2x+1)^8} =$ |
| $\frac{(2x+1)^4(x+2)^8}{(x+2)^4(2x+1)^8} =$ |
| $\frac{uvw^3(u+1)^2(2u+1)^3}{u^2v^{10}x^3(1+u)^5} =$ |
| $\left(\frac{a^5b^8a}{c^3d^3}\right)\left(\frac{acd^4}{a^2b^2}\right) = \frac{a^5b^8aacd^4}{c^3d^3a^2b^2} = \frac{a^7b^8cd^4}{c^3d^3a^2b^2} = \frac{a^5b^6d}{c^2}$ |
| $\left(\frac{x^5b^8}{c^3y^3}\right)\left(\frac{acy^{14}}{x^5a^2b^2}\right) =$ |
| $\frac{(x+1)^3(5x+2)^5}{(x+1)^8(5x+2)^2} =$ |
| $\frac{(x+1)^2(x+3)^3w^2}{w^5(x+1)^5(x+3)^3} =$ |
| $\frac{(1+2x)^2(3+x)^3w^{12}}{w^5(2x+1)^5(x+3)^3} =$ |
| $\frac{(x+1)^2(x+3)^3w^2}{w^5(x+1)^5(3+x)^3} \times \frac{(x+3)^3w^2}{w^5(1+x)^{25}(x+3)^8} =$ |

6.1 Identities involving fractions

Working with fractions can be tricky when the numerator and denominator involve variables. First we list important identities you should know. All of them are basic algebra rules listed earlier or can be derived from those rules by using substitution.

Fraction basics:

$$\frac{a}{1} = a$$

$$\frac{a}{b} = a \left(\frac{1}{b} \right) = \left(\frac{1}{b} \right) a$$

$$\frac{a}{a} = a \left(\frac{1}{a} \right) = \left(\frac{1}{a} \right) a = 1$$

Fraction addition: same denominators $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$;

general rule: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

Fraction subtraction: same denominators $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$;

general rule: $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$

Fraction multiplication: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ **Fraction division:** $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$

Fraction negation: $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ **Fraction cancellation** $\frac{ab}{ac} = \frac{b}{c}$

Implied parentheses in vertical fraction notation:

Let E and F be any expressions. Then $\frac{E}{F} = \frac{(E)}{(F)}$.

Warning: The corresponding statement is false for fractions written in calculator notation. **When you use calculator notation, you usually need to place parentheses around the denominator.** For example $\frac{pq}{rs} = pq/(rs)$, which is not the same as pq/rs .

Exercise 6.1.1 Use order of operations to show that $pq/rs = pqs/r$.

6.2 Multiplying and dividing fractions

Fraction multiplication is easy to describe: $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$, also written $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$

Often we omit the times sign and write $\frac{a}{b} \frac{c}{d} = \frac{ac}{bd}$

This is easy to remember but almost too simple: to multiply fractions, you just connect the fraction bars. Another way to write the multiplication law is

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$$

Here we use the standard convention that the pair of consecutive parentheses “)(“ really means “) \times (.”

Warning: The similar looking formula $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$ is 100% wrong!

The following examples explain how to substitute using the basic canceling law.

Example 6.2.1 Use the Fundamental Substitution Principle (FSP) in the following examples. When you substitute for a variable, use parentheses!

a) Use the fraction multiplication law to rewrite $\frac{(x+1)^2}{(x+2)^3} \frac{x+3}{x+4}$

Solution : Substitute $(x+1)^2$ for a , $(x+2)^3$ for b , $x+3$ for c , and $x+4$ for d in $\frac{a}{b} \frac{c}{d} = \frac{ac}{bd}$

to get $\frac{(x+1)^2}{(x+2)^3} \frac{(x+3)}{(x+4)} = \frac{(x+1)^2(x+3)}{(x+2)^3(x+4)}$

FSP requires parentheses around $(x+1)^2$ and $(x+2)^3$, but we omit them because PEMDAS says that all expressions contain implied parentheses around powers.

b) Find $\frac{u^2}{w} \frac{w}{v}$ Here parentheses aren't needed: $\frac{u^2}{w} \frac{w}{v} = \frac{u^2 w}{wv}$

Furthermore, the answer can be rewritten by using the commutative law together with the fraction multiplication law written in reverse:

$$\frac{u^2}{w} \frac{w}{v} = \frac{u^2 w}{wv} = \frac{wu^2}{wv} = \frac{w}{w} \frac{u^2}{v} = 1 \left(\frac{u^2}{v} \right) = \frac{u^2}{v}$$

This is really an example of “cancellation,” which will be discussed in the next section.

To divide by a fraction, multiply by its reciprocal. The quotient of u divided by v is expressed in three ways: $\frac{u}{v} = u / v = u \div v$. The definition of division says that

$$\frac{u}{v} = u \cdot \left(\frac{1}{v}\right) \text{ where } \frac{1}{v}, \text{ called the } \mathbf{inverse} \text{ of } v, \text{ satisfies } v \cdot \left(\frac{1}{v}\right) = \left(\frac{1}{v}\right) \cdot v = 1$$

In particular, the inverse of a fraction $\frac{c}{d}$ is its reciprocal $\frac{d}{c}$, since $\frac{c}{d} \cdot \frac{d}{c} = \frac{cd}{dc} = \frac{cd}{cd} = 1$

Thus we have two basic division laws:

Expression divided by fraction: $\frac{u}{c/d} = u \div \left(\frac{c}{d}\right) = u \cdot \left(\frac{d}{c}\right) = \frac{ud}{c}$

Fraction divided by fraction: $\frac{a/b}{c/d} = \frac{a}{b} \div \left(\frac{c}{d}\right) = \frac{a}{b} \cdot \left(\frac{d}{c}\right) = \frac{ad}{bc}$

The large parentheses are not necessary, but make it easier to see what's going on. The dots that indicate multiplication are optional.

In the last case the answer is written as a product divided by a product. Simplifying such expressions is the goal of the next section.

6.3 Simplifying fractions: the cancellation law

Perhaps the most important fraction identity is the **cancellation law**: $\frac{ac}{bc} = \frac{a}{b}$

Substituting any expressions whatsoever for each of the variables a, b, c yields an identity. For example, substitute $x + 2$ for a ; uvw for b , and xyz for c to obtain

$$\frac{(x+2)(xyz)}{(uvw)(xyz)} = \frac{(x+2)}{uvw}, \text{ usually written as } \frac{(x+2)xyz}{uvwxyz} = \frac{x+2}{uvw}.$$

Most students learn to cancel this example by putting a slash through the identical factors xyz on top and bottom of the fraction. That's fine, but these notes will omit the slash because it's easy to get into bad habits and misuse the slash when canceling isn't legal.

The cancellation law is an identity. Like other identities, it can be written using any variables. For example, substituting u for a , v for b , and w for c in $\frac{ac}{bc} = \frac{a}{b}$ gives $\frac{uw}{vw} = \frac{u}{v}$.

From now on we will restate algebra laws with whatever variables are convenient.

The cancellation law gives rise to very basic identities using powers of variables. The following examples show how to cancel such powers. The basic ingredient in most problems is the following rule.

Rule for canceling powers: If numerator and denominator both contain powers of the same variable, then factor the higher power and cancel the lower power.

For example, if a^8 appears in the numerator and a^3 appears in the denominator (or vice versa), rewrite $a^8 = a^{8-3}a^3 = a^5a^3$ and **cancel the lower power** a^3 from numerator

and denominator. In symbols: $\frac{a^3x}{a^8y} = \frac{a^3x}{a^3a^5y} = \frac{x}{a^5y}$.

Officially, what happened is that we substituted u for a^3 for u ; x for v ; and a^5y for w in the cancellation law $\frac{uv}{uw} = \frac{v}{w}$ to obtain $\frac{(a^3)(x)}{(a^3)(a^5y)} = \frac{(x)}{(a^5y)} = \frac{x}{a^5y}$. **Don't think of all**

this detail when you cancel: just write $\frac{a^3x}{a^8y} = \frac{x}{a^5y}$.

The important thing is to remember that **cancellation means: rewrite the fraction by canceling (crossing out, omitting) a common factor from numerator and denominator.**

If numerator or denominator consists entirely of a single variable raised to a power, be careful. When the power in the denominator is higher, be sure not to cancel the

numerator: $\frac{a^3}{a^8} = \frac{a^3}{a^3a^5} = \frac{1}{a^5}$, **NOT** a^5 (which is wrong) or $\frac{1}{a^5}$ (which is nonsense).

Similarly $\frac{a^3}{a^8y} = \frac{a^3 \cdot 1}{a^3a^5y} = \frac{1}{a^5y}$.

When the power in the numerator is higher, the answer need not be written as a fraction:

$$\frac{a^8}{a^3} = \frac{a^5a^3}{a^3} = \frac{a^5}{1} = a^5.$$

When you do a more complicated example, proceed as follows.

Procedure for reducing a fraction in which numerator and denominator are products of powers.

For each base that appears in both numerator and denominator, subtract the lower exponent from both exponents. The result will be to:

- replace the lower power by 1 and to
- replace the higher power's exponent by the difference of the exponents.

Here's an easy example: $\frac{a^3 b^5 c^{10}}{a^5 b^7 c^8} = \frac{1 \cdot 1 \cdot c^{10-8}}{a^{5-3} b^{7-5} \cdot 1} = \frac{c^2}{a^2 b^2}$. You should try to do this in one step and write just $\frac{a^3 b^5 c^{10}}{a^5 b^7 c^8} = \frac{c^2}{a^2 b^2}$.

The following table of additional examples includes more detail than is necessary. It is included for reference only, to show how canceling fractions is just one more application of the Fundamental Substitution Principle. Make sure that you can quickly write each right answer without writing down the details of the substitution.

Example 6.3.1

| Simplify the expression below | Solution on the top line is obtained by substituting in the identity on the bottom line. | Substitution used |
|--|--|--|
| $\frac{a^5 b}{a^2 c} = ?$ | $\frac{a^5 b}{a^2 c} = \frac{(a^2)(a^3 b)}{(a^2)(c)} = \frac{a^3 b}{c}$ $\frac{uv}{uw} = \frac{v}{w}$ | $\begin{cases} a^2 \leftrightarrow u \\ a^3 b \leftrightarrow v \\ c \leftrightarrow w \end{cases}$ |
| $\frac{(x+2)^5(x^2+23)}{(x+2)^2(x^2+7)} = ?$ | $\frac{(x+2)^5(x^2+23)}{(x+2)^2(x^2+7)} = \frac{(x+2)^3(x^2+23)}{(x^2+7)}$ $\frac{a^5 b}{a^2 c} = \frac{a^3 b}{c}$ | $\begin{cases} x+2 \leftrightarrow a \\ x^2+23 \leftrightarrow b \\ x^2+7 \leftrightarrow c \end{cases}$ |
| $\frac{a^m}{a^n} = ? \quad (m > n)$ | $\frac{a^m}{a^n} = \frac{a^{m-n} a^n}{a^n} = \frac{a^{m-n}}{1} = a^{m-n}$ $\frac{uv}{v} = \frac{u}{1} = u$ | $\begin{cases} a^{m-n} \leftrightarrow u \\ a^n \leftrightarrow v \end{cases}$ |
| $\frac{a^m}{a^n} = ? \quad (m < n)$ | $\frac{a^m}{a^n} = \frac{a^m}{a^m a^{n-m}} = \frac{1}{a^{n-m}}$ | |
| $\frac{a^5 b^7}{a^3 b^{10}}$ | $\frac{a^5 b^7}{a^3 b^{10}} = \frac{a^2}{1} \frac{1}{b^3} = \frac{a^2}{b^3}$ | |
| $\frac{a^4}{(a^5)^4}$ | $\frac{a^4}{(a^5)^4} = \frac{a^4}{a^{20}} = \frac{1}{a^{16}}$ $(a^m)^n = a^{mn}$ | $\begin{aligned} m &\leftrightarrow 5 \\ n &\leftrightarrow 4 \end{aligned}$ |
| $\frac{(x^2+3)^4}{((x^2+3)^5)^4}$ | $\frac{(x^2+3)^4}{((x^2+3)^5)^4} = \frac{(x^2+3)^4}{(x^2+3)^{5 \times 4}} = \frac{(x^2+3)^4}{(x^2+3)^{20}} = \frac{1}{(x^2+3)^{16}}$ | $a \leftrightarrow x^2+3$ See last example |

| | | |
|---|---|--|
| $\frac{\sqrt{x}(x^2+3)^4}{x^2(x^2+3)^7}$ | $\frac{\sqrt{x}(x^2+3)^4}{x^2(x^2+3)^7} = \frac{x^{1/2}(x^2+3)^4}{x^2(x^2+3)^7}$ $= \frac{1}{x^{2-1/2}} \frac{1}{(x^2+3)^{7-4}} = \frac{1}{x^{3/2}} \frac{1}{(x^2+3)^3} = \frac{1}{x\sqrt{x}(x^2+3)^3}$ | |
| $\frac{2000a^5b^5c^5}{1628a^3b^{10}c^{15}}$ | $\frac{2000a^5b^5c^5}{1628a^3b^{10}c^{15}} = \frac{2000}{1628} \frac{a^{5-3}}{1} \frac{1}{b^{10-5}} \frac{1}{c^{15-5}} = \frac{500a^2}{407b^5c^{10}}$ $\frac{2000}{1628} = \frac{4 \times 500}{4 \times 407} = \frac{500}{407} = \frac{2^2 5^3}{11 \times 37} \text{ doesn't reduce}$ | |

Exercise 6.3.1 Reduce the following fractions without indicating the substitution that you used. Note that a power of a variable that appears in numerator or denominator, but not both, is left undisturbed.

| |
|---|
| Example: $\frac{a^8b^{10}c^{12}d^4}{a^4b^{12}c^6e} = \frac{a^{8-4}c^{12-6}d^4}{b^{12-10}e} = \frac{a^4c^6d^4}{b^2e}$. Try to skip the middle step! |
| $\frac{a^8b^{10}c^{12}d^4}{a^{12}b^{12}c^6e} =$ |
| $\frac{a^8b^{10}c^{12}d^4}{a^{12}c^{16}e} =$ |
| $\frac{u^2v^2w^2}{u^5v^5w^5} =$ |
| $\frac{20u^{12}v^{12}w^{12}}{35u^5v^5w^5} =$ |
| $\frac{x^5(x+2)^7}{x^3(x+2)^{10}}$ |

Exercise 6.3.2 Fill in the table at the beginning of Chapter 6.

6.4 Factor numerator and denominator before you cancel

Students court disaster when they misuse the cancellation law. You need to know the following material in order to understand when canceling is, and is not, legal.

Procedure for simplifying a fraction:

- Factor the numerator and denominator.
- Cancel common factors of numerator and denominator.

The previous section covered only the second step of the above procedure, because numerators and denominators were given in factored form.

Definitions involving factoring. Let E , F , and G be polynomial expressions.

E is a **factor** of G means that G can be written as a product $(E)(F)$, where F is another expression.

E is a **common factor** of F and G provided E is a factor of F and E is a factor of G .

E is **prime** if its only factors are 1, -1, E , and $-E$.

A **reduced fraction** is a fraction whose numerator and denominator are polynomial expressions with no common factor other than -1 and 1.

A **fraction is in standard form** if it is reduced and if its numerator and denominator are each written as an integer multiplied by a product of (powers of) prime factors.

Simplify a fraction means: reduce and rewrite the fraction in standard form.

Example 6.4.1

- x is a factor of xyz because $xyz = (x)(yz)$.
- $3xy$ is a common factor of $6x^2y$ and $9xy^2$ because $6x^2y = (3xy)(2x)$ and $9x^2y = (3xy)(3y)$.
- 7, $3x+1$, and x^2+1 are all prime expressions. 7 is a prime number.
- $\frac{3xa+ya}{3x+1}$ is a reduced fraction. When you simplify you get
- $\frac{a(3x+y)}{3x+1}$, which is a fraction in standard form.

The cancellation law $\frac{ac}{bc} = \frac{a}{b}$ allows canceling a *common factor* from numerator and denominator of a fraction. Nothing else can be cancelled.

For example, in the fraction $\frac{4x}{x^2+xy} = \frac{4x}{(x+y)(x)}$

- x is a factor of the numerator because the numerator is equal to $(4)(x)$;
- x is a factor of the denominator because the denominator is equal to $(x+y)(x)$;
- x is a common factor of numerator and denominator because it is a factor of each.

Therefore we can substitute 4 for a , $x+y$ for b , and x for c in $\frac{ac}{bc} = \frac{a}{b}$ to rewrite the given

fraction as $\frac{(4)(x)}{(x+y)(x)} = \frac{4}{x+y}$.

Don't cancel if the cancellation law doesn't apply. For example, the attempted

cancellation $\frac{x + yz}{xyz} = \frac{1 + yz}{yz}$ is wrong, because x isn't a factor of the numerator. Other

fractions that can't be simplified include:

$$\frac{x + y}{2x + y}$$

$$\frac{x}{x + y}$$

$$\frac{4 + x}{2 + x}$$

$$\frac{ax + bx + c}{dx}$$

One of the reasons that students make errors when cancelling is that they don't try to factor before they cancel. As a result, they cancel a piece of the numerator that isn't a factor. Here are some more possible **errors**:

$$\frac{(x + 3)^2}{(x + 5)^2} = \frac{x + 3}{x + 5} \quad \frac{x}{xy} = y \quad \frac{\cos 2x}{2} = \cos x.$$

Worst of all is $\frac{\sin x}{n} = \text{six}$. (just kidding!).

If you aren't sure whether you are allowed to cancel, play it safe and write down

explicitly both the cancellation law $\frac{ax}{bx} = \frac{a}{b}$ and the three expressions that are being substituted for a , b , and x .

Study carefully the table below.

Example 6.4.2

| | | |
|----------------------------------|---|--|
| $\frac{2a + c}{4a + c} = ?$ | Doesn't simplify: numerator and denominator do not have a common factor. | |
| $\frac{2a + 4c}{4a + 6c} = ?$ | $\frac{2a + 4c}{4a + 6c} = \frac{2(a + 2c)}{2(2a + 3c)} = \frac{(a + 2c)}{(2a + 3c)}$ | |
| $\frac{x^2 + y}{2x^2 + y} = ?$ | Doesn't simplify: numerator and denominator do not have a common factor. | |
| $\frac{x^3 + xy}{2x^3 + xy} = ?$ | $\frac{x(x^2 + y)}{x(2x^2 + y)} = \frac{x^2 + y}{2x^2 + y}$ | |

| | | |
|---|--|---|
| $\frac{\left(\frac{(x+1)^3(x+2)^5}{(x+1)^2(x+2)^7} \right)}{\left((x+2)^4 \right)^2}$ | $\begin{aligned} & \frac{(x+1)^3(x+2)^5 + (x+1)^2(x+2)^7}{\left((x+1)^4 \right)^2} \\ &= \frac{a^3b^5 + a^2b^7}{(a^4)^2} = \frac{a^2b^5(a+b^2)}{a^8} = \frac{b^5(a+b^2)}{a^6} \\ &= \frac{(x+2)^5[(x+1) + (x+2)^2]}{(x+1)^6} \\ &= \frac{(x+2)^5[x^2 + 5x + 5]}{(x+1)^6} \end{aligned}$ | <p>Used substitution</p> $\begin{cases} a = x+1 \\ b = x+2 \end{cases}$ |
| $\frac{\frac{1}{2a}b^3 - 3ab^2}{(b^3)^2}$ | $\begin{aligned} & \left(\frac{2a}{2a} \right) \left(\frac{\frac{1}{2a}b^3 - 3ab^2}{(b^3)^2} \right) \\ &= \frac{(2a) \left(\frac{1}{2a}b^3 - 3ab^2 \right)}{(2a)(b^6)} \\ &= \frac{(2a) \left(\frac{1}{2a}b^3 \right) - (2a)(3ab^2)}{(2a)(b^6)} \\ &= \frac{b^3 - 6a^2b^2}{2ab^6} \\ &= \frac{b^2(b - 6a^2)}{2ab^6} \\ &= \frac{b - 6a^2}{2ab^4} \end{aligned}$ | <p>First step is multiplying by $\frac{2a}{2a}$ to get rid of the fraction in the numerator. This problem is a warm-up for the next problem.</p> |

| | | |
|---|--|---|
| $\frac{\left(\frac{1}{2\sqrt{x}}(x+3)^3 - 3\sqrt{x}(x+3)^2 \right)}{\left((x+3)^3 \right)^2}$ | $\frac{\left(2\sqrt{x} \left(\frac{1}{2\sqrt{x}}(x+3)^3 - 3\sqrt{x}(x+3)^2 \right) \right)}{\left(2\sqrt{x} \right) \left((x+3)^3 \right)^2}$ $= \frac{(x+3)^3 - (2\sqrt{x})(3\sqrt{x}(x+3)^2)}{(2\sqrt{x})(x+3)^6}$ $= \frac{(x+3)^3 - 6x(x+3)^2}{2\sqrt{x}(x+3)^6}$ <p>Now factor. Don't multiply!</p> $= \frac{(x+3)^2[x+3-6x]}{2\sqrt{x}(x+3)^6}$ $= \left(\frac{\sqrt{x}}{\sqrt{x}} \right) \frac{[3-5x]}{2\sqrt{x}(x+3)^4}$ $= \frac{3-5x}{2\sqrt{x}(x+3)^4}$ | <p>Substitute \sqrt{x} for a and $x+3$ for b in the last example.</p> <p>This problem arises in first semester calculus when you find the derivative of $\frac{\sqrt{x}}{(x+3)^3}$</p> |
|---|--|---|

Exercise 6.4.1: Simplify the follow fractions or write DNS (Does Not Simplify). Start by making sure that the numerator and denominator are factored completely.

| | |
|----|------------------------------------|
| a) | $\frac{x^3 + 4x^2}{3x + x^2}$ |
| b) | $\frac{240}{256}$ |
| c) | $\frac{2x + 4}{(x + 2)^2}$ |
| d) | $\frac{2x + 4}{x^2 + 4x + 4}$ |
| e) | $\frac{3x + 6}{2x^2 + 8x + 8}$ |
| f) | $\frac{xyz}{xyw}$ |
| g) | $\frac{xyz}{xy + w}$ |
| h) | $\frac{abc + bcd}{abd + bx}$ |
| i) | $\frac{4x + 2y + 2z}{2x + y + 2z}$ |

| | |
|----|---|
| j) | $\frac{uvw^2}{2uvw}$ |
| k) | $\frac{ab+ac+ad}{ax+ay}$ |
| l) | $\frac{12a^3b^5+18a^2b^7+24ab^2}{a^8b}$ |
| m) | $\frac{x^2(x+2)^2+x^3(x+2)}{(x+2)^4}$ |
| n) | $\frac{\frac{1}{2a^3}b^7-3a^4b^2}{(b^2)^4}$ |
| o) | $\frac{\frac{(2x+1)^7}{2(x+2)^3}-3(x+2)^4(2x+1)^2}{((2x+1)^2)^4}$ |
| p) | $\frac{\frac{(2x+1)^7}{2\sqrt{x}}-3x^4(2x+1)^2}{(x^3(2x+1)^2)^4}$ |

6.5 More on factoring.

The only factoring topic discussed so far was pulling out the GCF of a sum of terms. Two other important topics are factoring integers and factoring quadratic polynomials.

Factoring a quadratic polynomial: Factoring the polynomial ax^2+bx+c with whole number coefficients a , b , and c is easiest when $a=1$. Then the polynomial x^2+bx+c is called *monic*. We want to rewrite it as follows: $x^2+bx+c=(x+r)(x+s)$. Notice that

$$=x^2+xs+rx+rs$$

$$=x^2+sx+rx+rs$$

$$=x^2+(r+s)x+rs$$

In summary, we want to find integers r and s such that $x^2+bx+c=x^2+(r+s)x+rs$. Certainly this will be the case if $r+s=b$ and $rs=c$. In other words, to factor x^2+bx+c as $(x+r)(x+s)$ find whole numbers r and s with sum b and product c .

To find r and s , first write down pairs of integers whose product is c . Then look for a pair that adds to b . If there is no such pair, then x^2+bx+c doesn't factor.

Here's a short cut: if you find a pair with $r + s = -b$ instead of b , just reverse the signs of r and s : then $-r + (-s) = b$.

To factor an arbitrary quadratic expression, first pull out the GCF of its coefficients. You don't need to find the GCF explicitly: just pull out common factors until you can't continue. For example,

$$42x^2 + 210x + 252 = 2(21x^2 + 105x + 126) = 2 \cdot 3(7x^2 + 35x + 42) = 2 \cdot 3 \cdot 7(x^2 + 5x + 6)$$

The last polynomial is monic and factors easily. Therefore

$$42x^2 + 210x + 252 = 42(x + 2)(x + 3)$$

If the leading coefficient of $ax^2 + bx + c$ is not 1, the trial and error procedure shown above is more complicated and more annoying. A systematic procedure that avoids trial and error is discussed in Chapter 9.

Example 6.5.1 Factoring monic quadratic polynomials

| Factor $x^2 + bx + c$ given below, with solution shown | Possible pairs (r,s) that multiply to c | Successful pair with $r + s = b$ |
|--|---|---|
| $x^2 + 7x + 12 = (x + 3)(x + 4)$ | (1,12); (2,6); (3,4) | $3 + 4 = 7$ |
| $x^2 - 7x + 12 = (x - 3)(x - 4)$ | (-1,-12); (-2,-6); (-3,-4) | $-3 + -4 = -7$ |
| $x^2 - x - 12 = (x - 4)(x + 3)$ | (-1,12); (-2,6); (-3,4) | $-3 + 4 = 1$, which is $-b$ Multiply by -1 to get the successful pair $3 + -4 = -1$ |
| $x^2 + x + 12$ doesn't factor | (1,12) (2,6) (3,4) | None of these pairs add to 1. |
| $-x^2 + 2x + 15$ $= -1(x^2 - 2x - 15)$ $= -(x + 3)(x - 5)$ | (1,-15), (3, -5), (5,-3), (15,-1) | When the leading coefficient is -1, factor it out before you begin. Then $3 + -5 = -2$ |

Factoring whole numbers.

A systematic method for reducing fractions of integers is to first factor numerator and denominator as products of powers of prime integers, or "primes," for short. These are positive integers m with exactly two positive integer factors: m and 1. Note that 1 has only one factor and is therefore not prime.

The first few primes are 2, 3, 5, 7, 11, 13. To factor an integer, factor out 2 as many times as you can, then factor out 3 as many times as you can, and continue up the list of primes. Stop when the square of the prime being tested is larger than the number you are trying to factor. For example to test whether 97 is prime, first try 2,3,5, and 7: none are factors of 97. The next prime to try is 11, but its square is 121, which is greater than 97, so stop trying. It follows that 97 is prime.

The most systematic way of reducing a fraction is to factor both numerator and denominator using the method just shown, and then canceling common factors.

Example 6.5.2

| |
|--|
| $300 = 2 \cdot 150 = 2 \cdot 2 \cdot 75 = 2 \cdot 2 \cdot 3 \cdot 25 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 = 2^2 \cdot 3 \cdot 5^2$. Another way: $300 = 10 \cdot 30 = 2 \cdot 5 \cdot 2 \cdot 15 = 2 \cdot 5 \cdot 2 \cdot 3 \cdot 5 = 2^2 \cdot 3 \cdot 5^2$ |
| $\frac{30(x^2 + 2x + 1)}{35(x^2 + 3x + 2)} = \frac{5 \cdot 6(x+1)(x+1)}{5 \cdot 7(x+1)(x+2)} = \frac{6(x+1)}{7(x+2)}$ |
| $\frac{30x^2 + 60x + 30}{35x^2 + 105x + 70} = \frac{30(x^2 + 2x + 1)}{35(x^2 + 3x + 2)}$ Now continue as in the above example. |
| $\begin{aligned} & \frac{60(x^2 + 3x + 2)(x+1)^2}{(2x^2 + 10x + 8)(x^2 + 5x + 6)} \\ &= \frac{60(x+2)(x+1)(x+1)^2}{2(x^2 + 5x + 4)(x+3)(x+2)} \\ &= \frac{2 \cdot 30(x+2)(x+1)^3}{2(x+1)(x+4)(x+3)(x+2)} \\ &= \frac{30(x+1)^2}{(x+4)(x+3)} \end{aligned}$ |

Factoring out signs. Use the fundamental identity $a - b = -1(b - a)$

$$\text{Thus } \frac{a-b}{b-a} = \frac{-1(b-a)}{b-a} = -1 \text{ but } \frac{(a-b)^2}{(b-a)^2} = \frac{[-1(b-a)]^2}{(b-a)^2} = \frac{(-1)^2(b-a)^2}{(b-a)^2} = 1$$

Rewrite factors as standard form polynomials before you try to cancel.

Example 6.5.3

| |
|--|
| $\frac{(2x+1)^2}{(1+2x)^4} = \frac{(2x+1)^2}{(2x+1)^4} = \frac{1}{(2x+1)^2}$ |
| $\frac{(2x-1)^2}{(1-2x)^5} = \frac{(2x-1)^2}{(-1 \cdot (2x+1))^5} = \frac{(2x+1)^2}{(-1)^5(2x+1)^5} = \frac{(2x+1)^2}{(-1)(2x+1)^5} = \frac{-1}{(2x+1)^3}$ |
| $\frac{(-2x-1)^2}{(1+2x)^5} = \frac{((-1)(2x+1))^2}{(2x+1)^5} = \frac{(-1)^2(2x+1)^2}{(2x+1)^5} = \frac{(2x+1)^2}{(2x+1)^5} = \frac{1}{(2x+1)^3}$ |

Exercise 6.5.1 Reduce the following fractions completely: factor numerator and denominator, then cancel common factors.

| | |
|----|---|
| a) | $\frac{100}{250} =$ |
| b) | $\frac{1024}{729} =$ |
| c) | $\frac{4900}{320} =$ |
| d) | $\frac{x^2 + 3x + 2}{x^2 + 5x + 6} =$ |
| e) | $\frac{x^2 + 8x + 15}{2x^2 + 10x + 50}$ |
| f) | $\frac{2x^2 - 16x + 30}{3x^2 - 18x + 15}$ |
| g) | $\frac{x^2 - 3x - 4}{(1 + x)^4} =$ |
| h) | $\frac{(x^2 + 2x + 1)^2(x + 3)}{x^2 + 4x + 3} =$ |
| i) | $\frac{(x - 2)}{(2 - x)^2} =$ |
| j) | $\frac{(x^2 + 5x + 6)(x^2 - 5x + 6)}{(x^2 - 9)(x^2 + 9)} =$ |
| k) | $\frac{9 - x^2}{2x^2 - 18} =$ |
| l) | $\frac{9 - 4x^2}{8x^2 - 18} =$ |
| m) | $\frac{(x - y)^2}{(2x - 2y)^2} =$ |
| n) | $\frac{(x - y)^2}{(2y - 2x)^2} =$ |
| o) | $\frac{(2x^2 + 4x + 2)^2(4x + 12)}{8x^2 + 32x + 24} =$ |
| p) | $\frac{\frac{1}{x+2} + \frac{1}{x}}{\frac{1}{x^2 + 2x}} =$ |

CHAPTER 7: ADDING RATIONAL EXPRESSIONS

7.1 Substituting in the fraction addition law

Adding fractions with the same denominator is easy, since $\frac{x}{a} + \frac{y}{a} = \frac{x+y}{a}$.

Warning: The following is a **major error**: $\frac{a}{x+y} = \frac{a}{x} + \frac{a}{y}$.

For fractions with different denominators, the basic fraction addition law is

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

Example 7.1.1 Use the fraction addition law $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ to rewrite $\frac{5}{x+2} + \frac{7}{(x+3)^2}$ as a single fraction.

Solution: You want to match

$$\frac{5}{x+2} + \frac{7}{(x+3)^2}$$

with the addition law

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

To do so, substitute 5 for a ; $x+2$ for b ; 7 for c ; and $(x+3)^2$ for d :

$$\frac{5}{x+2} + \frac{7}{(x+3)^2} = \frac{(5)((x+3)^2) + (x+2)(7)}{(x+2)((x+3)^2)}$$

Clearly there are unnecessary parentheses on the right hand side. When you substitute an expression E for a variable u , you don't always need to place parentheses around E . Here are a few situations where the parentheses can and should be omitted.

- E is also a variable;
- E is a parenthesized expression;
- E is a power but u is not raised to a power.

In the above example, it follows that $(x+3)^2$, which was substituted for d , does not need to be placed in parentheses. We could have written

$$\frac{5}{x+2} + \frac{7}{(x+3)^2} = \frac{(5)(x+3)^2 + (x+2)(7)}{(x+2)(x+3)^2}$$

This is a correct answer to the question, which did not ask that the result be simplified.

Exercise 7.1.1: Prove the addition law for three fractions: $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{adf + bcf + bde}{bdf}$

Exercise 7.1.2: Rewrite the following expressions by substituting in the

identity $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$. Rewrite the answer as a fraction in standard form.

| | |
|--|-------------------------|
| a] $\frac{2}{3} + \frac{3}{4} =$ | |
| b] $2 + \frac{3}{4} =$ | Hint: $2 = \frac{2}{1}$ |
| c] $x + \frac{3}{4} =$ | |
| d] $\frac{x}{2} + \frac{3}{4} =$ | |
| e] $\frac{1}{2} + \frac{1}{3} =$ | |
| f] $\frac{1}{a} + \frac{1}{b} =$ | |
| g] $\frac{pq}{pr} + \frac{p^2}{q^2} =$ | |
| h] $\frac{x+2}{x+3} + \frac{x+3}{x+4} =$ | |
| i] $\frac{a^2}{b} + \frac{b}{a^2} =$ | |
| j] $x + \frac{y^2}{x} =$ | |
| k] $\frac{a}{b} - \frac{c}{d} =$ | |
| l] $\frac{1}{4} - \frac{1}{3} =$ | |
| m] $\frac{1}{b} - \frac{1}{a} =$ | |
| n] $\sqrt{x} + \frac{1}{\sqrt{x}} =$ | |
| o] $x + \frac{1}{x} =$ | |
| p] $x + \frac{1}{\sqrt{x}} =$ | |
| q] $x^5 + \frac{1}{x^5} =$ | |

7.2 Adding fractions: find common denominator, build, add, reduce

First we justify the basic fraction addition law. $\frac{x}{a} + \frac{y}{b} = \frac{xb + ya}{ab}$.

Start with the left hand expression.

$$\frac{x}{a} + \frac{y}{b}$$

Rewrite the expression by multiplying each fraction by 1.

$$= \frac{x}{a} \left(\frac{b}{b} \right) + \frac{y}{b} \left(\frac{a}{a} \right)$$

Multiplication comes before addition.

$$= \frac{xb}{ab} + \frac{ya}{ba}$$

Use the commutative law to rewrite the second denominator.

$$= \frac{xb}{ab} + \frac{ya}{ab}$$

Add fractions with the same denominator ab .

$$= \frac{xb + ya}{ab}$$

In practice, the fraction addition law can be inefficient. For example:

$$\frac{3}{1000} + \frac{1}{2000} = \frac{3 \cdot 2000 + 1000 \cdot 1}{1000 \cdot 2000} = \frac{7000}{2000000}$$

This answer is correct but needs to be reduced to $\frac{7}{2000}$. A better method for adding fractions is to begin by finding their least common denominator, as will be explained in the next section.

Our goal is to explain in detail the following procedure for adding fractions.

Procedure for adding fractions : CD, build, add, reduce

- Find a common denominator (CD) for the fractions.
- Build the fractions so that they have that common denominator.
- Add the built-up fractions.
- Reduce the result.

7.3 Finding the least common denominator

To add fractions, you need to rewrite them with the same (“common”) denominator. The product of their denominators can be used, but that product can be very complicated. For example, if you use the product of denominators as the common denominator when you

find $\frac{3}{1000} + \frac{1}{2000}$, you would have to rewrite both fractions with common denominator

$1000 \cdot 2000 = 2000000$. That would be a waste of time, since you can easily rewrite the first fraction as $\frac{6}{2000}$. Thus $\frac{3}{1000} + \frac{1}{2000} = \frac{6}{2000} + \frac{1}{2000} = \frac{7}{2000}$, an easy calculation.

It's often best to find the smallest or simplest common denominator of two or more fractions. That denominator is called the least common denominator (LCD) of the fractions.

The LCD can also be described as the smallest (or simplest) expression that has each denominator as a factor. Here's a bit more detail:

Definitions involving LCD and LCM. Assume that E , a , and b are polynomials

- E is a **multiple** of a means that there is a polynomial b with $E = ab$.
- E is a **common multiple** of a and b means that E is a multiple of a and E is a multiple of b .
- E is a **least common multiple** of expressions a and b means
 - E is a common multiple of a and b and
 - Any other common multiple of a and b is a multiple of E .
- The **least common denominator (LCD) of fractions** is the LCM of their denominators. We'll find LCD's of fractions in the next section.

Example 7.3.1 x^3y^2 is a common multiple of xy^2 and x^2y , since $x^3y^2 = (x^2)(xy^2) = (xy)(x^2y)$. The LCM of xy^2 and x^2y is x^2y^2 . Note that $x^3y^2 = (x^2y^2)(x)$ and so x^3y^2 is a multiple of the LCM.

Example 7.3.2

| | | |
|-----------------------------------|---|---|
| Find LCM of a, b, c | LCM is $abc = a \cdot bc = b \cdot ac = c \cdot ab$ | The LCM of distinct variables is their product. |
| Find LCM of a^4, b^3, c^{17} | LCM is $a^4b^3c^{17} = a^4 \cdot b^3c^{17}$ $= b^3 \cdot a^4c^{17} = c^{17} \cdot a^4b^3$ | The LCM of powers of distinct variables is the product of those powers. |

In the above examples, the LCM of expressions with no common factors is the product of the expressions.. However, when the expressions do have common factors, the LCM will be simpler than the product of the expressions.

Example 7.3.3

| | | |
|---------------------------|--|--|
| Find LCM of ab and bc | LCM is abc since $\begin{cases} abc = c \cdot ab \\ abc = a \cdot bc \end{cases}$ | Since b is a common factor of ab and bc , it shouldn't be repeated in the LCM. |
|---------------------------|--|--|

| | | |
|---------------------------------|--|---|
| Find LCM of ab^4 and cb^4 | LCM is ab^4c $ab^4c = c \cdot ab^4 = a \cdot cb^4$ | Since b^4 is a common factor of ab^4 and cb^4 , it isn't repeated in the LCM. |
| Find LCM of a^3b^4 and cb^7 | LCM is a^3b^7c $a^3b^7c = cb^3 \cdot a^3b^4 = a^3 \cdot cb^7$ | <i>The LCM is the product of the highest power of variables that appear in any of the expressions.</i> In this example: highest power of a is a^3 , highest power of b is b^7 , highest power of c is c . The LCM is the product a^3b^7c |

The rule stated in the last box and in more detail at the bottom of this page applies not only to variables, but to prime factors of the expressions. Recall that an expression E is prime if its only factors are ± 1 and $\pm E$. Here are some examples listing prime factors of expressions.

Example 7.3.4

- 2 and 3 are the prime factors of 54, which can be written as $2 \cdot 3^3$. Other factors (1, 6, 9, 18, 27, 54) are not prime.
- 2, 3, and x are the prime factors of $54x^3$, which can be written as $2 \cdot 3^3 x^3$. There are no other prime factors.
- 2, 3, x , and y are the prime factors of $54x^3y^{17} = 2 \cdot 3^3 x^3 y^{17}$.
- 2, 3, and $x + 1$ are the prime factors of $54(x + 1)^3 = 2 \cdot 3^3 (x + 1)^3$.
- 2, 3, 5, and $x + 3$ are the prime factors of $54(5x + 15)^3 = 2 \cdot 3^3 \cdot (5(x + 3))^3 = 2 \cdot 3^3 \cdot 5^3 \cdot (x + 3)^3$
- x , $x + 1$, and $x - 1$ are the prime factors of $x^3(x^2 - 1)^2 = x^3((x + 1)(x - 1))^2 = x^3(x + 1)^2(x - 1)^2$

To find the LCM of two or more expressions:

Step 1: Factor each expression as a product of powers of primes.

Step 2: For each prime that appears in any of the factorizations, write down the *highest* power of that prime in any of the factorizations.

Step 3: The LCM is the product of the powers written down in Step 2.

Example 7.3.5

| Find LCM of the following expressions: | Expressions rewritten as products of prime powers | LCM of the expressions |
|--|--|--|
| Find LCM of 12, 30, 64, 15 | $12 = 2^2 \cdot 3 = 2^2 \cdot 3$ $30 = 2 \cdot 3 \cdot 5$ $64 = 2^6$ $15 = 3 \cdot 5$ | Highest power of 2 is 2^6 Highest power of 3 is 3 Highest power of 5 is 5 LCM = product of highest powers = $2^6 \cdot 3 \cdot 5 = 64 \cdot 15 = 960$ |
| Find LCM of $12x^2y^3z^2$ $30xyz$ $64x^9$ $15y^4z^5$ | $12x^2y^3z^2 = 2^2 \cdot 3 \cdot y^3z^2$ $30xyz = 2 \cdot 3 \cdot 5xyz$ $64x^9 = 2^6 \cdot x^9$ $15y^4z^5 = 3 \cdot 5 \cdot y^4z^5$ | Use the previous example to see that the LCM of the coefficients is 960. Highest power of x is x^9 Highest power of y is y^4 Highest power of z is z^5 LCM is $2^6 \cdot 3 \cdot 5x^9y^4z^5$ $= 960x^9y^4z^5$ |
| Find LCM of $(x+1)^3(x+2)^4$ $(x^2+2x+1)(x+2)^4$ $(x^2+4x+4)(x+3)$ $y(x^2+3x+2)$ | Expressions factor as: $(x+1)^3(x+2)^4$ $(x+1)^2(x+2)^4$ $(x+2)^2(x+3)$ $y(x+1)(x+2)$ | Highest power of $x+1$ is $(x+1)^3$; highest power of $x+2$ is $(x+2)^4$; highest power of $x+3$ is $x+3$. LCM is $(x+1)^3(x+2)^4(x+3)y$ |
| Find LCM of $12(x+1)^3(x+2)^4$ $(30x^2+60x+30)(x+2)^4$ $64(x^2+4x+4)$ $15x^2+45x+30$ | Expressions factor as: $12(x+1)^3(x+2)^4$ $30(x+1)^2(x+2)^4$ $64(x+2)^2$ $15(x+1)(x+2)$ | LCM is the LCM of the coefficients 12, 30, 64, 15 multiplied by the LCM of the variable factors of the expressions. Use results of the last two examples to see that the LCM is $960(x+1)^3(x+2)^4(x+3)y$ |

Exercise 7.3.1 Find the LCM for each of the following list of expressions:

| |
|---|
| a) 24,36,18,32 |
| b) a^3b^4c , $a^2b^5c^7$, $abcd^2$ |
| c) $12x^2$, $20x^3$, $35x^2$ |
| d) $14x^{10}(x+1)^2$, $20(x+1)^3$, $35x^3(x+1)^2$ |
| e) $(x+1)^3(x+2)^4(x+3)$, $(x+1)^2(x+2)(x+1)^7$, $(x+1)(x+3)^2$ |
| f) $24(x+1)^3(x+2)^4(x+3)$, $36(x+1)^2(x+2)(x+1)^7$, $32(x+1)(x+3)^2$ |

$$g) \quad 24x^2 + 72x + 48, \quad 36x^2 + 72x + 36, \quad 32(x^2 + 4x + 4)^2$$

7.4 Building Fractions

This second step in the fraction addition procedure is called “build.” The idea is to multiply each fraction by 1 in order to rewrite it with a chosen common denominator.

Example 7.4.1 Rewrite $\frac{3}{4}$ as a fraction with denominator 20.

Solution:
$$\frac{3}{4} = \left(\frac{3}{4}\right)\left(\frac{?}{?}\right) = \frac{3 \times ?}{4 \times ?} = \frac{3 \times ?}{20}$$
$$4 \times ? = 20 \rightarrow ? = 20/4 = 5$$

In other words, to rewrite $\frac{3}{4}$ as a fraction with denominator 20, multiply top and bottom by 20/4, which is 5. Thus $\frac{3}{4} = \frac{3 \times 5}{4 \times 5} = \frac{15}{20}$

In this example it was easy to rewrite the new denominator 20 as 5 times the original denominator. That step can be hard in some cases. Study the examples below.

Example 7.4.2

| | | |
|---|--|--|
| <p>Rewrite $\frac{7}{60}$ as a fraction with denominator 720</p> | $\frac{7}{60} = \left(\frac{7}{60}\right)\left(\frac{?}{?}\right) = \frac{7(?)}{720}$ $(60)(?) = 720$ $? = 720/60 = 12$ $\frac{7}{60} = \frac{7 \cdot 12}{60 \cdot 12} = \frac{84}{720}$ | <p>You could also find ? by factoring 60 and 720 as follows</p> $60 = 2^2 \cdot 3 \cdot 5$ $720 = 2^4 \cdot 3^2 \cdot 5 = 60 \times ?$ <p>? = $2^2 \times 3 = 12$ is formed as the product of factors of 720 that are missing from the factors of 60. See next example.</p> |
|---|--|--|

Example 7.4.3

| | | |
|---|--|--|
| Rewrite $\frac{1}{a}$ as a fraction with denominator abc | $\frac{1}{a} = \frac{1 \cdot bc}{a \cdot bc} = \frac{bc}{abc}$ | To solve $a(?) = abc$, set ? equal to the factors of abc other than a . |
| Rewrite $\frac{7}{a^2}$ as a fraction with denominator a^3b^4 | $\frac{7}{a^2} = \frac{7(?)}{a^2(?)} = \frac{7 \cdot ab^4}{a^2 \cdot ab^4} = \frac{7ab^4}{a^3b^4}$ | ? is the original numerator 7 times the (product of) all factors of the built-up denominator a^3b^4 that are missing from the original denominator a^2 . |
| Rewrite $\frac{1}{\sqrt{x}}$ as fraction with denominator x^2 . | $\left(\frac{1}{\sqrt{x}}\right)\left(\frac{?}{?}\right) = \frac{?}{x^2}$ $\sqrt{x}(?) = x^2 = xx$ $? = x\sqrt{x}$ $\frac{1}{\sqrt{x}} = \frac{1 \cdot x\sqrt{x}}{\sqrt{x} \cdot x\sqrt{x}} = \frac{x\sqrt{x}}{x^2}$ | The point here is that $\sqrt{x}\sqrt{x} = x$ and so $\sqrt{x}(x\sqrt{x}) = xx = x^2$. Another method is shown in the second part of Example 7.4.4 immediately below. |

Example 7.4.4

| | | |
|---|---|---|
| Rewrite xyz as a fraction with denominator a^3b^4 | $xyz = \frac{xyz a^3 b^4}{a^3 b^4}$ | The original expression xyz is not a fraction. In this easy case, the built-up fraction's numerator is xyz times the desired denominator a^3b^4 . |
| Rewrite as fractions with denominator x^2 . \sqrt{x} x x^k $\frac{1}{\sqrt{x}}$ | $\sqrt{x} = \frac{x^2 \sqrt{x}}{x^2}$ $x = \frac{x^2 x}{x^2} = \frac{x^3}{x^2}$ $x^k = \frac{x^k x^2}{x^2} = \frac{x^{k+2}}{x^2}$ $\frac{1}{x^{1/2}} = \frac{1 \cdot x^{3/2}}{x^{1/2} x^{3/2}} = \frac{x^{3/2}}{x^2} = \frac{x\sqrt{x}}{x^2}$ | First three examples: use same procedure as in previous example. Last example: solve $x^{3/2} x^? = x^2$ by setting $? = 2 - \frac{1}{2} = \frac{3}{2}$ |

Example 7.4.5

| | | |
|---|---|---|
| $\frac{x}{a} = \frac{?}{abc}$ $\frac{y}{b} = \frac{?}{abc}$ $\frac{z}{c} = \frac{?}{abc}$ $\frac{u}{bc} = \frac{?}{abc}$ $\frac{xy}{a} = \frac{?}{abc}$ | $\frac{x}{a} = \frac{xbc}{abc}$ $\frac{y}{b} = \frac{yac}{abc}$ $\frac{z}{c} = \frac{azb}{abc}$ $\frac{u}{bc} = \frac{ua}{abc}$ $\frac{xy}{a} = \frac{xybc}{abc}$ | <p>Important simple examples. The first three arise in the addition problem</p> $\frac{x}{a} + \frac{y}{b} + \frac{z}{c}$ $= \frac{xbc}{abc} + \frac{yac}{abc} + \frac{zab}{abc}$ $= \frac{xbc + yac + zab}{abc}$ |
|---|---|---|

7.5 Adding fractions: find LCD, build, add, reduce

We have so far completed the first two steps of the following general procedure for adding fractions.

- Find a common denominator (CD) for the fractions
- Build up each fraction so that it has that common denominator
- Add the built-up fractions
- Reduce the result by factoring numerator and denominator and canceling.

In most cases, the work is easiest if you find the least common denominator in the first step. Therefore we summarize this procedure as: **LCD, build, add, reduce**. Remember that the LCD of the fractions is the LCM of their denominators.

In the examples below, we state the problem and find the LCD in the left column, then build fractions and add in the second column.

Example 7.5.1

| | |
|--|---|
| Find $\frac{1}{12} + \frac{3}{20}$ $12 = 2^2 \cdot 3$ $20 = 2^2 \cdot 5$ $LCD = 2^2 \cdot 3 \cdot 5$ | $LCD = 2^2 \cdot 3 \cdot 5$ $60 = (2^2 \cdot 3) \cdot 5 = 12 \cdot 5$ $60 = (2^2 \cdot 5) \cdot 3 = 20 \cdot 3$ $\frac{1}{12} + \frac{3}{20}$ $= \frac{1 \cdot 5}{12 \cdot 5} + \frac{3 \cdot 3}{20 \cdot 3}$ $= \frac{5}{60} + \frac{9}{60} = \frac{14}{60} = \frac{7}{30}$ |
|--|---|

Example 7.5.2

| | |
|---|---|
| Find $\frac{1}{12} + \frac{7}{30} + \frac{1}{64}$. Find LCD: $12 = 2^2 \cdot 3$ $30 = 2 \cdot 3 \cdot 5$ $64 = 2^6$ LCD is $2^6 \cdot 3 \cdot 5 = 960$ | $960 = 2^6 \cdot 3 \cdot 5$ $12 = 2^2 \cdot 3 \rightarrow 960 = (2^2 \cdot 3)(2^4 \cdot 5) = 12 \cdot 80$ $30 = 2 \cdot 3 \cdot 5 \rightarrow 960 = (2 \cdot 3 \cdot 5)(2^5) = 30 \cdot 32$ $64 = 2^6 \rightarrow 960 = (2^6)(3 \cdot 5) = 64 \cdot 15$ $\frac{1}{12} + \frac{7}{30} + \frac{1}{64}$ $= \frac{1 \cdot 80}{12 \cdot 80} + \frac{7 \cdot 32}{30 \cdot 32} + \frac{1 \cdot 15}{64 \cdot 15}$ $= \frac{80}{960} + \frac{224}{960} + \frac{15}{960}$ $= \frac{80 + 224 + 15}{960} = \frac{319}{960}$ Answer is a reduced fraction since $319 = 11 \cdot 29$ and $960 = 2^6 \cdot 3 \cdot 5$ have no common prime factors. |
|---|---|

In each of the next two examples, notice at the line marked (!!!) how important it is to place parentheses around the numerator of a fraction that is subtracted from another fraction. At that step you are really using the formula $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$ and so you need to use parentheses when you substitute for the variables!

Example 7.5.3

| | |
|---|---|
| Find | |
| $\frac{1}{3x+3} + \frac{2}{x+2} - \frac{2x+1}{6x^2+18x+12}$ | $\frac{1}{3x+3} + \frac{2}{x+2} - \frac{2x+1}{(6x+6)(x+2)}$ |
| $3x+3 = 3(x+1)$ | $= \frac{1}{3(x+1)} + \frac{2}{x+2} - \frac{2x+1}{2 \cdot 3(x+1)(x+2)}$ |
| $6x^2+18x+12$ | $= \frac{1 \cdot 2 \cdot (x+2)}{3(x+1) \cdot 2 \cdot (x+2)} + \frac{2 \cdot 2 \cdot 3(x+1)}{(x+2) \cdot 2 \cdot 3(x+1)} - \frac{2x+1}{2 \cdot 3(x+1)(x+2)}$ |
| $= 6(x^2+2x+3)$ | $= \frac{2x+4+12x+12-(2x+1)}{2 \cdot 3(x+1)(x+2)} \quad !!!$ |
| $= 2 \cdot 3(x+1)(x+2)$ | $= \frac{14x+16-2x-1}{2 \cdot 3(x+1)(x+2)}$ |
| LCD is $2 \cdot 3(x+1)(x+2)$ | $= \frac{12x+15}{6(x+1)(x+2)}$ |

Example 7.5.4

| | |
|--|---|
| Find | $LCD = x^2(x+1)(x+2)$ |
| $\frac{2}{x^2+x} - \frac{x+1}{x^3+2x^2}$ | $x^2(x+1)(x+2) = [x(x+1)][x(x+2)]$ |
| $x^2+x = x(x+1)$ | $x^2(x+1)(x+2) = [x^2(x+2)](x+1)$ |
| $x^3+2x^2 = x^2(x+2)$ | $\frac{2}{x^2+x} - \frac{x+1}{x^3+2x^2}$ |
| $LCD = x^2(x+1)(x+2)$ | $= \frac{2}{x(x+1)} \cdot \frac{x(x+2)}{x(x+2)} - \frac{x+1}{x^2(x+2)} \cdot \frac{x+1}{x+1}$ |
| | $= \frac{2x(x+2)}{x^2(x+1)(x+2)} - \frac{(x+1)(x+1)}{x^2(x+2)(x+1)}$ |
| | $= \frac{2x^2+4x}{x^2(x+1)(x+2)} - \frac{x^2+2x+1}{x^2(x+1)(x+2)}$ |
| | $= \frac{2x^2+4x-(x^2+2x+1)}{x^2(x+1)(x+2)} \quad (!!!)$ |
| | $= \frac{2x^2+4x-x^2-2x-1}{x^2(x+1)(x+2)}$ |
| | $= \frac{x^2+2x-1}{x^2(x+1)(x+2)}$ |

Exercise 7.5.1: Write each of the following sums as a simplified fraction

| |
|--|
| a) $\frac{1}{100} + \frac{7}{300} + \frac{3}{700}$ |
| b) $\frac{x}{a} + \frac{y}{a^2} + \frac{z}{a^3}$ |
| c) $\frac{x}{ab} + \frac{y}{a^2} + \frac{z}{a^3bc}$ |
| d) $\frac{x}{x+1} - x$ |
| e) $\frac{x}{(x+1)^2} + \frac{x}{(x+1)^3}$ |
| f) $\frac{2}{x+3} + \frac{1}{x+2}$ |
| g) $\frac{x+3}{x+2} - \frac{3x-4}{x}$ |
| h) $\frac{y}{x^2+2x} + \frac{y}{x^2+3x} + \frac{z}{(x+3)(x+2)}$ |
| i) $\frac{x+3}{x^2} - \frac{3x+4}{x}$ |
| j) $\frac{2}{x^2+3x} + \frac{1}{x^2}$ |
| k) $\frac{x+3}{x^2+x} - \frac{3x+4}{x^2+3x}$ |
| l) $\frac{x+3}{(x+1)^2} - \frac{4-2x}{x+1}$ |
| m) $\frac{x+3}{x} - \frac{x-3}{3-x}$ |

Exercise 7.5.2: Use two different methods to rewrite each of the following products or powers as a single fraction and compare the results of the two methods.

- Method 1:
 - Multiply out by using FOIL.
 - Add the resulting fractions to get a single fraction.
 - Simplify the resulting fraction
- Method 2:
 - Rewrite each factor as a fraction
 - Multiply fractions
 - Simplify the resulting fraction.

| |
|--|
| a) $\left(\frac{2}{3} + \frac{3}{4}\right)\left(\frac{2}{3} + \frac{3}{4}\right) =$ |
| b) $\left(2 + \frac{3}{4}\right)\left(1 + \frac{3}{4}\right) =$ |
| c) $\left(x + \frac{3}{4}\right)\left(x + \frac{3}{4}\right) =$ |
| d) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 =$ |
| e) $\left(x + \frac{1}{\sqrt{x}}\right)^2 =$ |
| f) $\left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right) =$ |
| g) $\left(\frac{1}{a} + \frac{1}{b}\right)^2 =$ |
| h) $\left(x + \frac{1}{x\sqrt{x}}\right)^2 =$ |
| i) $\left(x + \frac{1}{\sqrt{x}}\right)\left(x + \frac{1}{x}\right)$ |
| j) $\left(x^5 + \frac{1}{x^5}\right)^2$ |

7.6 Simplifying Nested Fractions

On occasion you will encounter a fraction whose numerator and/or denominator contains a fraction. These are often called “nested fractions.” Some examples are

$$\frac{x + \frac{1}{x}}{x - \frac{1}{x}} \quad \text{and} \quad \frac{1 + \frac{1}{x+1}}{1 + \frac{1}{x-1}}.$$

In each case, you start with a big fraction whose numerator or denominator contains one or more little fractions, which we call “the little fractions in the nest.”

To make a nested fraction simpler, recall that the cancellation law, like every identity, is really a pair of rewriting rules.

When you cancel, you rewrite the fraction $\frac{ab}{ac}$ in simpler form as $\frac{b}{c}$.

When you build fractions (as in section 7.4), you start with $\frac{b}{c}$ and rewrite it as $\frac{ab}{ac}$.

Let’s call this second rewriting method “canceling in reverse.” It really says:

You can rewrite a fraction by multiplying both numerator and denominator by the same expression.

It turns out that canceling in reverse can be used to get rid of the little fractions that appear in the numerator or denominator of an expression with nested fractions.

Procedure for simplifying nested fractions.

Multiply numerator and denominator by the LCD of the little fractions in the nest.

Example 7.6.1 Rewrite $\frac{3 + \frac{1}{4}}{\frac{1}{2} - \frac{4}{x}}$ as a reduced fraction.

Solution: Multiply numerator and denominator by $4x$, which is the desired LCD.

$$\frac{4x\left(3 + \frac{1}{4}\right)}{4x\left(\frac{1}{2} - \frac{4}{x}\right)} = \frac{4x(3) + 4x\left(\frac{1}{4}\right)}{4x\left(\frac{1}{2}\right) - 4x\left(\frac{4}{x}\right)} = \frac{12x + x}{2x - 16} = \frac{13x}{2(x - 8)}$$

Example 7.6.2 Rewrite $\frac{x + \frac{1}{x}}{x - \frac{1}{x}}$ as a reduced fraction.

Solution: Multiply both numerator and denominator by x .

$$\frac{x + \frac{1}{x}}{x - \frac{1}{x}} = \frac{(x)\left(x + \frac{1}{x}\right)}{(x)\left(x - \frac{1}{x}\right)} = \frac{x(x) + x\left(\frac{1}{x}\right)}{x(x) - x\left(\frac{1}{x}\right)} = \frac{x^2 + 1}{x^2 - 1}.$$

The fraction doesn't reduce, since $x^2 + 1$ doesn't factor.

Example 7.6.3 Rewrite $\frac{1 + \frac{1}{x+1}}{1 + \frac{1}{x-1}}$ as a reduced fraction.

Solution: Multiply top and bottom by the LCD of the fractions $\frac{1}{x+1}$ and $\frac{1}{x-1}$. In this case the LCD is the product $(x+1)(x-1)$. Then

$$\begin{aligned} & \frac{(x+1)(x-1)\left(1 + \frac{1}{x+1}\right)}{(x+1)(x-1)\left(1 + \frac{1}{x-1}\right)} \\ &= \frac{(x+1)(x-1)(1) + (x+1)(x-1)\left(\frac{1}{x+1}\right)}{(x+1)(x-1)(1) + (x+1)(x-1)\left(\frac{1}{x-1}\right)} \\ & \frac{(x+1)(x-1)(1) + (x-1)}{(x+1)(x-1)(1) + (x+1)} \text{ Don't multiply out! Factor the top and bottom to see if you can cancel.} \\ &= \frac{(x-1)[(x+1) + 1]}{(x+1)[(x-1) + 1]} \\ &= \frac{(x-1)(x+2)}{(x+1)x} \end{aligned}$$

Exercise 7.6.1 Rewrite each nested fraction as a fraction in standard form.

| | |
|----|---|
| a) | $\frac{\frac{1}{x} + \frac{2}{y}}{\frac{2}{x} + \frac{1}{y}}$ |
| b) | $\frac{\frac{1}{2} + \frac{2}{3}}{\frac{1}{3} + \frac{1}{4}}$ |
| c) | $\frac{\frac{1}{x+2} + \frac{2}{x-2}}{\frac{2}{x+2} + \frac{1}{x-2}}$ |
| d) | $\frac{\frac{1}{a} + \frac{2}{b}}{\frac{3}{c} + \frac{4}{d}}$ |
| e) | $\frac{\frac{1}{x^2-1} + \frac{1}{x-1}}{\frac{1}{x^2-1} + \frac{1}{x+1}}$ |
| f) | $\frac{\frac{1}{x+3} + \frac{1}{2x-6}}{\frac{1}{x^2-9}}$ |
| g) | $\frac{\frac{1}{a} + ab^2}{\frac{1}{b} + ba^2}$ |