Introductory description of **The Dynamics of Topologically Generic Homeomorphisms** 2003 Memoirs AMS **783**

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The world is no longer flat. Once, we hoped to smooth and straighten at least our local problems by blowing up the scale until the picture looked approximately linear. In dynamics, this hope has proved vain with the discovery of the fractal quality of many natural objects, their complexities reproduced with every magnification. Wild sets have escaped the Wisconsin Zoo of Topological Pathologies and indecomposable continua roam the earth.

Meanwhile, these geometric complexities have been matched by the dynamic instabilities which have acquired the label chaos. Unlike periodic or quasi-periodic motion the recurrent behavior on a hyperbolic subset is sufficiently delicate to call into question the meaning of predictability even in rather simple appearing deterministic systems.

In this paper, we extend earlier work by Kennedy and Hurley to give a fairly complete picture of the dynamic behavior of a generic homeomorphism on a compact, piecewise linear manifold (with no boundary). If X is a compact metric space then $H(X)$, the group of homeomorphisms on X, is a completely metrizable space with respect to the topology of uniform convergence. We describe a G_δ subset $\hat{H}_{\text{res}}(X)$ of $H(X)$ and show that it is dense when X is a compact piecewise linear manifold (and a fortiori when X is a compact smooth manifold) of dimension at least two. A homeomorphism f in this residual set satisfies a number of peculiar properties.

(1) If A is an attractor for f then A contains infinitely many repellors for f. In fact, the interior of A, never empty, is exactly the union of the basins of repulsion for the repellors contained in A. The topological boundary of A, ∂A, is a quasi-attractor, that is, it is the intersection of a sequence of attractors, but is not itself an attractor (as ∂A has empty interior). The reverse is true for repellors. So there are uncountably many distinct chains $A_1 \supset R_1 \supset A_2 \supset R_2 ...$ with $A_1 = A$, the A_i's attractors and the R_i's repellors.

(2) Conley's chain recurrent set, which in general contains the set of nonwandering points, is for $f \in \hat{H}_{\text{res}}$ equal to the nonwandering set. It is a Cantor space and in it the set of periodic points is a dense subset of first category, i.e. its complement in the chain recurrent set is residual.

(3) For a homeomorphism a basic set B is, in general, a maximal subset such that any two points of B can be joined by ϵ chains for any positive ϵ. The basic sets form a closed decomposition of the chain recurrent set. A basic set is called terminal if it is a quasi-attractor. For f in $\hat{H}_{\text{res}}(X)$ there is a dense set of points in the chain recurrent set which are contained in basic

sets on which f restricts to a system with factor a subshift of finite type. On the other hand, the restriction of f to a terminal basic set is either a single periodic orbit or is conjugate to a so-called adding machine, a translation on a profinite group like the two-adic integers. The adding machine points in terminal basic sets form a residual subset of the chain recurrent set, disjoint from the subshift type points as well as the periodic points. Finally, the set of points whose omega limit set is a terminal adding machine basic set is a residual subset of X.

The infinity of tiny attractors studded with uncountably many basic sets of various types is exactly the sort of geometric complexity which was the object of our first paragraph's lament. It is hard to imagine what such maps look like. However, we can provide a one dimensional example which illustrates the conditions in (1), at least.

Let K be a Cantor set in $I = [0,1]$ with $0, 1 \in K$. Let $f : I \to I$ be a homeomorphism fixing exactly the points of K. For example, let L be a smooth real-valued map on I vanishing exactly on K and let f be the time one map of the flow for $dx/dt = L(x)$. So $I \backslash K$ consists of infinitely many disjoint, invariant, open intervals on each of which f moves either up or down, e.g. depending whether L was positive or negative on the interval. Assume that between any two up intervals there is a down interval and vice versa. Now identify the points 0 and 1 to get a homeomorphism of the circle. If we remove any up interval and any down interval the circle breaks into two closed intervals which are an attractor- repellor pair for f. A little contemplation verifies the conditions in (1). The chain recurrent set is the set of fixed points K and the basic sets are the individual points of K. We will later see that among orientation preserving maps of the circle with a fixed point this is the generic picture. However, because the dimension is one, we don't see the complex behavior on the individual basic sets described by the conditions in (3) above.

At this point we made a discovery which astonished us until it was interpreted for us by our elderly, imaginary, topologically inclined aunt: "Let your homeomorphisms be wild. It will make them stable."

One of the weaker interpretations of chaos, sensitive dependence on initial conditions, implies that no point is Lyapunov stable, i.e. is an equicontinuity point. There is a much stronger condition called chain continuity which is studied in [3]. A point is a chain continuity point exactly when its omega limit set is a terminal basic set on which the map is either a periodic orbit or an adding machine. Thus, we have:

(4) For $f \in \hat{H}_{\text{res}}(X)$ the set of chain continuity points is a residual subset of X and intersects the chain recurrent set in a residual subset of the latter.

Thus, the typical homeomorphism is geometrically complicated but very far from chaotic. In fact, most points satisfy a condition much stronger than Lyapunov stability.