Introduction to

Recurrence in Topological Dynamics: Furstenberg Families and Ellis Actions

1997 Plenum Press

Ethan Akin
Mathematics Department
The City College
137 Street and Convent Ave.
New York, N. Y. 10031

If $f : X \to X$ is a continuous map and $x \in X$, we say that y is a limit point for the associated dynamical system with initial value x, or just y is an ω limit point of x, when y is a limit point of the orbit sequence $\{f^n(x) : n \in T\}$ where T is the set of nonnegative integers. This means that the sequence enters every neighborhood of y infinitely often. That is, for any open set U containing y, the entrance time set $N(x, U) = \{n \in T : f^n(x) \in U\}$ is infinite.

It is often useful to keep track of just how frequently these entrance times occur. In his book *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Hillel Furstenberg used families of subsets of T to keep track of the frequencies. A family $\mathcal{F}$ for T is just a collection of subsets, i.e. a subset of the power set $\mathcal{P}$ of T, which is hereditary upwards. That is, if $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ then $F_2 \in \mathcal{F}$. A family is proper if it is a proper subset of $\mathcal{P}$, i.e. $\mathcal{F} \neq \emptyset, \mathcal{P}$. In view of heredity this says that $\mathcal{F}$ is proper when $T \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. For a proper family $\mathcal{F}$ we say that y is an $\mathcal{F}$ ω limit point of x if $N(x, U) \in \mathcal{F}$ for all neighborhoods U of y. In Chapter 2 we present the elementary theory–implicit in Furstenberg's work–for such families and in Chapters 3 and 4 apply it to dynamical systems, in general, and to topologically transitive systems, in particular.

This family approach goes back at least to Gottschalk and Hedlund (1955) who in their Chapter 3 introduced *admissible subset* collections, in order to unify several notions of recurrence, exactly our purpose.

When the state space X is compact, the semigroup theory of Robert Ellis can be applied. For our purposes this is best described here as an extension of the action of T on X to an action of βT, the Stone-Čech compactification of T, on X. For $x \in X$ we can define the orbit map $\varphi_x : T \to X$ by $\varphi_x(n) = f^n(x)$. As T is discrete this is a continuous map which therefore extends to a map $\Phi_x : \beta T \to X$. For $p \in \beta T$ we write $p(x) = \Phi_x(p)$, thus regarding the elements of βT as functions on X. Ellis observed that βT has a natural semigroup structure satisfying $(pq)(x) = p(q(x))$. However, this hybrid between the algebra and the topology on βT appears at first to be rather mulish. Right translation is continuous but left translation is not, i.e. $p \mapsto pq$ is continuous, $q \mapsto pq$ is not. Similarly, $p \mapsto p(x)$ is continuous but p itself, $x \mapsto p(x)$, is usually not. Perhaps unsurprisingly the composition in βT, while it extends addition on the dense subset T, is not commutative. Despite these infelicities the semigroup structure on βT has proved very fruitful in the study of dynamical systems.

The reader should be aware that Ellis uses right rather than left actions and so his semigroup structure on βT is the reverse of mine. His left translations are continuous and his compatibility equation reads $(xq)p = x(qp)$. Thus, the view in Chapters 6 and 7 is the mirror image of the Ellis way.

We can regard the points of βT as ultrafilters on T. A filter is just a proper family which is closed under the operation of intersection. An ultrafilter is a maximal filter. Thus, the two approaches meet. The Furstenberg theory applies to ultrafilters as it does to all families, and some of the Ellis constructions are nicely expressed in a family way. Furthermore, the family approach is more general. Various family constructions do not preserve the filter property. On the other hand, the semigroup structure with its associative law and collections of idempotents reveal properties about certain special families which would not be otherwise apparent.

We now outline the contents of the chapters which follow.

1. Monoid Actions: Once you abandon compactness assumptions you discover that various dynamic notions like equicontinuity and chain recurrence are really uniform space notions. The very definitions require a uniform structure. The associated topological spaces, i.e. the completely regular spaces, appear in full generality when you take arbitrary products and subsets. Each dynamical system in this book is a *uniform action* of an abelian *uniform monoid* on a uniform space, written $\varphi : T \times X \to X$. The initial chapter presents the easy set-up work to make sense of these phases.

An abelian topological group has a unique translation invariant uniform structure obtained from the neighborhoods of the identity. An abelian uniform monoid is a submonoid T of an abelian topological group, i.e. T is closed under addition and zero is in T but inverses might not be. Also, T satisfies a mild technical condition, the *Interior Condition*, on the set of tails. For $t \in T$ the associated tail T_t is the image of T under translation by t, $T_t = \{s + t : s \in T\}$. Any discrete abelian monoid, e.g. the nonnegative integers, $\mathbf{Z}_+$, and any abelian topological group, e.g. $\mathbf{R}$, is uniform monoid. The nonnegative reals, $\mathbf{R}_+$ under addition is also uniform.

N. B. After Chapter 1 all monoids are assumed abelian unless otherwise mentioned. All topologies are assumed Hausdorff.

An action is a function $\varphi : T \times X \to X$ such that the time t maps defined by $f^t(x) = \varphi(t, x)$ satisfy the composition property $f^{t_1} \circ f^{t_2} = f^{t_1+t_2}$. For φ to be a uniform action we assume, first, that each $f^t : X \to X$ is a uniformly continuous map of the uniform space X. Hence, the adjoint associate $\varphi^{\#}$ of φ which associates $t \mapsto f^t$ is a homomorphism from T to $\mathcal{C}^u(X;X)$ the space of uniformly continuous maps on X. The second condition of a uniform action is that the homomorphism $\varphi^{\#}$ is continuous, and hence is uniformly continuous, when $\mathcal{C}^u(X;X)$ is given the uniformity of uniform convergence. Thus, φ is a uniform action of each f^t is uc and $t_i \to 0$ in T implies that $f^{t_i} \to 1_X$ uniformly on X. It then follows that φ is a continuous map (though not uniformly continuous), i.e. φ is a topological action. When T is discrete the second condition is trivial. In particular, a uniform action of $T = \mathbf{Z}_+$ is just given by the iterates of a uc map $f = f^1$ on X. If T is uniform and X is compact then any topological action $\varphi : T \times X \to X$ is uniform. Recall that a compact space has a unique uniformity consisting of all neighborhoods of the diagonal.

The use of monoids allows us to apply the theory to semiflows and noninvertible maps. More importantly, even for a homeomorphism f on a compact space X it is useful to distinguish between the $\mathbf{Z}_+$ action using f, the reverse action which is the $\mathbf{Z}_+$ action using f^{-1}, and the extended $\mathbf{Z}$ action which includes them both. The limit point set for x associated with these are, respectively, the ω limit set, the α limit set and the closure of the entire orbit of x. For a monoid we move out towards infinity using the tails, T_t. If T is a group then $T_t = T$ for all t.

Finally, while our theory is motivated by the cases $T = \mathbf{Z}_+, \mathbf{R}_+, \mathbf{Z}$ and $\mathbf{R}$ there is an least one other example worth mentioning here, namely $T = \mathbf{Z}^*$, the positive integers under multiplication (discrete uniformity). If X is a compact topological group, eg. the unit circle in $\mathbf{C}$, then $\mathbf{Z}^*$ acts on X via exponentiation. That the $\mathbf{Z}^*$ action on the circle is strongly mixing will prove useful, once the definitions are in place to make sense of the statement.

2. Furstenberg Families: For a uniform monoid T a *family* $\mathcal{F}$ is a subset of $\mathcal{P}$, the power set of T, which is hereditary upwards. $\mathcal{F}$ is a proper family when $\emptyset \not\in \mathcal{F}$ and $T \in \mathcal{F}$. The *dual* $k\mathcal{F}$ is $\{F : F$ meets F_1 for every $F_1 \in \mathcal{F}\}$, or, equivalently, $F \in k\mathcal{F}$ iff $T \setminus F \not\in \mathcal{F}$. For any family $\mathcal{F}$, $k\mathcal{F}$ is a family and $kk\mathcal{F} = \mathcal{F}$. Clearly, $k\mathcal{P} = \emptyset$ and so $k\mathcal{F}$ is proper iff $\mathcal{F}$ is. The largest proper family is $\mathcal{P}_+ = \mathcal{P} \setminus \{\emptyset\}$ whose dual, $k\mathcal{P}_+$, is $\{T\}$.

A *filter* is a proper family which is closed under intersection. A *filterdual* is a family whose dual is a filter. $\mathcal{F}$ is a filterdual iff it satisfies what Furstenberg calls the *Ramsey Property:* $F_1 \cup F_2 \in \mathcal{F} \Rightarrow F_1 \in \mathcal{F}$ or $F_2 \in \mathcal{F}$. An *ultrafilter* is a maximal filter, or, equivalently, a selfdual filter.

Using the action of T on itself by translation we define $g^t : T \to T$ the translation map by t whose image is the tail T_t. Call a family $\mathcal{F}$ *translation invariant* if for all $t \in T$, $F \in \mathcal{F}$ iff $g^{-t}(F) \in \mathcal{F}$ (where $g^{-t}(F)$ denotes the preimage $(g^t)^{-1}(F)$). $\mathcal{F}$ is *thick* if $F \in \mathcal{F}$ and $t_1, \dots, t_k \in T$ imply $\cap_{i=1}^k g^{-t_i}(F) \in \mathcal{F}$. For any family $\mathcal{F}$, $\gamma\mathcal{F}$ denotes the smallest translation invariant family containing $\mathcal{F}$ and $\tilde{\gamma}\mathcal{F}$ the largest translation invariant family contained in $\mathcal{F}$, so that $k\gamma\mathcal{F} = \tilde{\gamma}k\mathcal{F}$. Define $\tau\mathcal{F}$ to be the largest thick family contained in $\mathcal{F}$. Observe that a translation invariant filter is automatically thick.

The family $\tilde{\gamma}\mathcal{P}_+$, denoted $\mathcal{B}_T$, is the largest translation invariant proper family. $F \in \mathcal{B}_T$ iff $g^{-t}(F) \neq \emptyset$ for all $t \in T$. Its dual, $k\mathcal{B}_T = \gamma k\mathcal{P}_+$ is the family generated by the tails. $F \in k\mathcal{B}_T$ iff $g^{-t}(F) = T$ for some $t \in T$. $k\mathcal{B}_T$ is the smallest translation invariant proper family. It is a filter and so $\mathcal{B}_T$ is a filterdual. Notice that if T is a group then $\mathcal{B}_T = \mathcal{P}_+$ and $k\mathcal{B}_T = k\mathcal{P}_+ = \{T\}$.

In the case, $T = \mathbf{Z}_+$, $\mathcal{B}_T$ is the family of infinite subsets and the dual $k\mathcal{B}_T$ is the family of cofinite subsets. The family $\tau\mathcal{B}_T$ is called the family of thick sets of $\mathbf{Z}_+$, $F \in \tau\mathcal{B}_T$ if F has arbitrarily long runs, i.e. for every $N \in \mathbf{Z}_+$ there exists $t \in \mathbf{Z}_+$ such that, $t, t+1, \dots, t+N \in F$. The dual, $k\tau\mathcal{B}_T$ consists of the syndetic or relatively dense sets. $F \in k\tau\mathbf{B}_T$ iff there exists N such that every interval of length N meets F, i.e. for every $t \in \mathbf{Z}_+$, $\{t, t+1, \dots, t+N\} \cap F \neq \emptyset$. $\tau k\tau\mathcal{B}_T$ consists of what we will call *replete* sets (Furstenberg uses "replete" as a synonym for thick, a waste of a fine word). $F \in \tau k\tau\mathcal{B}_T$ if for every N the positions where length N runs begin form a syndetic set. All of these families are translation invariant and $\tau k\tau\mathcal{B}_T$ is a filter.

Translation invariant filters are quite useful. In general, if $\mathcal{F}$ is a filter then $\mathcal{F}$ is contained in some translation invariant filter iff $\mathcal{F} \subset \tau\mathcal{B}_T$.

Using the uniform structure on T, we call $\mathcal{F}$ an *open family* if every $F \in \mathcal{F}$ is a uniform neighborhood of some other element of $\mathcal{F}$, i.e. there exists $F_1 \in \mathcal{F}$ and V in the uniformity $\mathcal{U}_T$ such that $F \supset V(F_1)$. One of the purposes of the Interior Condition on a uniform monoid is to ensure that the filter $k\mathcal{B}_T$ generated by the tails is an open family. Of course, if T is discrete and so the diagonal $1_T \in \mathcal{U}_T$ then every family is open.

3. Recurrence: For $\varphi : T \times X \to X$ a uniform action and subsets A, B of X we define the *meeting time set* $N(A, B) = \{t \in T : f^t(A) \cap B \neq \emptyset\}$.

If we identify each map f^t with its graph we can define the relation f^F for $F \subset T$ to be $\cup\{f^t : t \in F\}$. Thus, F meets $N(A,B)$ iff $f^t(A) \cap B \neq \emptyset$ for some t in F and so iff $f^F(A) \cap B \neq \emptyset$.

Define $\mathcal{N}(A, u[B])$ to be the family of subsets of T generated by all $N(A,U)$ where U is a uniform neighborhood of B, and $\mathcal{N}(u[A], u[B])$ the family generated by all $N(W,U)$ where W and U are uniform neighborhoods of A and B, respectively. Each of these is, when proper, an open filter.

Given a family $\mathcal{F}$ and a nonempty subset A of X we define $\omega_{\mathcal{F}}\varphi[A] = \cap\{\overline{f^F(A)} : F \in k\mathcal{F}\}$. A point $y \in \omega_{\mathcal{F}}\varphi[A]$ iff $N(A,U) \in \mathcal{F}$ for every neighborhood U of y, i.e. iff $\mathcal{N}(A, u[y]) \subset \mathcal{F}$. In particular, we define the relation $\omega_{\mathcal{F}}\varphi \subset X \times X$ by $\omega_{\mathcal{F}}\varphi(x) = \omega_{\mathcal{F}}\varphi[x] = \cap\{\overline{f^F(x)} : F \in k\mathcal{F}\}$. When X is compact and $\mathcal{F}$ is a filterdual then $\omega_{\mathcal{F}}\varphi[A]$ is nonempty and if U is an open set containing $\omega_{\mathcal{F}}\varphi[A]$ then U contains $\overline{f^F(A)}$ for some $F \in k\mathcal{F}$.

We define the closed relation $\Omega_{\mathcal{F}}\varphi$ to be $\cap\{\overline{f^F} : F \in k\mathcal{F}\}$ taking the closure in $X \times X$. Two points x, y satisfy $y \in \Omega_{\mathcal{F}}\varphi(x)$ iff $N(W,U) \in \mathcal{F}$ for every neighborhood W and U of x and y respectively, i.e. iff $\mathcal{N}(u[x], u[y]) \subset \mathcal{F}$. Equivalently, $\Omega_{\mathcal{F}}\varphi(x) = \cap \omega_{\mathcal{F}}\varphi[W]$, intersecting over all neighborhoods W of x.

Finally, we say that x $\mathcal{F}$ *adheres* to a set B if $\mathcal{N}(x, u[B]) \subset \mathcal{F}$, i.e. $N(x,U) \in \mathcal{F}$ for every uniform neighborhood U of B. If B is compact, then the sufficient condition $\overline{f^F(x)} \cap B \neq \emptyset$ for all $F \in k\mathcal{F}$ is necessary as well. So if B is compact and $\mathcal{F}$ is a filterdual then x $\mathcal{F}$ adheres to B iff $\omega_{\mathcal{F}}\varphi(x) \cap B \neq \emptyset$. For any family $\mathcal{F}$, $\omega_{\mathcal{F}}\varphi(x) = \{y : x\mathcal{F} \text{ adheres to } y\}$. In particular, $x \in \omega_{\mathcal{F}}\varphi(x)$ iff x $\mathcal{F}$ adheres to x. We call such a point $\mathcal{F}$ *recurrent*.

The usual notions of $\omega\varphi$ and $\Omega\varphi$ (cf. Akin (1993)) correspond to $\mathcal{F} = \mathcal{B}_T$ and so we will drop the subscript in that case.

To describe the meaning of these concepts in certain important cases recall that an action φ is called *minimal* if for B a nonempty, closed subset of X, $f^t(B) \subset B$ for all $t \in T$ (i.e. B is + invariant) implies $B = X$. A closed, nonempty, + invariant subset is called a *minimal subset* if the restriction of the action to B is a minimal action. Every compact, nonempty, + invariant subset of X contains a minimal subset. The closure of the union of all minimal subsets of X is called the *mincenter* of X.

We call x a *fixed point* for φ if $f^t(x) = x$ for all $t \in T$, or, equivalently, if $\{x\}$ is a + invariant, and hence, minimal subset.

Now assume that $\varphi : T \times X \to X$ is a uniform action with X compact. Let $x \in X$ and B be a closed subset of X. If $\mathcal{F}$ is a filterdual then x $\mathcal{F}$

adheres to B iff $\omega_{\mathcal{F}}\varphi(x)$ meets B and x $k\mathcal{F}$ adheres to B iff $\omega_{\mathcal{F}}\varphi(x) \subset B$. In particular, x $\mathcal{B}_T$ adheres to B iff $\omega\varphi(x) \cap B \neq \emptyset$ and x $k\mathcal{B}_T$ adheres to B iff $\omega\varphi(x) \subset B$. $k\tau k\tau\mathcal{B}_T$ is a filterdual and $\omega_{k\tau k\tau\mathcal{B}_T}\varphi(x)$ is the mincenter of $\omega\varphi(x)$. So x $k\tau k\tau\mathcal{B}_T$ adheres to B iff B meets the mincenter of $\omega\varphi(x)$ and x $\tau k\tau\mathcal{B}_T$ adheres to B iff B contains the mincenter of $\omega\varphi(x)$.

Furthermore, under this compactness hypothesis, x $k\tau\mathcal{B}_T$ adheres to B iff B meets every minimal subset of $\omega\varphi(x)$, while x $\tau\mathcal{B}_T$ adheres to B iff B contains some minimal subset of $\omega\varphi(x)$. In particular, x is $k\tau\mathcal{B}_T$ recurrent iff x is contained in some minimal subset of X (which is then, necessarily, $\omega\varphi(x)$) and x is $k\tau k\tau\mathcal{B}_T$ recurrent iff the minimal subsets of $\omega\varphi(x)$ are dense in $\omega\varphi(x)$.

4. Transitive and Central Systems: A uniform action $\varphi : T \times X \to X$ is called $\mathcal{F}$ *central*, for a family $\mathcal{F}$, if $1_X \subset \Omega_{\mathcal{F}}\varphi$, i.e. $\mathcal{N}(u[x], u[x]) \subset \mathcal{F}$ for all $x \in X$. This means that for every nonempty open set U, the return time set $N(U, U)$ is in $\mathcal{F}$. The action is called $\mathcal{F}$ *transitive* if $X \times X = \Omega_{\mathcal{F}}\varphi$, i.e. $\mathcal{N}(u[x], u[y]) \subset \mathcal{F}$ for all $x, y \in X$. This means that for every pair U, W of nonempty open sets $N(W, U) \in \mathcal{F}$. φ is called *central* (or *transitive* when it is $\mathcal{B}_T$ central (resp. $\mathcal{B}_T$ transitive). When φ is central it is a dense action, that is, $f^t(X)$ is a dense subset of X for every $t \in T$. In the compact case this means, of course, that $f^t(X) = X$ for all t.

Suppose X is a complete, separable metric space, e.g. compact metric space, and that φ is a uniform action, with T separable as well. If φ is central then the set of recurrent points, $|\omega\varphi| = \{x : x \in \omega\varphi(x)\}$, is a residual subset of X. If φ is transitive then the set of transitive points, $\mathrm{Trans}_{\varphi} = \{x : \omega\varphi(x) = X\}$, is a residual subset of X. For a central, uniform action φ with a separable T on any compact space X the set of recurrent points is always dense, but the analogous result for transitivity is false. If T is separable and the continuous image $f^T(x)$ is dense in X then X is separable and so has cardinality at most 2^c, that of $\beta\mathbf{Z}_+$. However, transitive actions occur on spaces of arbitrarily large cardinality.

The action φ is called *weak mixing* when the product action $\varphi \times \varphi$ on $X \times X$ is transitive. Furstenberg's beautiful *Intersection Lemma* yields that a uniform action φ is weak mixing exactly when it is $\tau\mathcal{B}_T$ transitive. For a translation invariant family $\mathcal{F}$ we call φ $\mathcal{F}$ mixing when it satisfies the following equivalent conditions: (1) $\varphi \times \varphi$ is $\mathcal{F}$ transitive, (2) φ is $\tau\mathcal{F}$ transitive, (3) φ is $\mathcal{F}$ transitive and weak mixing, (4) φ is $\mathcal{F}$ central and weak mixing,

(5) φ is $\mathcal{F}_1$ transitive for some translation invariant filter $\mathcal{F}_1 \subset \mathcal{F}$. It follows that if φ is an $\mathcal{F}$ mixing action on X then the product action induced on an arbitrary product X^I is also $\mathcal{F}$ mixing, and so is, a fortiori, transitive.

We call φ *strong mixing* when it is $k\mathcal{B}_T$ transitive ($= k\mathcal{B}_T$ mixing as $k\mathcal{B}_T$ is a filter), topologically ergodic or just *ergodic* when it is $k\tau\mathcal{B}_T$ transitive and *ergodic mixing* when it is both ergodic and weak mixing, which is equivalent to $\tau k\tau\mathcal{B}_T$ mixing. Many transitive systems are in fact ergodic and the gap between the two notions consists of peculiar systems.

A uniform action $\tilde{\varphi} : T \times \tilde{X} \to \tilde{X}$ is called an *eversion* if $\tilde{X}$ is compact, the action is surjective ($\tilde{f}^t(\tilde{X}) = \tilde{X}$ for all t) and there is a fixed point $e \in \tilde{X}$ such that for every neighborhood U of e, the times $\{t : f^t(X \backslash U) \subset U\} \in \mathcal{B}_T$. If $\varphi : T \times X \to X$ is a surjective uniform action with X compact and $x \in X$ such that $x \notin \Omega_{k\tau\mathcal{B}_T}\varphi(x)$ (and so φ is not $k\tau\mathcal{B}_T$ central) then there is an eversion $\tilde{\varphi}$ with fixed point e and a continuous map h from X onto $\tilde{X}$ relating the actions such that $x \notin h^{-1}(e)$. It follows that a transitive but nonergodic system on a compact space has a transitive but nontrivial eversion as a factor.

One application of this machinery is an extension of a theorem of Kronecker: Let $\varphi : T \times X \to X$ be a uniform action with X compact metric and T separable. If φ is weak mixing then there exists a Cantor subset A of X, i.e. a compact, perfect, zero-dimensional subset, such that $\{f^t|A : t \in T\}$ is dense in $\mathcal{C}(A; X)$. Thus, every continuous map from A into X can be uniformly approximated by the special maps $f^t|A : A \to X$. The existence of so-called Kronecker subsets of the circle arise from the application of this result to the strongly mixing action of $T = \mathbf{Z}^*$ on the circle by $(n, z) \mapsto z^n$.

5. Compactifications: For a uniform space X let $\mathcal{B}(X)$ denote the Banach algebra of bounded, real-valued continuous functions on X with the sup norm. Let $\mathcal{B}^u(X)$ denote the closed subalgebra of uniformly continuous functions in $\mathcal{B}(X)$.

For any closed subalgebra E of $\mathcal{B}(X)$ let j_E denote the map from X to the dual space E^* associating to $x \in X$ evaluation at x, i.e. $j_E(x)(u) = u(x)$ for $u \in E$. The map $j_E : X \to E^*$ is continuous when E^* is given the weak* topology. Let X_E denote the closure of the image $j_E(X)$. The set X_E consists of all algebra maps from E to $\mathbf{R}$ and X_E is compact with the topology induced from E^*. Furthermore, the induced map $j_E^* : \mathcal{B}(X_E) \to \mathcal{B}(X)$ is a B algebra isometry with image E. This Gelfand theory classifies the compactifications of X, i.e. continuous maps from X to a compact space, via the closed subalgebras of $\mathcal{B}(X)$. Associated with $\mathcal{B}(X)$ is the Stone-Čech compactification βX and with $\mathcal{B}^u(X)$ is the uniform Stone-Čech compactification denoted $\beta_u(X)$. The latter is especially important for our purposes. The map $j_u : X \to \beta_u(X)$ $(j_u \equiv j_{\mathcal{B}^u(X)})$ is uniformly continuous and is a topological *embedding*, i.e. a homeomorphism of X onto its image in $\beta_u(X)$. It is, however, a uniform isomorphism onto its image only when X is totally bounded. The points of $\beta_u X$ can be identified with the maximal open filters on X.

The space X_E is metrizable iff the algebra E is separable. If X is a separable metric space with bounded metric d then a special metrizable compactification, the *Gromov compactification*, is obtained by using the algebra E_d generated by the functions $\{d(x) : x \in X\}$ where $d(x)(x_1) = d(x, x_1)$.

If φ acts uniformly on X and E is a closed subalgebra of $\mathcal{B}(X)$ then each $f^t : X \to X$ factors through j_E exactly when the algebra maps $f^{t*} : \mathcal{B}(X) \to \mathcal{B}(X)$ all preserve E, i.e. when E is φ invariant. If, in addition, $E \subset \mathcal{B}^u(X)$ then φ extends to a uniform action $\varphi_E : T \times X_E \to X_E$ such that $j_E : X \to X_E$ maps the action φ to φ_E.

If φ is $\mathcal{F}$ central (or $\mathcal{F}$ transitive) and $E \subset \mathcal{B}^u(X)$ is φ invariant then the compactified flow φ_E on X_E is $\mathcal{F}$ central (resp. $\mathcal{F}$ transitive). Conversely, if φ_E is $\mathcal{F}$ central (or $\mathcal{F}$ transitive) for any compactification with $j_E : X \to X_E$ an embedding or, when T is separable, for all compactifications with E separable (and hence with X_E metrizable) then φ is $\mathcal{F}$ central (resp. $\mathcal{F}$ transitive).

6. Ellis Semigroups: Addition, $T \times T \to T$, can be regarded as the translation action of T on itself. For a uniform monoid this translation action is uniform and so extends to a uniform action $T \times \beta_u T \to \beta_u T$ where $\beta_u T$ is the uniform Stone-Čech compactification. Fixing the second coordinate yields a uniformly continuous map from T to $\beta_u T$ which extends to $\beta_u T$. The result is an associative composition $\beta_u T \times \beta_u T \to \beta_u T$, $(p, q) \to pq$. However, only the right translation $i_q(p) = pq$ is continuous on $\beta_u T$.

For a uniform action $T \times X \to X$ with X compact, fixing the second coordinate yields a uniformly continuous map of T to X which extends to $\beta_u T$. The result is an action of the semigroup $\beta_u T$ on the space X but by not necessarily continuous maps. $\Phi_x : \beta_u T \to X$ defined by $\Phi_x(p) = px$ is continuous for each x and $(pq)(x) = p(qx)$, extending the associative law on

$\beta_u T$.

Starting with this example, Ellis studied compact semigroups. An *Ellis semigroup* S is a, usually nonabelian, semigroup with a compact topology such that each right translation is continuous on S. For any compact space X, the function space X^X is an Ellis semigroup under composition of maps with the product topology. An *Ellis action* of an Ellis semigroup S on a compact space X is a, usually not continuous, function $\varphi : S \times X \to X$ such that the adjoint associate $\varphi^{\#}$ is a continuous semigroup homomorphism from S to X^X. Its image, denoted S_φ, is an Ellis subsemigroup of X^X called the *enveloping semigroup* of φ.

An element e of S is called *idempotent* when $e^2 = e$. Namakura's Lemma says that any compact semigroup contains idempotents. For example, if φ is an Ellis action on X and $x \in X$ then $\mathrm{Iso}_x = \{p \in S : px = x\}$ is a closed subsemigroup if it is nonempty. It then follows that idempotents fixing x exist. A point x is called S *recurrent* if the isotropy set Iso_x is nonempty.

7. Semigroups and Families: In $\beta_u T$ the semigroup structure can be described using family ideas. For $p \in \beta_u T$ we pull back by the embedding $j_u : T \to \beta_u T$ the filter of neighborhoods of p. We obtain $\mathcal{F}_p$ a maximal open filter of subsets of T. The point px is the point in the singleton set $\omega_{\mathcal{F}_p}\varphi(x) = \omega_{k\mathcal{F}_p}\varphi(x)$.

In general, for any open filter $\mathcal{F}$ of subsets of T, the hull $H(\mathcal{F}) = \{p \in \beta_u T : \mathcal{F} \subset \mathcal{F}_p\}$ is a closed subset of $\beta_u T$ and the compact subset $\{px : p \in H(\mathcal{F})\}$ is $\omega_{k\mathcal{F}}\varphi(x)$.

Recall that a filter $\mathcal{F}$ is thick if $F \in \mathcal{F}$ and $t \in T$ imply $g^{-t}(F) \in \mathcal{F}$. Define $F \in \mathcal{F}$ to be $\mathcal{F}$ *semiadditive* for a filter $\mathcal{F}$ if $t \in F$ implies $g^{-t}(F) \in \mathcal{F}$. A filter $\mathcal{F}$ is called semiadditive if it is generated by $\mathcal{F}$ semiadditive sets, i.e. $F \in \mathcal{F}$ implies $F \supset F_1$ with $F_1 \in \mathcal{F}$ an $\mathcal{F}$ semiadditive set.

For any open filter $\mathcal{F}$, let H be the hull of $\mathcal{F}$. $\mathcal{F}$ is semiadditive iff $\omega_{k\mathcal{F}}\mu[H] \subset H$ where μ is the translation action of T on $\beta_u T$. Notice that this implies $\omega_{k\mathcal{F}}\mu(H) = \{pq : p, q \in H\}$ is contained in H and so H is a closed subsemigroup. $\omega_{k\mathcal{F}}\mu(H) = \omega_{k\mathcal{F}}\mu[H]$ in the particular case where H is a singleton. As a corollary we see that the maximal open filter $\mathcal{F}_p$ is semiadditive iff p is an idempotent.

Assume the hull $H(\mathcal{F})$ is a subsemigroup, e.g. $\mathcal{F}$ is semiadditive or translation invariant. If $\varphi : T \times X \to X$ is a uniform action with X compact and $x \in X$, then x is recurrent for the filterdual $k\mathcal{F}$, i.e. $x \in \omega_{k\mathcal{F}}\varphi(x)$, iff x is

$H(\mathcal{F})$ recurrent, i.e. the isotropy set $\mathrm{Iso}_x \subset \beta_u T$ meets $H(\mathcal{F})$. There then exists an idempotent e such that $\mathcal{F} \subset \mathcal{F}_e \subset k\mathcal{F}$.

The family $\cup\{\mathcal{F}_e : e \in \beta_u T \setminus T \text{ and } e^2 = e\}$ is the filterdual generated by the so-called IP sets.

8. Equicontinuity: For a uniform action $\varphi : T \times X \to X$ Lyapunov stability of a point $x \in X$, or of its orbit, is equicontinuity at x of the family of functions $\{f^t : t \in T\}$. To get the version associated with a family $\mathcal{F}$, we define for $V \in \mathcal{U}_X$ and $F \subset T$, $V_\varphi^F = \cap_{t \in F}(f^t \times f^t)^{-1}(V)$, and $Eq_{V,\varphi}^F = \{x \in X : (x,x) \in \text{Int } V_\varphi^F\}$. Then let $Eq_{V,\varphi}^{\mathcal{F}} = \cup_{F \in \mathcal{F}} Eq_{V,\varphi}^F$ and $Eq_\varphi^{\mathcal{F}} = \cap_{V \in \mathcal{U}_X} Eq_{V,\varphi}^{\mathcal{F}}$.

We call x an $\mathcal{F}, V$ *equicontinuity point* if x is in the open set $Eq_{V,\varphi}^{\mathcal{F}}$ and an $\mathcal{F}$ *equicontinuity point* if it is in $Eq_\varphi^{\mathcal{F}}$, i.e. if it is an $\mathcal{F}, V$ equicontinuity point for all V in $\mathcal{U}_X$. Thus, $x \in Eq_\varphi^{\mathcal{F}}$ if for every $V \in \mathcal{U}_X$ there is a neighborhood U of x and $F \in \mathcal{F}$ such that $(f^t(x_1), f^t(x_2)) \in V$ for all $(t, x_1, x_2) \in F \times U \times U$.

The action is called $\mathcal{F}$ *equicontinuous* if $X = Eq_\varphi^{\mathcal{F}}$, i.e. $X = Eq_{V,\varphi}^{\mathcal{F}}$ for all $V \in \mathcal{U}_X$. The action is $\mathcal{F}$ *almost equicontinuous* if for each $V \in \mathcal{U}_X$ the open set $Eq_{V,\varphi}^{\mathcal{F}}$ is dense in X. So for an $\mathcal{F}$ almost equicontinuous action on a complete, metrizable space X the set $Eq_\varphi^{\mathcal{F}}$ of $\mathcal{F}$ equicontinuity points is a dense G_δ. For the case $\mathcal{F} = k\mathcal{B}_T$, the filter generated by the tails, we drop the superscript $\mathcal{F}$ and refer simply to equicontinuity point, almost equicontinuity etc.

For a translation invariant family $\mathcal{F}$, if φ is $k\mathcal{F}$ transitive and $\mathcal{F}$ almost equicontinuous then φ satisfies the a priori stronger condition of almost equicontinuity. In fact, for every $V \in \mathcal{U}_X$, $Eq_{V,\varphi}^T$ is open and dense and $Eq_\varphi^{\mathcal{F}} = Eq_\varphi = \mathrm{Trans}_\varphi = \{x : \omega\varphi(x) = X\}$.

If φ is an almost equicontinuous action on a compact space with $\mathrm{Trans}_\varphi \neq \emptyset$ but which is not minimal and so is not equicontinuous then φ is not topologically ergodic and so has as factor a topologically transitive, nontrivial eversion. The eversion factor can be chosen almost equicontinuous as well. Such peculiar actions do, in fact, exist.