Math 37500, Final Exam

Dec. 20th, 2017

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
Please show all work!

Name: ______________________________

DO NOT OPEN THE QUESTION BOOKLET UNTIL THE EXAM BEGINS

Official use only:

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	4	4	5	3	4	4	3	3	4	34
Score:										

1. Let $X_1, \ldots, X_{20}$ be independent Poisson random variables with $\lambda = 1$. In parts (a) and (b) below, find bounds on $\mathbb{P}\left(\sum_{j=1}^{20} X_j > 30\right)$ using:

   (a) (2 points) Markov's inequality.

   (b) (2 points) Chebyshev's inequality.

2. (4 points) Let X and Y be jointly continuous RV's with joint PDF

$$f_{X,Y}(x,y) = \frac{e^{-y}}{y}\ , \qquad 0 < x < y, \quad 0 < y < \infty\ .$$

   Find $\mathbb{E}[X \mid Y = y]$.

3. Let X be a random variable having the moment generating function

$$M_X(t) = \frac{1}{2} + \frac{1}{3}e^{-4t} + \frac{1}{6}e^{5t} .$$

(a) (3 points) Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ by differentiating the MGF.

(b) (2 points) What is the PMF of X?

4. (3 points) A team of three is chosen randomly from an office with 2 men and 4 women. Let X be the number of women on the team. Find the PMF of X.

5. (4 points) Suppose X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} xy\,, & 0 < x < 1\,,\ 0 < y < 2\,; \\ 0\,, & \text{otherwise.} \end{cases}$$

Find $\mathbb{P}(X + Y < 1)$.

6. (4 points) Lucy flips a fair coin 5 times every morning, for 30 days straight. Let X be the number of mornings over these 30 days on which all 5 flips are tails. Use the Poisson approximation to give an estimate for the probability $\mathbb{P}(X = 2)$.

7. (3 points) An urn contains one 6-sided die, two 8-sided dice, three 10-sided dice, and four 20-sided dice. One die is chosen at random from the urn and then rolled.

   What is the (conditional) probability that the die rolled was the 20-sided die, given that the outcome of the roll was seven?

8. (3 points) Let X_1, X_2, and X_3 be independent standard normal random variables. Let $Y_1 = X_1 + X_2$ and $Y_2 = 2X_2 + X_3$. Find $\mathrm{Cov}(Y_1, Y_2)$.

9. (4 points) Let U be a uniform $(0,1)$ random variable. Let

$$X = -\frac{1}{2}\ln(1-U)\ .$$

Find the PDF f_X of X.

FORMULA SHEET

Densities

- Normal (mean μ, variance σ^2): $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$
- Exponential (parameter λ):

$$f(x) = \begin{cases} \lambda \exp(-\lambda x), & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

  Mean $1/\lambda$, variance $1/\lambda^2$.

- Gamma (parameters α, λ):

$$f(x) = \begin{cases} \frac{\lambda(x\lambda)^{\alpha-1}\exp(-\lambda x)}{\Gamma(\alpha)}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

  (here, $\Gamma(a) = \int_0^\infty y^{a-1} e^{-y}\, \mathrm{d}y$). Mean α/λ, variance α/λ^2.

- Uniform (parameters $a < b$):

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \ , \\ 0, & \text{otherwise.} \end{cases}$$

  Mean $(a+b)/2$, variance $(b-a)^2/12$.

Mass Functions

- Poisson (parameter λ): $p(k) = 0$ except when $k = 0, 1, 2, \ldots$, where

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

  Mean and variance both equal λ.

- Binomial (parameters n, p): $p(k) = 0$ except for $k = 0, 1, \ldots, n$, where

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

  Mean np, variance $np(1-p)$.

- Geometric (parameter p): $p(k) = 0$ except when $k = 1, 2, \ldots$, where

$$p(k) = (1-p)^{k-1} p$$

  Mean $1/p$, variance $(1-p)/p^2$.

- Negative binomial (parameters r, p):

$$p(k) = \binom{i-1}{r-1} p^r (1-p)^{i-r} \ , \quad i = r,\, r+1,\, r+2, \ldots$$

Mean r/p, variance $r(1-p)/p^2$.

- Hypergeometric (parameters m, n, N with $m \le n$ and $n \le N$):

$$p(k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}} \ , \quad k = 0, 1, 2 \ldots$$

Mean nq, variance $(N-n)(nq)(1-q)/(N-1)$, where $q = m/N$.

Other formulas

- The binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- De Morgan's laws:

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c \ ; \qquad \left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$$

- Inclusion-exclusion:

$$\mathbb{P}\left(\cup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \ldots < i_r} \mathbb{P}\left(E_{i_1} \cap \ldots \cap E_{i_r}\right)$$

- Bayes' formula: if $F_1, \ldots, F_n$ are mutually exclusive events such that $F_1 \cup \ldots \cup F_n = \mathcal{S}$, then

$$\mathbb{P}(F_j \mid E) = \frac{\mathbb{P}(E \mid F_j)\mathbb{P}(F_j)}{\sum_{i=1}^{n} \mathbb{P}(E \mid F_i)\mathbb{P}(F_i)}$$