7.2 The Gauss and Weingarten maps

Our second way to define the curvature of an oriented surface is to consider its unit normal vector $\vec{N}$. The way that $\vec{N}$ varies reflects the way in which the surface curves: $\vec{N}$ varies rapidly near a point at which the surface is highly curved and slowly where the surface is slightly curved.

7.2 The Gauss and Weingarten maps

Describe if the following surfaces curve rapidly or slightly.

- EXAMPLE: The Euclidean plane $(x, y, 0)$ in $\mathbb{R}^3$.
- EXAMPLE: The circle $x^2 + y^2 + z^2 = a^2$ in $\mathbb{R}^3$.
- EXAMPLE: The cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

7.2 The Gauss and Weingarten maps

The Gauss map $\mathcal{G}$ is the map from the surface S to the unit sphere S^2 that assigns to any point p of S the point $\vec{N}_p$ in S^2. The rate at which $\vec{N}$ varies across S is measured by the derivative of $\mathcal{G}$:

$$D_p\mathcal{G} : T_pS \to T_{\mathcal{G}(p)}S^2.$$

The Weingarten map is defined as $\mathcal{W}_{p,S} = -D_p\mathcal{G}$. (the minus sign will make some future formulas simpler).

7.2 The Gauss and Weingarten maps

- EXAMPLE: Find the Weingarten map of the Euclidean plane $(x, y, 0)$ in $\mathbb{R}^3$.
- EXAMPLE: Find the Weingarten map of the circle $x^2 + y^2 + z^2 = a^2$ in $\mathbb{R}^3$.
- EXAMPLE: Find the Weingarten map of the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

7.2 The Gauss and Weingarten maps

Proposition 7.2.2 The second fundamental form is related to the Weingarten map as follows:

$$L du(\vec{v}) du(\vec{w}) + M du(\vec{v}) dv(\vec{w}) + M du(\vec{w}) dv(\vec{v}) + N dv(\vec{v}) dv(\vec{w}) = \langle\langle v, w\rangle\rangle$$

when $\langle\langle v, w\rangle\rangle = \langle \mathcal{W}_{p,S}(v), w\rangle$.

7.3 Normal and geodesic curvatures

It is obvious that the shape of a surface influences the curvature of curves on the surface. For example, a curve on the plane or on the cylinder can have zero curvature everywhere, but this is not possible for curves on a sphere since no segment of a straight line can lie on a sphere. Thus, a natural way to investigate how much a surface curves is to look at the curvature of curves on the surface.

7.3 Normal and geodesic curvatures

If $\gamma'(t)$ is a unit speed curve on S, then $\gamma'(t)$ is perpendicular to the normal $\vec{N}$ of S, so $\gamma', \vec{N}$, and $\vec{N} \times \gamma'(t)$ are mutually perpendicular unit vectors. Moreover, since γ is unit-speed, $\gamma''(t)$ is perpendicular to $\gamma'(t)$, and hence is a linear combination of $\vec{N}$ and $\vec{N} \times \gamma'$:

$$\gamma'' = \kappa_n \vec{N} + \kappa_g \vec{N} \times \gamma'.$$

The scalars κ_n and κ_g are called the normal curvature and the geodesic curvature respectively.

7.3 Normal and geodesic curvatures

- Proposition 7.3.2 : $\kappa_n = \gamma'' \cdot \vec{N} = \kappa \cos\phi, \kappa_g = \gamma'' \cdot (\vec{N} \times \gamma') = \pm\kappa \sin\phi, \kappa^2 = \kappa_n^2 + \kappa_g^2$ when ϕ is the angle between the normal to S $\vec{N}$ and the principal normal $\vec{n}$ of γ.
- Proposition 7.3.3 (most important): when $\gamma = \sigma(u(t), v(t))$

$$\kappa_n = Lu'^2 + 2Mu'v' + Nv'^2 = \langle\langle \gamma', \gamma' \rangle\rangle.$$

7.3 Normal and geodesic curvatures

Proposition 7.3.3 reveals that the normal curvature does not depend on the unit speed curve γ only on its direction γ' in T_pS. Understanding that the normal curvature only depends on direction, we can use Proposition 7.3.2 together with normal sections to approximate the normal curvature at a point $p \in S$.

- Use normal sections to find the maximum and minimum normal curvatures of the sphere $x^2 + y^2 + z^2 = a^2$ and then of the cylinder $x^2 + y^2 = a^2$.

9 Geodesics

Geodesics are the curves in a surface that a bug living in the surface would perceive to be straight. Curvature κ for curves in the plane as studied in section 2.2 is a special case of geodesic curvature for curves on a surface. When suitably parametrized, curves on a surface with zero geodesic curvature are called *geodesics*. They are the analogs of straight lines in the plane.

9 Geodesics

If we drive along a "straight" road we do not have to turn the wheel of our car to the right or left (this is what we mean by "straight"). However the road is not a straight line as the surface of the earth is nearly spherical. If the road is represented by a curve $\gamma(t)$, its acceleration, γ'', will be nonzero, but we perceive the curve as being straight because the *tangential* component of γ'' is zero, in other words γ'' is perpendicular to the surface—such curves are important to our studies and are called *geodesics*.

9 Geodesics

In modern geometry then we have a new concept of a line: A line is a curve γ on a surface S with zero tangential acceleration, which means it has no recognizable acceleration to being living on the surface. This is how we defined a line in the Euclidean plane after we learned calculus. Before we learned calculus, one way to identify a line between two points P and Q was to say that the line is the shortest path (= curve) between the points. It turns out that in the modern geometry we will study that these two ideas are the same: Shortest paths between points and zero tangential acceleration are the same.

9.1 Definition and basic properties

- Recall the decomposition of γ'' into parallel and perpendicular parts
$$\gamma'' = \kappa_n \vec{N} + \kappa_g \vec{N} \times \gamma'$$
when γ is on the surface S and is parametrized by arclength.
- EXAMPLE: lines $\gamma(x) = (x, mx + b)$ in the plane.
- EXAMPLE: $\gamma(t) = (\cos u_0 \cos t, \cos u_0 \sin t, \sin u_0)$ on the unit sphere.
- EXAMPLE: helix $\gamma(t) = (\cos t, \sin t, at + b)$ on the cylinder $x^2 + y^2 = 1$.

9.1 Definition and basic properties

- Proposition 9.1.2 : Any geodesic is constant speed.
- Proposition 9.1.3 A unit-speed curve on a surface is a geodesic if and only if its geodesic curvature is zero everywhere
- Proposition 9.1.4 : Any part of a straight line on a surface is a geodesic.
- Proposition 9.1.5 : Straight lines in Euclidean plane are geodesics.

9.1 Definition and basic properties

- Proposition 9.1.6 : Any normal section of a surface is a geodesic.
- This proposition can be used to show that the meridians on a surface of revolution are geodesics. In particular the great circles on a sphere are geodesics.

9.1 Definition and basic properties

Use the decomposition

$$\gamma'' = \kappa_n \vec{N} + \kappa_g \vec{N} \times \gamma'.$$

to show that the great circles of the sphere are geodesics.

9.2 Geodesic Equations

Theorem 9.2.1 : A curve γ on a surface S is a geodesic iff for any part $\gamma(t) = \sigma(u(t), v(t))$ of γ satisfies the following two second order nonlinear ODE:

- $u''E + \frac{1}{2}u'^2 E_u + u'v'E_v + v''F + (v)'^2 \left(F_v - \frac{1}{2}G_u\right) = 0.$
- $v''G + \frac{1}{2}v'^2 G_v + u'v'G_u + u''F + (u')^2 \left(F_u - \frac{1}{2}E_v\right) = 0.$

9.2 Geodesic Equations

Theorem 9.2.1 : (PROOF) these are the same differential equations as in our textbook but written in a slightly different form.

- We have $\gamma' = u'\sigma_u + v'\sigma_v$
- and then

  $$\gamma'' = u''\sigma_u + u'(u'\sigma_{uu} + v'\sigma_{uv}) + v''\sigma_v + v'(u'\sigma_{uv} + v'\sigma_{vv})$$

  by the chain rule.
- Now the two ODEs arrive by knowing that γ'' must be perpendicular to σ_u and σ_v, i.e. the ODEs are $\gamma'' \cdot \sigma_u = 0$ and $\gamma'' \cdot \sigma_v = 0$.

9.2 Geodesic Equations

The two geodesic equations, like most nonlinear differential equations, are usually difficult or impossible to solve explicitly. However the existence and uniqueness theorem proves there are many solutions of such systems of nonlinear ODE.

- Existence and Uniqueness Theorem for ODE: there is a unique geodesic through any given point of a surface in any given tangent direction.
- The proof of the existence theorem is theoretical. It belongs in an ODE course, but is usually avoided in an undergraduate ODE course.
- The geodesic equations allow us to study geodesics in a patch. We can then use σ to map these plane curves to the surface S.

9.2 Geodesic Equations

EXAMPLE: There is one case in which the geodesic equations are simple to solve: in our favorite surface the Euclidean plane the geodesics are lines. The geodesics in an arbitrary surface S are the analogs of straight lines in the plane. We are studying modern geomtery by following the implications of the geometry determined on the surface S when the "lines" are geodesics. We have already discussed the strange implications of spherical geometry when the "lines" are great circles.