Name ________________________________

1. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and
   (a) (4 points) Find the $QR$ factorization of $A$.
   (b) (3 points) Use the $QR$ factorization from $(a)$ to solve the least squares problem $Ax = b$.
   (c) (3 points) Verify that the residual, $r = b - Ax$, from part $(b)$ is orthogonal to the column space of $A$ by computing $A^T r$. Explain.

2. (12 points) (a) Prove that $x^3 - x - 1 = 0$ has a solution in the interval $[1, 2]$.
   (b) Find an FPI sequence $x_{k+1} = g(x_k)$ that converges to the solution $x^3 - x - 1 = 0$ in the interval $[1, 2]$. Verify that your FPI converges for any initial guess $x_0$ is in $[1, 2]$.
   (c) Do you believe your FPI in part $(b)$ will converge faster than the bisection method on $x^3 - x - 1$ on the interval $[1, 2]$? Explain.

3. (8 points) Give short answers to each of the following.
   (a) Roughly what is the overall cost in flops of decomposing a $60 \times 60$ matrix $A$ into $A = LU$.
   (b) State two disadvantages of computing the inverse of $A$ to solve $Ax = b$ rather than using the LU decomposition approach when $A$ is a square invertible matrix.
   (c) What is the fundamental difference between error and residual in terms of computability?
   (d) State one advantage and one disadvantage of using Newton's method to solve a scalar equation $f(x) = 0$.

4. (4 points) Show that nested evaluation of $p(2)$ when $p(x) = -3x^4 + 9x^3 + 4x^2 - 5x + 11$ requires 8 elementary operations.

5. (16 points) Consider the function $g(x) = x^2 + \frac{3}{16}$.
   (a) Find the fixed points of $g$.
   (b) For which of the fixed points from part $(a)$ an you be sure that FPI $x_{k+1} = g(x_k)$ will converge? Briefly justify your answer. You may assume that the initial guess is sufficiently close to the fixed point.
   (c) Find $x_2$ if $x_0 = 0$.
   (d) For the point you found in $(b)$ roughly how many iterations will be required to reduce the convergence error by 100?

6. (4 points) Let $A$ and $T$ be two nonsingular $n \times n$ matrices. Furthermore suppose we have the decomposition
$$TA = LU$$
   when $L$ is unit lower triangular and $U$ is upper triangular. Write an algorithm that will solve $Ax = b$.

7. (4 points) Write code to implement back substitution: Given an upper triangular matrix U and right hand side b, solve $Ux = b$ for $x$. Approximately how many flops are needed in your algorithm?

8. (8 points) Show in detail how to obtain the Trapezoid rule:
$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{2}\left(f(a) + f(b)\right)$$
   using the Lagrange interpolating polynomial.

9. (12 points) (a) Write the Lagrange interpolation polynomial $p_2(x)$ of degree 2 approximating the function $f(x) = x^4$ at the points $x = 0, x = 2$ and $x = 3$.
   (b) Find an upper bound for the error $|f(1) - p_2(1)|$ using the interpolation error formula:
$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\dots(x - x_n).$$
   Justify your bound.
   (c) Find the actual error $e = f(1) - p_2(1)$. Explain how the actual error compares with your bound from part $(b)$.

10. (10 points) (a) Prove that Newton's method applied to $f(x) = -2x + 7$ converges in one step for any initial guess $x_0$.
   (b) Apply two steps of Newton's method to approximate a solution to $x^3 + x - 2 = 0$ when $x_0 = 0$.

11. (10 points) (a) Solve the system
$$\begin{aligned} x + 2y &= 7 \\ 3x + 4y &= 9 \end{aligned}$$
   using $PA = LU$ factorization with partial pivoting and then carrying out the two-step back substituion. (No credit will be given for any other method, although partial credit will still be given).
   (b) Estimate the number of flops needed in part $(a)$. HINT: $PA = LU$ requires the same order of flops as $LU$ decomposition.

12. (10 points) (a) Develop a second order method to approximate $f'(x)$ that uses the data $f(x+h), f(x)$, and $f(x - 2h)$ only.
   (b) Use the three-point centered difference
$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$
   when $f(x) = \frac{1}{x}, x_0 = 1$ and $h = \frac{1}{2}$.
   (c) Find the order of the the three-point centered difference formula from part $(b)$. Justify your answer.

This part of the exam is to be completed with a computer after handing in the first part of the exam.

1. (9 points) Let $y(t)$ be the solution to the IVP $y' = t - y, y(0) = 1$.
   (a) Use the Euler method and step size $h = 0.05$ to an approximate value of $y(3.2)$. Round your answer to 4 digits after the decimal.
   (b) Use the Trapezoid method
$$\begin{aligned} Y &= y_i + hf(t_i, y_i) \\ y_{i+1} &= y_i + \frac{h}{2}\left(f(t_i, y_i) + f(t_{i+1}, Y)\right) \end{aligned}$$
   and step size $h = 0.2$ to approximate $y(3.2)$. Round your answer to 4 digits after the decimal.
   (c) What do you think will happen to the error in parts (a) and (b) if the step size is reduced to $h = 0.01$ for both parts? Explain.

2. (4 points) Find five successive approximations to $\sqrt{2}$ using the BABYLON FPI program, an algorithm realized by a FOR loop with a single line that reads

```
x=g(x)=(x + 2/x) / 2
```

(You should start with $x = 1$). Round to four decimal places after the decimal point in each approximation.

3. (8 points) Approximate the definite integral $\int_0^1 6x^3 \mathrm{d}x$ (round each answer to 4 decimal places after decimal point) using
   (a) the composite trapezoid method with 8 subintervals.
   (b) Simpson's rule with 4 subintervals.
   (c) the midpoint rule, $h\sum_{i=1}^{r} f(a + (i - \frac{1}{2})h)$, with 6 subintervals.
   (d) Compare approximations in $(a), (b)$, and $(c)$ with the exact answer.