Math 323 — **Midterm** — April 18, 2024

Please PRINT your name on the cover of your exam book(s), indicating if you are using more than one. Write clearly and cross-out work not to be graded. Make your arguments rigorous and brief. Partial credit will be given. Total: 100 pts.

1. Give **definitions** of ANY FOUR of the following five symbols or **boldface** words. (20 pts.)
   (a) The real-valued function f is **continuous at** $x_o \in dom(f)$.
   (You may use either sequences or $\epsilon - \delta$.)
   (b) The real-valued function f is **uniformly continuous** on $S \subset dom(f)$.
   (c) The **series** $\sum a_n$ **satisfies the Cauchy criterion**. (Do not presume any definitions associated with sequences.)
   (d) The series of reals $\sum_{n=1}^{\infty} a_n = S$, where $S \in \mathbb{R}$.
   Do NOT presume the definition of partial sum.
   (e) $\lim_{x\to a^+} f(x) = -\infty$, when $a \in \mathbb{R}$ and f is a real-valued function on $\mathbb{R}$.
   (You may use either sequences or "$\epsilon - \delta$".)

2. **Use the DEFINITIONS to prove** THREE of the following four statements: (30 pts.)
   (a) If (s_n) is sequence of reals such that $s_n > 0$ for all n, and $\lim(1/s_n) = +\infty$, then $\lim s_n = 0$.
   (b) Cauchy sequences are bounded.
   (c) $f(x) = \frac{x}{x+1}$ is uniformly continuous on $[0, 2]$.
   (d) If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then the sequence $(f(s_n))$ is a Cauchy sequence.

3. **Find** each of the following: (10 pts.)
   (a) $\liminf s_n$, $\limsup s_n$, and a convergent subsequence (s_{n_k}), if $s_n = \sin(n\pi/2)$.
   (b) $\liminf r_n$, $\limsup r_n$, and the set of subsequential limits, when r_n is an enumeration of all the rational numbers in the interval $(0, 1]$.
   (c) A function $f(x)$ that is continuous, but not uniformly continuous, on $[1, \infty)$;
   (d) a convergent series $\sum b_n$ such that $\sum b_n^2$ diverges;
   (e) $\lim t_n$, when $t_1 = 1$ and $t_{n+1} = (t_n^2 + 2)/2t_n$ for $n \geq 1$.

EXAM CONTINUES, please TURN OVER.

4. **Prove** ANY ONE of the following two: (10 pts.)

Theorem 1 *If f is a continuous function on $\mathbb{R}$ and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$, then there exists an x between a and b where $f(x) = 0$.*

(Hint: think Intermediate Value Theorem.)

Theorem 2 *If (s_n) is a bounded sequence and k is a non-negative real number, then $\limsup(ks_n) = k \limsup s_n$.*

5. **Prove** ANY TWO of the following three: (30 pts.)

Theorem 3 *If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.*

(Hint: assume not and use the Bolzano-Weierstrass Theorem.)

Theorem 4 *If f is continuous on the closed interval $[a, b]$, then f is a bounded function on $[a, b]$.*

(Hint: assume not and use the Bolzano-Weierstrass Theorem.)

Theorem 5 *Let f be continuous on $[0, \infty)$ and uniformly continuous on $[k, \infty)$ for some $k > 0$, then f is uniformly continuous on $[0, \infty)$.*

(Hint: this can be a one line proof.)