

Chapter 11: Probability Theory II

Combining Events, Independence, Conditional Probability, and Bayes' Theorem

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Chapter 10 gave us ways to find probabilities of basic events.

Chapter 11 asks how to combine those probabilities.

The main questions are:

- ▶ What is $\mathbb{P}(A \text{ or } B)$?
- ▶ What is $\mathbb{P}(A \text{ and } B)$?
- ▶ What changes if the events are independent?
- ▶ What changes if we know another event already occurred?

Chapter roadmap

By the end of this chapter, students should be able to:

- ① recognize mutually exclusive events;
- ② compute probabilities involving “or”;
- ③ recognize independent events;
- ④ compute probabilities involving “and”;
- ⑤ use complements with independent repeated events;
- ⑥ compute conditional probabilities;
- ⑦ test whether events are independent;
- ⑧ work with contingency tables;
- ⑨ interpret relative risk;
- ⑩ use Bayes’ theorem and probability trees.

Skill 1: What does “or” mean?

The event

A or B

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$A \cup B$.

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Unless stated otherwise, probability uses the **inclusive** meaning of “or”.

Mutually exclusive events

Events A and B are **mutually exclusive** if they cannot occur at the same time.

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$$\mathbb{P}(A \text{ and } B) = 0.$$

Example for one die:

$$A = \{\text{roll a 2}\}, \quad B = \{\text{roll a 5}\}.$$

A single roll cannot be both 2 and 5.

Example 1A: Mutually exclusive or not?

One die is rolled.

$$A = \{\text{even}\}, \quad B = \{\text{odd}\}.$$

Example 1A: Mutually exclusive or not?

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Can both happen on the same roll?

Example 1A: Mutually exclusive or not?

One die is rolled.

$$A = \{\text{even}\}, \quad B = \{\text{odd}\}.$$

Can both happen on the same roll?

No.

Example 1A: Mutually exclusive or not?

One die is rolled.

$$A = \{\text{even}\}, \quad B = \{\text{odd}\}.$$

Can both happen on the same roll?

No.

Therefore:

A and B are mutually exclusive.

Example 1B: Mutually exclusive or not?

One die is rolled.

$$A = \{\text{even}\}, \quad B = \{\text{greater than 3}\}.$$

Example 1B: Mutually exclusive or not?

One die is rolled.

$$A = \{\text{even}\}, \quad B = \{\text{greater than 3}\}.$$

List them:

$$A = \{2, 4, 6\},$$

$$B = \{4, 5, 6\}.$$

Example 1B: Mutually exclusive or not?

One die is rolled.

$$A = \{\text{even}\}, \quad B = \{\text{greater than } 3\}.$$

List them:

$$A = \{2, 4, 6\},$$

$$B = \{4, 5, 6\}.$$

They overlap at:

$$\{4, 6\}.$$

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One die is rolled.

$$A = \{\text{even}\}, \quad B = \{\text{greater than 3}\}.$$

List them:

$$A = \{2, 4, 6\},$$

$$B = \{4, 5, 6\}.$$

They overlap at:

$$\{4, 6\}.$$

So:

A and B are not mutually exclusive.

Law of exclusives

If A and B are mutually exclusive, then:

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B).$$

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For several pairwise mutually exclusive events:

$$\mathbb{P}(E_1 \text{ or } \cdots \text{ or } E_n) = \mathbb{P}(E_1) + \cdots + \mathbb{P}(E_n).$$

Example 1C: Blood-type style calculation

Suppose:

$$\mathbb{P}(O) = 0.45, \quad \mathbb{P}(B) = 0.10.$$

A person cannot simultaneously have type O and type B .

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Therefore:

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Suppose:

$$\mathbb{P}(O) = 0.45, \quad \mathbb{P}(B) = 0.10.$$

A person cannot simultaneously have type O and type B .

Therefore:

$$\mathbb{P}(O \text{ or } B) = 0.45 + 0.10.$$

$$\boxed{0.55}.$$

Example 1D: A die

Roll one fair die.

Find:

$$\mathbb{P}(1 \text{ or } 6).$$

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The events are mutually exclusive.

Example 1D: A die

Roll one fair die.

Find:

$$\mathbb{P}(1 \text{ or } 6).$$

The events are mutually exclusive.

$$\mathbb{P}(1 \text{ or } 6) = \frac{1}{6} + \frac{1}{6}.$$

Example 1D: A die

Roll one fair die.

Find:

$$\mathbb{P}(1 \text{ or } 6).$$

The events are mutually exclusive.

$$\mathbb{P}(1 \text{ or } 6) = \frac{1}{6} + \frac{1}{6}.$$

$$\boxed{\frac{1}{3}}.$$

Why simple addition can fail

Suppose:

$$A = \{\text{even}\}, \quad B = \{\text{greater than 2}\}$$

for one die roll.

Why simple addition can fail

Suppose:

$$A = \{\text{even}\}, \quad B = \{\text{greater than } 2\}$$

for one die roll.

Then:

$$\mathbb{P}(A) = \frac{1}{2}, \quad \mathbb{P}(B) = \frac{2}{3}.$$

Why simple addition can fail

Suppose:

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$$\mathbb{P}(A) = \frac{1}{2}, \quad \mathbb{P}(B) = \frac{2}{3}.$$

If we simply add:

$$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}.$$

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If we simply add:

$$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}.$$

But:

$$\frac{7}{6} > 1,$$

so something is wrong.

What went wrong?

For the same example:

$$A = \{2, 4, 6\},$$

$$B = \{3, 4, 5, 6\}.$$

What went wrong?

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The outcomes 4 and 6 appear in **both** events.

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The outcomes 4 and 6 appear in **both** events.

Adding $\mathbb{P}(A) + \mathbb{P}(B)$ counts those outcomes twice.

What went wrong?

For the same example:

$$A = \{2, 4, 6\},$$

$$B = \{3, 4, 5, 6\}.$$

The outcomes 4 and 6 appear in **both** events.

Adding $\mathbb{P}(A) + \mathbb{P}(B)$ counts those outcomes twice.

We must subtract the overlap once.

Law of disjunction

For any two events:

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B).$$

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For any two events:

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B).$$

In set notation:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Example 2A: One die

Let:

$$A = \{\text{even}\}, \quad B = \{\text{greater than } 2\}.$$

We have:

$$\mathbb{P}(A) = \frac{1}{2}, \quad \mathbb{P}(B) = \frac{2}{3}.$$

Example 2A: One die

Let:

$$A = \{\text{even}\}, \quad B = \{\text{greater than } 2\}.$$

We have:

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The overlap is:

$$A \text{ and } B = \{4, 6\}.$$

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We have:

$$\mathbb{P}(A) = \frac{1}{2}, \quad \mathbb{P}(B) = \frac{2}{3}.$$

The overlap is:

$$A \text{ and } B = \{4, 6\}.$$

So:

$$\mathbb{P}(A \text{ and } B) = \frac{2}{6} = \frac{1}{3}.$$

Example 2A: Finish

Use:

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B).$$

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$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B).$$

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Example 2A: Finish

Use:

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B).$$

$$= \frac{1}{2} + \frac{2}{3} - \frac{1}{3}.$$

$$= \frac{1}{2} + \frac{1}{3}.$$

$$\boxed{\frac{5}{6}}.$$

Example 2B: Exam results

Suppose:

$$\mathbb{P}(A) = 0.60$$

for passing Exam 1,

$$\mathbb{P}(B) = 0.75$$

for passing Exam 2,
and:

$$\mathbb{P}(A \text{ and } B) = 0.50.$$

Find the probability of passing at least one exam.

Example 2B: Solution

$$\mathbb{P}(A \text{ or } B) = 0.60 + 0.75 - 0.50.$$

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$$\boxed{0.85}.$$

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$$\mathbb{P}(A \text{ or } B) = 0.60 + 0.75 - 0.50.$$

$$\boxed{0.85}.$$

So:

$\boxed{85\% \text{ passed at least one exam.}}$

A useful diagnostic

If you are finding:

$$\mathbb{P}(A \text{ or } B),$$

ask:

A useful diagnostic

If you are finding:

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ask:

Can A and B happen together?

A useful diagnostic

If you are finding:

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ask:

Can A and B happen together?

If no:

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B).$$

A useful diagnostic

If you are finding:

$$\mathbb{P}(A \text{ or } B),$$

ask:

Can A and B happen together?

If no:

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B).$$

If yes:

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B).$$

Skill 2: Independence

Events A and B are **independent** if knowing that one occurred does not change the probability of the other.

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Events A and B are **independent** if knowing that one occurred does not change the probability of the other.

Informally:

The occurrence of one gives no information about the other.

Example 3A: Independent events

Toss two fair coins.

Let:

$$A = \{\text{first coin is heads}\},$$

$$B = \{\text{second coin is heads}\}.$$

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Knowing the first coin landed heads does not affect the second coin.

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Let:

$$A = \{\text{first coin is heads}\},$$

$$B = \{\text{second coin is heads}\}.$$

Knowing the first coin landed heads does not affect the second coin.

So:

A and B are independent.

Example 3B: Not independent

Draw one card from a standard deck.

Let:

$$A = \{\text{card is an ace}\},$$

$$B = \{\text{card is a spade}\}.$$

Example 3B: Not independent

Draw one card from a standard deck.

Let:

$$A = \{\text{card is an ace}\},$$

$$B = \{\text{card is a spade}\}.$$

These are actually independent for one draw because:

$$\mathbb{P}(A) = \frac{4}{52}, \quad \mathbb{P}(A \mid B) = \frac{1}{13}.$$

Example 3B: Not independent

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$$A = \{\text{card is an ace}\},$$

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These are actually independent for one draw because:

$$\mathbb{P}(A) = \frac{4}{52}, \quad \mathbb{P}(A \mid B) = \frac{1}{13}.$$

And:

$$\frac{4}{52} = \frac{1}{13}.$$

Example 3B: Not independent

Draw one card from a standard deck.

Let:

$$A = \{\text{card is an ace}\},$$

$$B = \{\text{card is a spade}\}.$$

These are actually independent for one draw because:

$$\mathbb{P}(A) = \frac{4}{52}, \quad \mathbb{P}(A \mid B) = \frac{1}{13}.$$

And:

$$\frac{4}{52} = \frac{1}{13}.$$

So learning that the card is a spade does not change the probability of an ace.

Mutually exclusive is NOT the same as independent

This distinction is essential.

Mutually exclusive is NOT the same as independent

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Mutually exclusive means:

They cannot happen together.

Independent means:

One does not affect the probability of the other.

Mutually exclusive is NOT the same as independent

This distinction is essential.

Mutually exclusive means:

They cannot happen together.

Independent means:

One does not affect the probability of the other.

These are very different ideas.

Why mutually exclusive events are usually dependent

Suppose:

$$\mathbb{P}(A) > 0, \quad \mathbb{P}(B) > 0,$$

and A and B are mutually exclusive.

Why mutually exclusive events are usually dependent

Suppose:

$$\mathbb{P}(A) > 0, \quad \mathbb{P}(B) > 0,$$

and A and B are mutually exclusive.

If A occurs, then B becomes impossible.

Why mutually exclusive events are usually dependent

Suppose:

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So knowing A occurred changes the probability of B to:

0.

Why mutually exclusive events are usually dependent

Suppose:

$$\mathbb{P}(A) > 0, \quad \mathbb{P}(B) > 0,$$

and A and B are mutually exclusive.

If A occurs, then B becomes impossible.

So knowing A occurred changes the probability of B to:

0.

Therefore nontrivial mutually exclusive events are not independent.

Example 3C: Identify the relationship

One die is rolled.

$$A = \{\text{roll a 1}\}, \quad B = \{\text{roll a 2}\}.$$

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Are they independent?

Example 3C: Identify the relationship

One die is rolled.

$$A = \{\text{roll a 1}\}, \quad B = \{\text{roll a 2}\}.$$

They cannot happen together.

So they are:

mutually exclusive.

Are they independent?

No. If A occurs, then B cannot occur.

Skill 3: “And” for independent events

If A and B are independent, then:

$$\mathbb{P}(A \text{ and } B) = \mathbb{P}(A)\mathbb{P}(B).$$

Skill 3: “And” for independent events

If A and B are independent, then:

$$\mathbb{P}(A \text{ and } B) = \mathbb{P}(A)\mathbb{P}(B).$$

This is the **product rule for independence**.

Example 4A: Two coin tosses

Find:

$$\mathbb{P}(H \text{ on first and } H \text{ on second}).$$

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The tosses are independent.

Example 4A: Two coin tosses

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$$\mathbb{P}(H \text{ on first and } H \text{ on second}).$$

The tosses are independent.

$$\mathbb{P}(HH) = \frac{1}{2} \cdot \frac{1}{2}.$$

Example 4A: Two coin tosses

Find:

$\mathbb{P}(H \text{ on first and } H \text{ on second}).$

The tosses are independent.

$$\mathbb{P}(HH) = \frac{1}{2} \cdot \frac{1}{2}.$$

$$\boxed{\frac{1}{4}}.$$

Example 4B: Two unrelated donors

Suppose:

$$\mathbb{P}(A \text{ blood type}) = 0.40.$$

Two unrelated donors are selected.

Assume their blood types are independent.

Example 4B: Two unrelated donors

Suppose:

$$\mathbb{P}(\text{A blood type}) = 0.40.$$

Two unrelated donors are selected.

Assume their blood types are independent.

Probability both are type A:

$$0.40(0.40).$$

Example 4B: Two unrelated donors

Suppose:

$$\mathbb{P}(A \text{ blood type}) = 0.40.$$

Two unrelated donors are selected.

Assume their blood types are independent.

Probability both are type A:

$$0.40(0.40).$$

$$\boxed{0.16}.$$

Example 4C: Two independent failures

Each valve fails with probability:

0.01.

Two valve failures are independent.

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0.01.

Two valve failures are independent.

Probability both fail:

$0.01(0.01)$.

Example 4C: Two independent failures

Each valve fails with probability:

$$0.01.$$

Two valve failures are independent.

Probability both fail:

$$0.01(0.01).$$

$$\boxed{0.0001}.$$

Example 4C: Two independent failures

Each valve fails with probability:

$$0.01.$$

Two valve failures are independent.

Probability both fail:

$$0.01(0.01).$$

$$0.0001.$$

That is:

$$1 \text{ chance in } 10,000.$$

More than two independent events

If:

$$E_1, E_2, \dots, E_n$$

are independent, then:

$$\mathbb{P}(E_1 \text{ and } \dots \text{ and } E_n) = \mathbb{P}(E_1) \cdots \mathbb{P}(E_n).$$

Example 4D: Four tails

A fair coin is tossed 4 times.

Find:

$$\mathbb{P}(TTTT).$$

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A fair coin is tossed 4 times.

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The tosses are independent.

Example 4D: Four tails

A fair coin is tossed 4 times.

Find:

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The tosses are independent.

$$\left(\frac{1}{2}\right)^4.$$

Example 4D: Four tails

A fair coin is tossed 4 times.

Find:

$$\mathbb{P}(TTTT).$$

The tosses are independent.

$$\left(\frac{1}{2}\right)^4.$$

$$\boxed{\frac{1}{16}}.$$

Example 4E: Three independent successes

Suppose the probability of success on each trial is:

0.70,

and trials are independent.

Find the probability of three successes in a row.

Example 4E: Three independent successes

Suppose the probability of success on each trial is:

$$0.70,$$

and trials are independent.

Find the probability of three successes in a row.

$$0.70^3.$$

Example 4E: Three independent successes

Suppose the probability of success on each trial is:

$$0.70,$$

and trials are independent.

Find the probability of three successes in a row.

$$0.70^3.$$

$$\boxed{0.343}.$$

Skill 4: “At least one” with independent trials

The direct approach often requires many cases.

Skill 4: “At least one” with independent trials

The direct approach often requires many cases.

The complement of:

at least one success

is:

no successes.

Skill 4: “At least one” with independent trials

The direct approach often requires many cases.

The complement of:

at least one success

is:

no successes.

So:

$$\mathbb{P}(\text{at least one success}) = 1 - \mathbb{P}(\text{no successes}).$$

Example 5A: At least one type-A donor

Suppose:

$$\mathbb{P}(A) = 0.40.$$

Two unrelated donors are selected.

Find the probability that at least one is type A.

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Suppose:

$$\mathbb{P}(A) = 0.40.$$

Two unrelated donors are selected.

Find the probability that at least one is type A.

Complement:

neither is type A.

Example 5A: At least one type-A donor

Suppose:

$$\mathbb{P}(A) = 0.40.$$

Two unrelated donors are selected.

Find the probability that at least one is type A.

Complement:

neither is type A.

For one donor:

$$\mathbb{P}(\text{not } A) = 0.60.$$

Example 5A: Finish

For two independent donors:

$$\mathbb{P}(\text{neither } A) = 0.60^2 = 0.36.$$

Example 5A: Finish

For two independent donors:

$$\mathbb{P}(\text{neither } A) = 0.60^2 = 0.36.$$

Therefore:

$$\mathbb{P}(\text{at least one } A) = 1 - 0.36.$$

Example 5A: Finish

For two independent donors:

$$\mathbb{P}(\text{neither } A) = 0.60^2 = 0.36.$$

Therefore:

$$\mathbb{P}(\text{at least one } A) = 1 - 0.36.$$

$$\boxed{0.64}.$$

Example 5B: At least one win

Suppose you win a game with probability:

0.40.

You play independently twice.

Find the chance of at least one win.

Example 5B: At least one win

Suppose you win a game with probability:

0.40.

You play independently twice.

Find the chance of at least one win.

Probability of losing one game:

0.60.

Example 5B: At least one win

Suppose you win a game with probability:

$$0.40.$$

You play independently twice.

Find the chance of at least one win.

Probability of losing one game:

$$0.60.$$

Probability of losing both:

$$0.60^2 = 0.36.$$

Example 5B: At least one win

Suppose you win a game with probability:

$$0.40.$$

You play independently twice.

Find the chance of at least one win.

Probability of losing one game:

$$0.60.$$

Probability of losing both:

$$0.60^2 = 0.36.$$

Thus:

$$1 - 0.36 = 0.64.$$

Example 5C: Ten attempts

Same game:

$$\mathbb{P}(\text{win}) = 0.40.$$

You play 10 independent times.

Example 5C: Ten attempts

Same game:

$$\mathbb{P}(\text{win}) = 0.40.$$

You play 10 independent times.

Probability of no wins:

$$0.60^{10}.$$

Example 5C: Ten attempts

Same game:

$$\mathbb{P}(\text{win}) = 0.40.$$

You play 10 independent times.

Probability of no wins:

$$0.60^{10}.$$

Therefore:

$$\mathbb{P}(\text{at least one win}) = 1 - 0.60^{10}.$$

Example 5C: Ten attempts

Same game:

$$\mathbb{P}(\text{win}) = 0.40.$$

You play 10 independent times.

Probability of no wins:

$$0.60^{10}.$$

Therefore:

$$\mathbb{P}(\text{at least one win}) = 1 - 0.60^{10}.$$

$$\boxed{\approx 0.994}.$$

Example 5D: Forty independent attempts

Suppose each attempt succeeds with probability:

0.01.

There are 40 independent attempts.

Example 5D: Forty independent attempts

Suppose each attempt succeeds with probability:

$$0.01.$$

There are 40 independent attempts.

Probability all 40 fail:

$$0.99^{40}.$$

Example 5D: Forty independent attempts

Suppose each attempt succeeds with probability:

$$0.01.$$

There are 40 independent attempts.

Probability all 40 fail:

$$0.99^{40}.$$

$$0.99^{40} \approx 0.669.$$

Example 5D: Forty independent attempts

Suppose each attempt succeeds with probability:

$$0.01.$$

There are 40 independent attempts.

Probability all 40 fail:

$$0.99^{40}.$$

$$0.99^{40} \approx 0.669.$$

Therefore:

$$\mathbb{P}(\text{at least one success}) \approx 1 - 0.669 = \boxed{0.331}.$$

Skill 5: Conditional probability

The notation:

$$\mathbb{P}(A \mid B)$$

means:

the probability that A occurs, given that B has occurred.

Skill 5: Conditional probability

The notation:

$$\mathbb{P}(A \mid B)$$

means:

the probability that A occurs, given that B has occurred.

The information that B occurred changes the sample space we consider.

Example 6A: Die roll with new information

A fair die is rolled.

You are told:

the result is at least 4.

Possible outcomes are now only:

$\{4, 5, 6\}$.

Example 6A: Die roll with new information

A fair die is rolled.

You are told:

the result is at least 4.

Possible outcomes are now only:

$\{4, 5, 6\}$.

What is the probability the result is odd?

Example 6A: Die roll with new information

A fair die is rolled.

You are told:

the result is at least 4.

Possible outcomes are now only:

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What is the probability the result is odd?

Only:

5

works.

Example 6A: Die roll with new information

A fair die is rolled.

You are told:

the result is at least 4.

Possible outcomes are now only:

$\{4, 5, 6\}$.

What is the probability the result is odd?

Only:

5

works.

$$\boxed{\frac{1}{3}}.$$

Conditional probability formula

For events A and B with $\mathbb{P}(B) > 0$:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}.$$

Conditional probability formula

For events A and B with $\mathbb{P}(B) > 0$:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}.$$

The denominator is the event we are **given**.

Conditional probability formula

For events A and B with $\mathbb{P}(B) > 0$:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}.$$

The denominator is the event we are **given**.

A useful reading:

“A given B” means restrict attention to B.

Example 6B: Use the formula

One die is rolled.

Let:

$$A = \{\text{odd}\},$$

$$B = \{\text{at least } 4\}.$$

Example 6B: Use the formula

One die is rolled.

Let:

$$A = \{\text{odd}\},$$

$$B = \{\text{at least } 4\}.$$

$$A \text{ and } B = \{5\}.$$

So:

$$\mathbb{P}(A \text{ and } B) = \frac{1}{6}.$$

Example 6B: Use the formula

One die is rolled.

Let:

$$A = \{\text{odd}\},$$

$$B = \{\text{at least } 4\}.$$

$$A \text{ and } B = \{5\}.$$

So:

$$\mathbb{P}(A \text{ and } B) = \frac{1}{6}.$$

Also:

$$\mathbb{P}(B) = \frac{3}{6} = \frac{1}{2}.$$

Example 6B: Finish

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}.$$

Example 6B: Finish

$$\begin{aligned}\mathbb{P}(A \mid B) &= \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)} \\ &= \frac{1/6}{1/2}.\end{aligned}$$

Example 6B: Finish

$$\begin{aligned}\mathbb{P}(A \mid B) &= \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)} \\ &= \frac{1/6}{1/2} \\ &= \frac{1}{6} \cdot 2.\end{aligned}$$

Example 6B: Finish

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}.$$

$$= \frac{1/6}{1/2}.$$

$$= \frac{1}{6} \cdot 2.$$

$$\boxed{\frac{1}{3}}.$$

Example 6C: Class data

In a class:

40%

are seniors,
and:

18%

are both seniors and biology majors.

Find:

$$\mathbb{P}(\text{biology major} \mid \text{senior}).$$

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$$= \frac{0.18}{0.40}.$$

Example 6C: Class data

In a class:

40%

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and:

18%

are both seniors and biology majors.

Find:

$\mathbb{P}(\text{biology major} \mid \text{senior}).$

$$= \frac{0.18}{0.40}.$$

$$\boxed{0.45}.$$

Example 6D: Reverse conditioning is different

Suppose:

$$\mathbb{P}(A \mid B) = 0.80.$$

Does that imply:

$$\mathbb{P}(B \mid A) = 0.80?$$

Example 6D: Reverse conditioning is different

Suppose:

$$\mathbb{P}(A \mid B) = 0.80.$$

Does that imply:

$$\mathbb{P}(B \mid A) = 0.80?$$

No.

Example 6D: Reverse conditioning is different

Suppose:

$$\mathbb{P}(A \mid B) = 0.80.$$

Does that imply:

$$\mathbb{P}(B \mid A) = 0.80?$$

No.

The denominator changes:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)},$$

while

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(A)}.$$

Skill 6: Independence using conditional probability

Events A and B are independent exactly when:

$$\mathbb{P}(A \mid B) = \mathbb{P}(A).$$

Skill 6: Independence using conditional probability

Events A and B are independent exactly when:

$$\mathbb{P}(A \mid B) = \mathbb{P}(A).$$

Equivalent test:

$$\mathbb{P}(A \text{ and } B) = \mathbb{P}(A)\mathbb{P}(B).$$

Example 7A: Independent

Suppose:

$$\mathbb{P}(A) = 0.30,$$

$$\mathbb{P}(B) = 0.50,$$

$$\mathbb{P}(A \text{ and } B) = 0.15.$$

Example 7A: Independent

Suppose:

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$$\mathbb{P}(B) = 0.50,$$

$$\mathbb{P}(A \text{ and } B) = 0.15.$$

Compute:

$$\mathbb{P}(A)\mathbb{P}(B) = 0.30(0.50) = 0.15.$$

Example 7A: Independent

Suppose:

$$\mathbb{P}(A) = 0.30,$$

$$\mathbb{P}(B) = 0.50,$$

$$\mathbb{P}(A \text{ and } B) = 0.15.$$

Compute:

$$\mathbb{P}(A)\mathbb{P}(B) = 0.30(0.50) = 0.15.$$

This matches the joint probability.

Example 7A: Independent

Suppose:

$$\mathbb{P}(A) = 0.30,$$

$$\mathbb{P}(B) = 0.50,$$

$$\mathbb{P}(A \text{ and } B) = 0.15.$$

Compute:

$$\mathbb{P}(A)\mathbb{P}(B) = 0.30(0.50) = 0.15.$$

This matches the joint probability.

Therefore:

A and B are independent.

Example 7B: Not independent

Suppose:

$$\mathbb{P}(A) = 0.40, \quad \mathbb{P}(B) = 0.50,$$

but:

$$\mathbb{P}(A \text{ and } B) = 0.30.$$

Example 7B: Not independent

Suppose:

$$\mathbb{P}(A) = 0.40, \quad \mathbb{P}(B) = 0.50,$$

but:

$$\mathbb{P}(A \text{ and } B) = 0.30.$$

If independent, we would need:

$$0.40(0.50) = 0.20.$$

Example 7B: Not independent

Suppose:

$$\mathbb{P}(A) = 0.40, \quad \mathbb{P}(B) = 0.50,$$

but:

$$\mathbb{P}(A \text{ and } B) = 0.30.$$

If independent, we would need:

$$0.40(0.50) = 0.20.$$

But:

$$0.30 \neq 0.20.$$

Example 7B: Not independent

Suppose:

$$\mathbb{P}(A) = 0.40, \quad \mathbb{P}(B) = 0.50,$$

but:

$$\mathbb{P}(A \text{ and } B) = 0.30.$$

If independent, we would need:

$$0.40(0.50) = 0.20.$$

But:

$$0.30 \neq 0.20.$$

Therefore:

not independent.

Example 7C: Conditional check

Suppose:

$$\mathbb{P}(A) = 0.60, \quad \mathbb{P}(A \mid B) = 0.60.$$

Example 7C: Conditional check

Suppose:

$$\mathbb{P}(A) = 0.60, \quad \mathbb{P}(A \mid B) = 0.60.$$

The probability of A is unchanged after learning B occurred.

Example 7C: Conditional check

Suppose:

$$\mathbb{P}(A) = 0.60, \quad \mathbb{P}(A \mid B) = 0.60.$$

The probability of A is unchanged after learning B occurred.

Therefore:

A and B are independent.

Skill 7: Read a contingency table

Suppose 200 students are classified by major and class year.

	Math	Other	Total
Senior	30	50	80
Not Senior	20	100	120
Total	50	150	200

Skill 7: Read a contingency table

Suppose 200 students are classified by major and class year.

	Math	Other	Total
Senior	30	50	80
Not Senior	20	100	120
Total	50	150	200

Every cell represents an intersection of two categories.

Example 8A: Marginal probability

Using the table:

	Math	Other	Total
Senior	30	50	80
Not Senior	20	100	120
Total	50	150	200

Find:

$$\mathbb{P}(\text{Senior}).$$

Example 8A: Marginal probability

Using the table:

	Math	Other	Total
Senior	30	50	80
Not Senior	20	100	120
Total	50	150	200

Find:

$\mathbb{P}(\text{Senior})$.

$$\frac{80}{200}.$$

Example 8A: Marginal probability

Using the table:

	Math	Other	Total
Senior	30	50	80
Not Senior	20	100	120
Total	50	150	200

Find:

$\mathbb{P}(\text{Senior})$.

$$\frac{80}{200}.$$

$$\boxed{0.40}.$$

Example 8B: Joint probability

Find:

$$\mathbb{P}(\text{Senior and Math}).$$

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Find:

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The relevant cell is:

30.

Example 8B: Joint probability

Find:

$\mathbb{P}(\text{Senior and Math}).$

The relevant cell is:

30.

Total students:

200.

Example 8B: Joint probability

Find:

$\mathbb{P}(\text{Senior and Math}).$

The relevant cell is:

30.

Total students:

200.

$$\boxed{\frac{30}{200} = 0.15}.$$

Example 8C: Conditional probability from a table

Find:

$$\mathbb{P}(\text{Math} \mid \text{Senior}).$$

Example 8C: Conditional probability from a table

Find:

$$\mathbb{P}(\text{Math} \mid \text{Senior}).$$

Restrict attention to seniors:

80.

Among those seniors, math majors:

30.

Example 8C: Conditional probability from a table

Find:

$$\mathbb{P}(\text{Math} \mid \text{Senior}).$$

Restrict attention to seniors:

80.

Among those seniors, math majors:

30.

$$\boxed{\frac{30}{80} = 0.375}.$$

Example 8D: Are Senior and Math independent?

We have:

$$\mathbb{P}(\text{Math}) = \frac{50}{200} = 0.25.$$

Example 8D: Are Senior and Math independent?

We have:

$$\mathbb{P}(\text{Math}) = \frac{50}{200} = 0.25.$$

And:

$$\mathbb{P}(\text{Math} \mid \text{Senior}) = \frac{30}{80} = 0.375.$$

Example 8D: Are Senior and Math independent?

We have:

$$\mathbb{P}(\text{Math}) = \frac{50}{200} = 0.25.$$

And:

$$\mathbb{P}(\text{Math} \mid \text{Senior}) = \frac{30}{80} = 0.375.$$

Since:

$$0.375 \neq 0.25,$$

we conclude:

Senior and Math are not independent.

Skill 8: Compare conditional risks

Suppose:

$D = \{\text{disease outcome}\},$

$E = \{\text{exposed group}\}.$

We may compare:

$$\mathbb{P}(D \mid E)$$

with:

$$\mathbb{P}(D \mid E^c).$$

Skill 8: Compare conditional risks

Suppose:

$D = \{\text{disease outcome}\},$

$E = \{\text{exposed group}\}.$

We may compare:

$$\mathbb{P}(D \mid E)$$

with:

$$\mathbb{P}(D \mid E^c).$$

This compares risk in the exposed and unexposed groups.

A common comparison is:

$$RR = \frac{\mathbb{P}(D \mid E)}{\mathbb{P}(D \mid E^c)}.$$

A common comparison is:

$$RR = \frac{\mathbb{P}(D \mid E)}{\mathbb{P}(D \mid E^c)}.$$

Interpretation:

- ▶ $RR = 1$: equal risk;
- ▶ $RR > 1$: greater risk in the exposed group;
- ▶ $RR < 1$: lower risk in the exposed group.

Example 9A: Relative risk

Suppose:

$$\mathbb{P}(D \mid E) = 0.012,$$

and:

$$\mathbb{P}(D \mid E^c) = 0.003.$$

Example 9A: Relative risk

Suppose:

$$\mathbb{P}(D \mid E) = 0.012,$$

and:

$$\mathbb{P}(D \mid E^c) = 0.003.$$

$$RR = \frac{0.012}{0.003}.$$

Example 9A: Relative risk

Suppose:

$$\mathbb{P}(D \mid E) = 0.012,$$

and:

$$\mathbb{P}(D \mid E^c) = 0.003.$$

$$RR = \frac{0.012}{0.003}.$$

$$\boxed{RR = 4}.$$

Example 9A: Relative risk

Suppose:

$$\mathbb{P}(D \mid E) = 0.012,$$

and:

$$\mathbb{P}(D \mid E^c) = 0.003.$$

$$RR = \frac{0.012}{0.003}.$$

$$\boxed{RR = 4}.$$

The exposed group has four times the risk of the unexposed group.

Example 9B: Interpret a ratio less than 1

Suppose:

$$RR = 0.60.$$

Example 9B: Interpret a ratio less than 1

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$$RR = 0.60.$$

This means the exposed group has:

60%

of the risk of the comparison group.

Example 9B: Interpret a ratio less than 1

Suppose:

$$RR = 0.60.$$

This means the exposed group has:

60%

of the risk of the comparison group.

Equivalently, its risk is about:

40%

lower.

Skill 9: Diagnostic-test vocabulary

Let:

$$D = \{\text{disease present}\},$$

$$D^c = \{\text{disease absent}\},$$

$$+ = \{\text{positive test}\},$$

$$- = \{\text{negative test}\}.$$

Skill 9: Diagnostic-test vocabulary

Let:

$D = \{\text{disease present}\},$

$D^c = \{\text{disease absent}\},$

$+$ = {positive test},

$-$ = {negative test}.

Important conditional probabilities include:

$$\mathbb{P}(+ \mid D), \quad \mathbb{P}(+ \mid D^c).$$

False positive and false negative

False positive rate:

$$\mathbb{P}(+ \mid D^c).$$

False positive and false negative

False positive rate:

$$\mathbb{P}(+ \mid D^c).$$

False negative rate:

$$\mathbb{P}(- \mid D).$$

False positive and false negative

False positive rate:

$$\mathbb{P}(+ \mid D^c).$$

False negative rate:

$$\mathbb{P}(- \mid D).$$

Therefore:

$$\mathbb{P}(+ \mid D) = 1 - \mathbb{P}(- \mid D),$$

and:

$$\mathbb{P}(- \mid D^c) = 1 - \mathbb{P}(+ \mid D^c).$$

The question patients usually care about

A test may have a very low false-positive rate.

But after receiving a positive test, the important question is:

$$\mathbb{P}(D \mid +).$$

The question patients usually care about

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But after receiving a positive test, the important question is:

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This is **not** the same as:

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The question patients usually care about

A test may have a very low false-positive rate.

But after receiving a positive test, the important question is:

$$\mathbb{P}(D \mid +).$$

This is **not** the same as:

$$\mathbb{P}(+ \mid D).$$

Reversing the conditioning changes the question completely.

Bayes' theorem

For a diagnostic test:

$$\mathbb{P}(D \mid +) = \frac{\mathbb{P}(D)\mathbb{P}(+ \mid D)}{\mathbb{P}(D)\mathbb{P}(+ \mid D) + \mathbb{P}(D^c)\mathbb{P}(+ \mid D^c)}.$$

Bayes' theorem

For a diagnostic test:

$$\mathbb{P}(D \mid +) = \frac{\mathbb{P}(D)\mathbb{P}(+ \mid D)}{\mathbb{P}(D)\mathbb{P}(+ \mid D) + \mathbb{P}(D^c)\mathbb{P}(+ \mid D^c)}.$$

The denominator counts all ways a positive test can occur.

Example 10A: Diagnostic test setup

Suppose:

$$\mathbb{P}(D) = 0.02.$$

False negative rate:

$$\mathbb{P}(- \mid D) = 0.01.$$

False positive rate:

$$\mathbb{P}(+ \mid D^c) = 0.03.$$

Example 10A: Diagnostic test setup

Suppose:

$$\mathbb{P}(D) = 0.02.$$

False negative rate:

$$\mathbb{P}(- \mid D) = 0.01.$$

False positive rate:

$$\mathbb{P}(+ \mid D^c) = 0.03.$$

Therefore:

$$\mathbb{P}(+ \mid D) = 0.99,$$

and:

$$\mathbb{P}(D^c) = 0.98.$$

Example 10A: Use 10,000 people

Imagine testing:

10,000

people.

Example 10A: Use 10,000 people

Imagine testing:

10,000

people.

Disease present:

$$10,000(0.02) = 200.$$

Disease absent:

$$10,000(0.98) = 9800.$$

Example 10A: Among those with disease

There are:

200

people with disease.

Example 10A: Among those with disease

There are:

200

people with disease.

Since:

$$\mathbb{P}(+ \mid D) = 0.99,$$

positive tests among them:

$$200(0.99) = 198.$$

Example 10A: Among those with disease

There are:

$$200$$

people with disease.

Since:

$$\mathbb{P}(+ \mid D) = 0.99,$$

positive tests among them:

$$200(0.99) = 198.$$

Negative tests among them:

$$200(0.01) = 2.$$

Example 10A: Among those without disease

There are:

9800

people without disease.

Example 10A: Among those without disease

There are:

9800

people without disease.

False-positive rate:

0.03.

So false positives:

$$9800(0.03) = 294.$$

Example 10A: Among those without disease

There are:

9800

people without disease.

False-positive rate:

0.03.

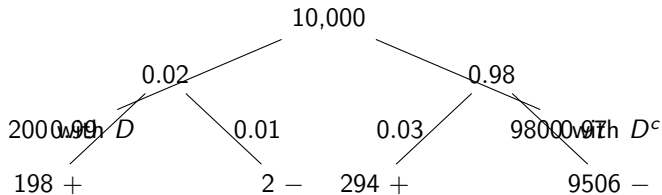
So false positives:

$$9800(0.03) = 294.$$

True negatives:

$$9800(0.97) = 9506.$$

Example 10A: Probability tree



Example 10A: Positive predictive value

Total positive tests:

$$198 + 294 = 492.$$

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$$198 + 294 = 492.$$

Of these, actual disease cases:

$$198.$$

Example 10A: Positive predictive value

Total positive tests:

$$198 + 294 = 492.$$

Of these, actual disease cases:

$$198.$$

Therefore:

$$\mathbb{P}(D \mid +) = \frac{198}{492}.$$

Example 10A: Positive predictive value

Total positive tests:

$$198 + 294 = 492.$$

Of these, actual disease cases:

$$198.$$

Therefore:

$$\mathbb{P}(D \mid +) = \frac{198}{492}.$$

$$\boxed{\mathbb{P}(D \mid +) \approx 0.402}.$$

Example 10A: Positive predictive value

Total positive tests:

$$198 + 294 = 492.$$

Of these, actual disease cases:

$$198.$$

Therefore:

$$\mathbb{P}(D \mid +) = \frac{198}{492}.$$

$$\boxed{\mathbb{P}(D \mid +) \approx 0.402}.$$

So a positive test corresponds to only about a 40% chance of disease.

Why the answer can be surprisingly low

The test is fairly accurate.

But the disease is rare:

2%

prevalence.

Why the answer can be surprisingly low

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Most people do not have the disease.

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Even a small false-positive rate applied to a very large healthy group can create many false positives.

Why the answer can be surprisingly low

The test is fairly accurate.

But the disease is rare:

2%

prevalence.

Most people do not have the disease.

Even a small false-positive rate applied to a very large healthy group can create many false positives.

This is the **base-rate effect**.

Example 10B: Negative result

Using the same 10,000 people:

Total negative tests:

$$2 + 9506 = 9508.$$

Example 10B: Negative result

Using the same 10,000 people:

Total negative tests:

$$2 + 9506 = 9508.$$

True negatives:

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Example 10B: Negative result

Using the same 10,000 people:

Total negative tests:

$$2 + 9506 = 9508.$$

True negatives:

$$9506.$$

So:

$$\mathbb{P}(D^c \mid -) = \frac{9506}{9508}.$$

Example 10B: Negative result

Using the same 10,000 people:

Total negative tests:

$$2 + 9506 = 9508.$$

True negatives:

$$9506.$$

So:

$$\mathbb{P}(D^c \mid -) = \frac{9506}{9508}.$$

$$\boxed{\approx 0.9998}.$$

A negative test is extremely reassuring in this example.

Example 10C: A simpler Bayes problem

Suppose:

$$\mathbb{P}(D) = 0.10,$$

$$\mathbb{P}(+ \mid D) = 0.90,$$

$$\mathbb{P}(+ \mid D^c) = 0.20.$$

Find:

$$\mathbb{P}(D \mid +).$$

Example 10C: Bayes calculation

$$\mathbb{P}(D \mid +) = \frac{(0.10)(0.90)}{(0.10)(0.90) + (0.90)(0.20)}.$$

Example 10C: Bayes calculation

$$\mathbb{P}(D \mid +) = \frac{(0.10)(0.90)}{(0.10)(0.90) + (0.90)(0.20)}.$$

Numerator:

$$0.09.$$

Denominator:

$$0.09 + 0.18 = 0.27.$$

Example 10C: Bayes calculation

$$\mathbb{P}(D \mid +) = \frac{(0.10)(0.90)}{(0.10)(0.90) + (0.90)(0.20)}.$$

Numerator:

$$0.09.$$

Denominator:

$$0.09 + 0.18 = 0.27.$$

$$\mathbb{P}(D \mid +) = \frac{0.09}{0.27} = \boxed{\frac{1}{3}}.$$

Skill 10: Bayes can be done without memorizing a formula

Suppose a 1000-person table is:

	+	−	Total
D	90	10	100
D^c	180	720	900
Total	270	730	1000

Skill 10: Bayes can be done without memorizing a formula

Suppose a 1000-person table is:

	+	−	Total
D	90	10	100
D^c	180	720	900
Total	270	730	1000

Then:

$$\mathbb{P}(D \mid +) = \frac{\text{disease and positive}}{\text{all positive}}.$$

Example 11A: Read Bayes directly from the table

From:

	+	−	Total
D	90	10	100
D^c	180	720	900
Total	270	730	1000

Example 11A: Read Bayes directly from the table

From:

	+	−	Total
D	90	10	100
D^c	180	720	900
Total	270	730	1000

$$\mathbb{P}(D \mid +) = \frac{90}{270}.$$

Example 11A: Read Bayes directly from the table

From:

	+	−	Total
D	90	10	100
D^c	180	720	900
Total	270	730	1000

$$\mathbb{P}(D \mid +) = \frac{90}{270}.$$

$$\boxed{\frac{1}{3}}.$$

Example 11A: Read Bayes directly from the table

From:

	+	−	Total
D	90	10	100
D^c	180	720	900
Total	270	730	1000

$$\mathbb{P}(D \mid +) = \frac{90}{270}.$$

$$\boxed{\frac{1}{3}}.$$

This matches the previous Bayes calculation.

Why tables and trees help

Bayes' theorem can look intimidating because several conditional probabilities appear at once.

Why tables and trees help

Bayes' theorem can look intimidating because several conditional probabilities appear at once. A table or tree converts the problem into:

count the relevant group, then divide.

Why tables and trees help

Bayes' theorem can look intimidating because several conditional probabilities appear at once. A table or tree converts the problem into:

count the relevant group, then divide.

For many students, building a hypothetical population is the safest method.

Skill 11: Which rule do I use?

Look for the wording.

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“or”

usually suggests an addition/disjunction rule.

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“and”

may suggest multiplication, but only under independence or with conditional probability.

Skill 11: Which rule do I use?

Look for the wording.

“or”

usually suggests an addition/disjunction rule.

“and”

may suggest multiplication, but only under independence or with conditional probability.

“given”

signals conditional probability.

The full “and” rule

From the definition of conditional probability:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}.$$

The full “and” rule

From the definition of conditional probability:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}.$$

Multiply both sides by $\mathbb{P}(B)$:

$$\boxed{\mathbb{P}(A \text{ and } B) = \mathbb{P}(B)\mathbb{P}(A \mid B)}.$$

The full “and” rule

From the definition of conditional probability:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}.$$

Multiply both sides by $\mathbb{P}(B)$:

$$\boxed{\mathbb{P}(A \text{ and } B) = \mathbb{P}(B)\mathbb{P}(A \mid B)}.$$

This works whether or not the events are independent.

Example 12A: Without replacement

A box contains:

3 red, 5 black

balls.

Two balls are drawn without replacement.

Find the probability both are red.

Example 12A: Solution

First draw red:

$$\frac{3}{8}.$$

Example 12A: Solution

First draw red:

$$\frac{3}{8}.$$

Given that one red was removed, second draw red:

$$\frac{2}{7}.$$

Example 12A: Solution

First draw red:

$$\frac{3}{8}.$$

Given that one red was removed, second draw red:

$$\frac{2}{7}.$$

Therefore:

$$\mathbb{P}(R_1 \text{ and } R_2) = \frac{3}{8} \cdot \frac{2}{7}.$$

Example 12A: Solution

First draw red:

$$\frac{3}{8}.$$

Given that one red was removed, second draw red:

$$\frac{2}{7}.$$

Therefore:

$$\mathbb{P}(R_1 \text{ and } R_2) = \frac{3}{8} \cdot \frac{2}{7}.$$

$$\boxed{\frac{3}{28}}.$$

Example 12B: With replacement

Same box:

3 red, 5 black.

Now replace the first ball before drawing again.

Example 12B: With replacement

Same box:

3 red, 5 black.

Now replace the first ball before drawing again.

The draws are independent.

Example 12B: With replacement

Same box:

3 red, 5 black.

Now replace the first ball before drawing again.

The draws are independent.

$$\mathbb{P}(RR) = \frac{3}{8} \cdot \frac{3}{8}.$$

Example 12B: With replacement

Same box:

3 red, 5 black.

Now replace the first ball before drawing again.

The draws are independent.

$$\mathbb{P}(RR) = \frac{3}{8} \cdot \frac{3}{8}.$$

$$\boxed{\frac{9}{64}}.$$

With versus without replacement

With replacement:

probabilities stay the same from draw to draw.

With versus without replacement

With replacement:

probabilities stay the same from draw to draw.

Without replacement:

probabilities change after each draw.

With versus without replacement

With replacement:

probabilities stay the same from draw to draw.

Without replacement:

probabilities change after each draw.

So without-replacement draws are generally not independent.

Mixed Example 1

Suppose:

$$\mathbb{P}(A) = 0.30, \quad \mathbb{P}(B) = 0.40,$$

and A, B are mutually exclusive.

Find:

$$\mathbb{P}(A \text{ or } B).$$

Mixed Example 1

Suppose:

$$\mathbb{P}(A) = 0.30, \quad \mathbb{P}(B) = 0.40,$$

and A, B are mutually exclusive.

Find:

$$\mathbb{P}(A \text{ or } B).$$

$$0.30 + 0.40 = \boxed{0.70}.$$

Mixed Example 2

Suppose:

$$\mathbb{P}(A) = 0.50, \quad \mathbb{P}(B) = 0.60,$$

$$\mathbb{P}(A \text{ and } B) = 0.20.$$

Find:

$$\mathbb{P}(A \text{ or } B).$$

Mixed Example 2

Suppose:

$$\mathbb{P}(A) = 0.50, \quad \mathbb{P}(B) = 0.60,$$

$$\mathbb{P}(A \text{ and } B) = 0.20.$$

Find:

$$\mathbb{P}(A \text{ or } B).$$

$$0.50 + 0.60 - 0.20.$$

Mixed Example 2

Suppose:

$$\mathbb{P}(A) = 0.50, \quad \mathbb{P}(B) = 0.60,$$

$$\mathbb{P}(A \text{ and } B) = 0.20.$$

Find:

$$\mathbb{P}(A \text{ or } B).$$

$$0.50 + 0.60 - 0.20.$$

$$\boxed{0.90}.$$

Mixed Example 3

Suppose:

$$\mathbb{P}(A) = 0.25, \quad \mathbb{P}(B) = 0.40,$$

and the events are independent.

Find:

$$\mathbb{P}(A \text{ and } B).$$

Mixed Example 3

Suppose:

$$\mathbb{P}(A) = 0.25, \quad \mathbb{P}(B) = 0.40,$$

and the events are independent.

Find:

$$\mathbb{P}(A \text{ and } B).$$

$$0.25(0.40) = \boxed{0.10}.$$

Mixed Example 4

Suppose:

$$\mathbb{P}(A) = 0.25, \quad \mathbb{P}(B) = 0.40,$$

and the events are independent.

Find:

$$\mathbb{P}(A \text{ or } B).$$

Mixed Example 4

Suppose:

$$\mathbb{P}(A) = 0.25, \quad \mathbb{P}(B) = 0.40,$$

and the events are independent.

Find:

$$\mathbb{P}(A \text{ or } B).$$

First:

$$\mathbb{P}(A \text{ and } B) = 0.25(0.40) = 0.10.$$

Mixed Example 4

Suppose:

$$\mathbb{P}(A) = 0.25, \quad \mathbb{P}(B) = 0.40,$$

and the events are independent.

Find:

$$\mathbb{P}(A \text{ or } B).$$

First:

$$\mathbb{P}(A \text{ and } B) = 0.25(0.40) = 0.10.$$

Then:

$$0.25 + 0.40 - 0.10.$$

Mixed Example 4

Suppose:

$$\mathbb{P}(A) = 0.25, \quad \mathbb{P}(B) = 0.40,$$

and the events are independent.

Find:

$$\mathbb{P}(A \text{ or } B).$$

First:

$$\mathbb{P}(A \text{ and } B) = 0.25(0.40) = 0.10.$$

Then:

$$0.25 + 0.40 - 0.10.$$

$$\boxed{0.55}.$$

Mixed Example 5

Suppose:

$$\mathbb{P}(A \text{ and } B) = 0.18,$$

and:

$$\mathbb{P}(B) = 0.30.$$

Find:

$$\mathbb{P}(A \mid B).$$

Mixed Example 5

Suppose:

$$\mathbb{P}(A \text{ and } B) = 0.18,$$

and:

$$\mathbb{P}(B) = 0.30.$$

Find:

$$\mathbb{P}(A \mid B).$$

$$\frac{0.18}{0.30}.$$

Mixed Example 5

Suppose:

$$\mathbb{P}(A \text{ and } B) = 0.18,$$

and:

$$\mathbb{P}(B) = 0.30.$$

Find:

$$\mathbb{P}(A \mid B).$$

$$\frac{0.18}{0.30}.$$

$$\boxed{0.60}.$$

Mixed Example 6

Suppose:

$$\mathbb{P}(A) = 0.60,$$

and:

$$\mathbb{P}(A \mid B) = 0.60.$$

What does this suggest?

Mixed Example 6

Suppose:

$$\mathbb{P}(A) = 0.60,$$

and:

$$\mathbb{P}(A \mid B) = 0.60.$$

What does this suggest?

Knowing B occurred did not change the probability of A .

Mixed Example 6

Suppose:

$$\mathbb{P}(A) = 0.60,$$

and:

$$\mathbb{P}(A \mid B) = 0.60.$$

What does this suggest?

Knowing B occurred did not change the probability of A .

Therefore:

A and B are independent.

Mixed Example 7

An event has probability:

0.20

on each of 5 independent trials.

Find the probability of at least one occurrence.

Mixed Example 7

An event has probability:

0.20

on each of 5 independent trials.

Find the probability of at least one occurrence.

No occurrence on one trial:

0.80.

Mixed Example 7

An event has probability:

0.20

on each of 5 independent trials.

Find the probability of at least one occurrence.

No occurrence on one trial:

0.80.

No occurrence in all five:

0.80^5 .

Mixed Example 7

An event has probability:

$$0.20$$

on each of 5 independent trials.

Find the probability of at least one occurrence.

No occurrence on one trial:

$$0.80.$$

No occurrence in all five:

$$0.80^5.$$

Therefore:

$$1 - 0.80^5 \approx 0.672.$$

Mixed Example 8

A disease has prevalence:

1%.

A test has:

95%

sensitivity and:

5%

false-positive rate.

Would you expect:

$$\mathbb{P}(D \mid +)$$

to be close to 95%?

Mixed Example 8

A disease has prevalence:

1%.

A test has:

95%

sensitivity and:

5%

false-positive rate.

Would you expect:

$$\mathbb{P}(D \mid +)$$

to be close to 95%?

No.

Mixed Example 8

A disease has prevalence:

1%.

A test has:

95%

sensitivity and:

5%

false-positive rate.

Would you expect:

$$\mathbb{P}(D \mid +)$$

to be close to 95%?

No.

The disease is rare, so false positives from the large healthy population can be numerous.

The chapter in one slide

General “or” rule:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Independent “and” rule:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

Conditional probability:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Independence test:

$$\mathbb{P}(A \mid B) = \mathbb{P}(A)$$

Bayes:

$$\mathbb{P}(D \mid +) = \frac{\mathbb{P}(D)\mathbb{P}(+ \mid D)}{\mathbb{P}(D)\mathbb{P}(+ \mid D) + \mathbb{P}(D^c)\mathbb{P}(+ \mid D^c)}$$

Decision tree for probability rules

When reading a problem:

- 1 Does it say **or**?

Decision tree for probability rules

When reading a problem:

- 1 Does it say **or**?
- 2 Does it say **and**?

Decision tree for probability rules

When reading a problem:

- ① Does it say **or**?
- ② Does it say **and**?
- ③ Are the events mutually exclusive?

Decision tree for probability rules

When reading a problem:

- ① Does it say **or**?
- ② Does it say **and**?
- ③ Are the events mutually exclusive?
- ④ Are the events independent?

Decision tree for probability rules

When reading a problem:

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- ⑤ Does the problem say **given**?

Decision tree for probability rules

When reading a problem:

- 1 Does it say **or**?
- 2 Does it say **and**?
- 3 Are the events mutually exclusive?
- 4 Are the events independent?
- 5 Does the problem say **given**?
- 6 Would a complement make “at least one” easier?

Decision tree for probability rules

When reading a problem:

- ① Does it say **or**?
- ② Does it say **and**?
- ③ Are the events mutually exclusive?
- ④ Are the events independent?
- ⑤ Does the problem say **given**?
- ⑥ Would a complement make “at least one” easier?
- ⑦ Would a table or probability tree make the conditioning clearer?

Common errors checklist

Before submitting an answer, check:

- ▶ Did I confuse mutually exclusive with independent?

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Before submitting an answer, check:

- ▶ Did I confuse mutually exclusive with independent?
- ▶ For “or,” did I subtract the overlap when needed?

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- ▶ Did I use the correct denominator in a conditional probability?

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Common errors checklist

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- ▶ For “and,” did I justify multiplication?
- ▶ Did I use the correct denominator in a conditional probability?
- ▶ Did I reverse $\mathbb{P}(A \mid B)$ and $\mathbb{P}(B \mid A)$?
- ▶ For repeated independent trials, did I use a complement for “at least one”?
- ▶ In a Bayes problem, did I include both true positives and false positives?

Final practice

Suppose:

$$\mathbb{P}(A) = 0.40, \quad \mathbb{P}(B) = 0.50,$$

and:

$$\mathbb{P}(A \text{ and } B) = 0.20.$$

Find:

- 1 $\mathbb{P}(A \text{ or } B)$;
- 2 $\mathbb{P}(A \mid B)$;
- 3 whether A and B are independent.

$$\mathbb{P}(A \text{ or } B) = 0.40 + 0.50 - 0.20.$$

$$\mathbb{P}(A \text{ or } B) = 0.40 + 0.50 - 0.20.$$

$$\boxed{0.70}.$$

$$\mathbb{P}(A \mid B) = \frac{0.20}{0.50}.$$

$$\mathbb{P}(A \mid B) = \frac{0.20}{0.50}.$$

$$\boxed{0.40}.$$

Compare:

$$\mathbb{P}(A \mid B) = 0.40$$

with:

$$\mathbb{P}(A) = 0.40.$$

Final practice: Part 3

Compare:

$$\mathbb{P}(A \mid B) = 0.40$$

with:

$$\mathbb{P}(A) = 0.40.$$

They are equal.

Final practice: Part 3

Compare:

$$\mathbb{P}(A \mid B) = 0.40$$

with:

$$\mathbb{P}(A) = 0.40.$$

They are equal.

Therefore:

A and B are independent.
