

Chapter 10: Probability Theory I

Foundations, Counting, and Basic Probability Rules

Why probability now?

Earlier chapters described data that had already been observed.

Why probability now?

Earlier chapters described data that had already been observed.

Probability asks about what **may happen** before the outcome is known.

Why probability now?

Earlier chapters described data that had already been observed.

Probability asks about what **may happen** before the outcome is known.

Examples:

- ▶ Will a coin land heads?
- ▶ Will a randomly selected voter support a candidate?
- ▶ What sum will appear when two dice are tossed?
- ▶ How often should an event occur in many repetitions?

Why probability now?

Earlier chapters described data that had already been observed.

Probability asks about what **may happen** before the outcome is known.

Examples:

- ▶ Will a coin land heads?
- ▶ Will a randomly selected voter support a candidate?
- ▶ What sum will appear when two dice are tossed?
- ▶ How often should an event occur in many repetitions?

Probability gives us a mathematical language for uncertainty.

Chapter roadmap

By the end of this chapter, students should be able to:

- 1 interpret probability using relative frequency;
- 2 describe a sample space and elementary outcomes;
- 3 use the classical/equal-likelihood model;
- 4 count outcomes with tree diagrams and the multiplication rule;
- 5 distinguish sampling with and without replacement;
- 6 compute event probabilities by counting favorable outcomes;
- 7 use the zero-one and frequency rules;
- 8 use complements, especially for “at least one” problems.

Skill 1: Relative frequency

Suppose an experiment is repeated many times.

If an event E occurs k times in n trials, then its observed relative frequency is

$$\boxed{\frac{k}{n}}.$$

Skill 1: Relative frequency

Suppose an experiment is repeated many times.

If an event E occurs k times in n trials, then its observed relative frequency is

$$\frac{k}{n}.$$

For large n , we use this relative frequency as an estimate of

$$\mathbb{P}(E).$$

The relative-frequency viewpoint

A probability such as

$$\mathbb{P}(E) = 0.60$$

can be interpreted informally as:

The relative-frequency viewpoint

A probability such as

$$\mathbb{P}(E) = 0.60$$

can be interpreted informally as:

If the same type of experiment were repeated many times under similar conditions, we would expect E to occur about

60%

of the time.

The relative-frequency viewpoint

A probability such as

$$\mathbb{P}(E) = 0.60$$

can be interpreted informally as:

If the same type of experiment were repeated many times under similar conditions, we would expect E to occur about

60%

of the time.

This is the **frequentist viewpoint**.

Example 1A: Estimate a probability from data

A treatment is given to 250 patients.

It is successful for 165 patients.

Estimate the probability of success.

Example 1A: Estimate a probability from data

A treatment is given to 250 patients.

It is successful for 165 patients.

Estimate the probability of success.

Relative frequency:

$$\frac{165}{250}.$$

Example 1A: Estimate a probability from data

A treatment is given to 250 patients.

It is successful for 165 patients.

Estimate the probability of success.

Relative frequency:

$$\frac{165}{250}.$$

$$\frac{165}{250} = 0.66.$$

Example 1A: Estimate a probability from data

A treatment is given to 250 patients.

It is successful for 165 patients.

Estimate the probability of success.

Relative frequency:

$$\frac{165}{250}.$$

$$\frac{165}{250} = 0.66.$$

So our estimate is

$$\mathbb{P}(\text{success}) \approx 0.66.$$

Example 1B: Survey data

A random sample of 1200 voters contains:

500

supporters of candidate A.

Estimate:

$\mathbb{P}(\text{random voter supports A}).$

Example 1B: Survey data

A random sample of 1200 voters contains:

500

supporters of candidate A.

Estimate:

$\mathbb{P}(\text{random voter supports A}).$

$$\frac{500}{1200} = 0.4167.$$

Example 1B: Survey data

A random sample of 1200 voters contains:

500

supporters of candidate A.

Estimate:

$\mathbb{P}(\text{random voter supports A}).$

$$\frac{500}{1200} = 0.4167.$$

So:

$$\boxed{\mathbb{P}(A) \approx 0.42}.$$

Why sample size matters

Suppose two studies estimate the same probability.

Study A:

$$6 \text{ successes out of } 10 \Rightarrow 0.60.$$

Study B:

$$6000 \text{ successes out of } 10000 \Rightarrow 0.60.$$

Why sample size matters

Suppose two studies estimate the same probability.

Study A:

$$6 \text{ successes out of } 10 \Rightarrow 0.60.$$

Study B:

$$6000 \text{ successes out of } 10000 \Rightarrow 0.60.$$

The numerical estimate is the same.

Why sample size matters

Suppose two studies estimate the same probability.

Study A:

$$6 \text{ successes out of } 10 \Rightarrow 0.60.$$

Study B:

$$6000 \text{ successes out of } 10000 \Rightarrow 0.60.$$

The numerical estimate is the same.

But the estimate from the much larger study is usually more stable and trustworthy.

Coin tossing and stabilization

For a fair coin, the theoretical probability of heads is

$$\frac{1}{2}.$$

Coin tossing and stabilization

For a fair coin, the theoretical probability of heads is

$$\frac{1}{2}.$$

In a small number of tosses, the observed proportion of heads may be far from 0.5.

Coin tossing and stabilization

For a fair coin, the theoretical probability of heads is

$$\frac{1}{2}.$$

In a small number of tosses, the observed proportion of heads may be far from 0.5. As the number of tosses becomes large, the relative frequency tends to fluctuate less dramatically around

$$0.5.$$

Coin tossing and stabilization

For a fair coin, the theoretical probability of heads is

$$\frac{1}{2}.$$

In a small number of tosses, the observed proportion of heads may be far from 0.5. As the number of tosses becomes large, the relative frequency tends to fluctuate less dramatically around

$$0.5.$$

Large samples reduce random fluctuation, although they do not eliminate it.

Example 1C: Small samples can look strange

Suppose a fair coin is tossed 10 times and gives 7 heads.

Example 1C: Small samples can look strange

Suppose a fair coin is tossed 10 times and gives 7 heads.
Observed relative frequency:

$$\frac{7}{10} = 0.70.$$

Example 1C: Small samples can look strange

Suppose a fair coin is tossed 10 times and gives 7 heads.
Observed relative frequency:

$$\frac{7}{10} = 0.70.$$

Does this prove

$$\mathbb{P}(H) = 0.70?$$

Example 1C: Small samples can look strange

Suppose a fair coin is tossed 10 times and gives 7 heads.

Observed relative frequency:

$$\frac{7}{10} = 0.70.$$

Does this prove

$$\mathbb{P}(H) = 0.70?$$

No.

A short run can differ substantially from the underlying probability.

Skill 2: Elementary outcomes

An **elementary outcome** is one basic possible result of an experiment.

Skill 2: Elementary outcomes

An **elementary outcome** is one basic possible result of an experiment.

Example: toss one die.

The elementary outcomes are:

1, 2, 3, 4, 5, 6.

Skill 2: Elementary outcomes

An **elementary outcome** is one basic possible result of an experiment.

Example: toss one die.

The elementary outcomes are:

1, 2, 3, 4, 5, 6.

The set of all elementary outcomes is called the

sample space.

Sample-space notation

For one fair die:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Sample-space notation

For one fair die:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

For one fair coin:

$$S = \{H, T\}.$$

Sample-space notation

For one fair die:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

For one fair coin:

$$S = \{H, T\}.$$

For two coin tosses:

$$S = \{HH, HT, TH, TT\}.$$

Equal-likelihood method

Suppose the sample space has

n

elementary outcomes and all are equally likely.

Equal-likelihood method

Suppose the sample space has

n

elementary outcomes and all are equally likely.

Then each elementary outcome has probability:

$$\boxed{\frac{1}{n}}.$$

Equal-likelihood method

Suppose the sample space has

n

elementary outcomes and all are equally likely.

Then each elementary outcome has probability:

$$\boxed{\frac{1}{n}}.$$

This is the **classical** or **equal-likelihood** model.

Example 2A: One fair die

For a fair six-sided die:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Example 2A: One fair die

For a fair six-sided die:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

There are:

$$n = 6$$

equally likely outcomes.

Example 2A: One fair die

For a fair six-sided die:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

There are:

$$n = 6$$

equally likely outcomes.

Therefore:

$$\mathbb{P}(3) = \frac{1}{6}.$$

Example 2B: Is “even” an elementary outcome?

For a die toss:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

The event “even” consists of:

$$\{2, 4, 6\}.$$

Example 2B: Is “even” an elementary outcome?

For a die toss:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

The event “even” consists of:

$$\{2, 4, 6\}.$$

So “even” is not one elementary outcome.

It is an **event** containing three elementary outcomes.

An **event** is any collection of elementary outcomes.

An **event** is any collection of elementary outcomes.

Example:

$$E = \{\text{even result on a die}\} = \{2, 4, 6\}.$$

An **event** is any collection of elementary outcomes.

Example:

$$E = \{\text{even result on a die}\} = \{2, 4, 6\}.$$

Example:

$$F = \{\text{result greater than 4}\} = \{5, 6\}.$$

Skill 3: Why counting matters

If outcomes are equally likely, then probability becomes a counting problem.

Skill 3: Why counting matters

If outcomes are equally likely, then probability becomes a counting problem.

We need to know:

- ① how many total outcomes are possible;
- ② how many of them are favorable to the event.

Skill 3: Why counting matters

If outcomes are equally likely, then probability becomes a counting problem.

We need to know:

- ① how many total outcomes are possible;
- ② how many of them are favorable to the event.

For small experiments, we can list outcomes.

For larger experiments, we need counting rules.

Example 3A: Toss two different coins

Suppose we toss a dime and a nickel.

Possible outcomes:

$$(H, H), (H, T), (T, H), (T, T).$$

Example 3A: Toss two different coins

Suppose we toss a dime and a nickel.

Possible outcomes:

$$(H, H), (H, T), (T, H), (T, T).$$

There are:

4

equally likely outcomes.

Example 3A: Toss two different coins

Suppose we toss a dime and a nickel.

Possible outcomes:

$$(H, H), (H, T), (T, H), (T, T).$$

There are:

$$4$$

equally likely outcomes.

Each has probability:

$$\frac{1}{4}.$$

Order matters

For two labeled coins:

(H, T)

means the first coin is heads and the second is tails.

Order matters

For two labeled coins:

(H, T)

means the first coin is heads and the second is tails.

But

(T, H)

means the first is tails and the second is heads.

Order matters

For two labeled coins:

$$(H, T)$$

means the first coin is heads and the second is tails.

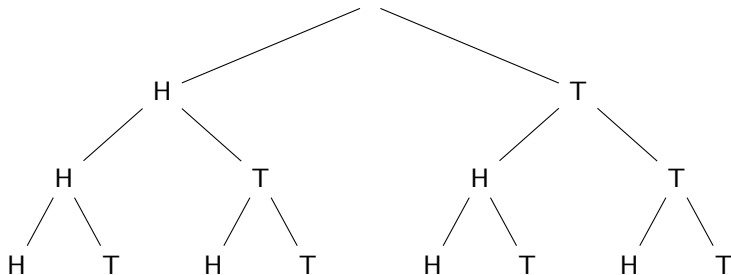
But

$$(T, H)$$

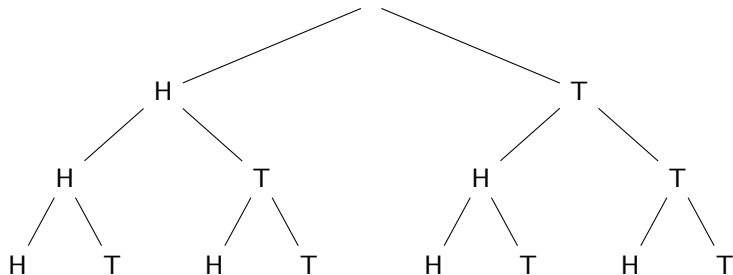
means the first is tails and the second is heads.

These are different elementary outcomes.

Tree diagram for three coin tosses



Tree diagram for three coin tosses



Each complete path corresponds to one elementary outcome.

Example 3B: Enumerate three coin tosses

The sample space is:

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}.$$

Example 3B: Enumerate three coin tosses

The sample space is:

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}.$$

So:

$$|S| = 8.$$

Example 3B: Enumerate three coin tosses

The sample space is:

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}.$$

So:

$$|S| = 8.$$

For fair coins, each sequence has probability:

$$\frac{1}{8}.$$

Skill 4: Multiplication rule

Suppose an experiment has several stages.

If there are:

m

choices at stage 1,

n

choices at stage 2,
and

p

choices at stage 3,

Skill 4: Multiplication rule

Suppose an experiment has several stages.

If there are:

m

choices at stage 1,

n

choices at stage 2,

and

p

choices at stage 3,

then the total number of outcomes is:

$$\boxed{mnp}.$$

Think “choices per stage”

A good routine:

- 1 Identify the stages.

Think “choices per stage”

A good routine:

- ① Identify the stages.
- ② Count the choices available at each stage.

Think “choices per stage”

A good routine:

- ① Identify the stages.
- ② Count the choices available at each stage.
- ③ Multiply.

Think “choices per stage”

A good routine:

- ① Identify the stages.
- ② Count the choices available at each stage.
- ③ Multiply.

Do not add the numbers of choices from different stages.

Example 4A: Four coin tosses

A coin is tossed 4 times.

At each stage there are:

2

possible outcomes.

Example 4A: Four coin tosses

A coin is tossed 4 times.

At each stage there are:

2

possible outcomes.

Therefore:

$$2 \cdot 2 \cdot 2 \cdot 2.$$

Example 4A: Four coin tosses

A coin is tossed 4 times.

At each stage there are:

2

possible outcomes.

Therefore:

$$2 \cdot 2 \cdot 2 \cdot 2.$$

$$= 2^4 = \boxed{16}.$$

Example 4A: Four coin tosses

A coin is tossed 4 times.

At each stage there are:

$$2$$

possible outcomes.

Therefore:

$$2 \cdot 2 \cdot 2 \cdot 2.$$

$$= 2^4 = \boxed{16}.$$

Each exact sequence has probability:

$$\boxed{\frac{1}{16}}.$$

Example 4B: Two dice

Two standard dice are tossed.

First die:

6

choices.

Second die:

6

choices.

Example 4B: Two dice

Two standard dice are tossed.

First die:

6

choices.

Second die:

6

choices.

Total:

$$6 \cdot 6 = \boxed{36}.$$

Example 4B: Two dice

Two standard dice are tossed.

First die:

6

choices.

Second die:

6

choices.

Total:

$$6 \cdot 6 = \boxed{36}.$$

Each ordered pair

(i, j)

has probability:

$$\boxed{\frac{1}{36}}.$$

The two-dice sample space

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The two-dice sample space

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

There are 36 equally likely ordered pairs.

Example 4C: Outfit choices

A student owns:

4 shirts, 3 pairs of pants, 2 pairs of shoes.

How many outfits can be formed by choosing one of each?

Example 4C: Outfit choices

A student owns:

4 shirts, 3 pairs of pants, 2 pairs of shoes.

How many outfits can be formed by choosing one of each?

$$4 \cdot 3 \cdot 2.$$

Example 4C: Outfit choices

A student owns:

4 shirts, 3 pairs of pants, 2 pairs of shoes.

How many outfits can be formed by choosing one of each?

$$4 \cdot 3 \cdot 2.$$

$$\boxed{24}.$$

Example 4D: Code numbers

A code has 5 digits.

Each position can contain any digit from 0 through 9.

Repetition is allowed.

Example 4D: Code numbers

A code has 5 digits.

Each position can contain any digit from 0 through 9.

Repetition is allowed.

Choices per position:

10, 10, 10, 10, 10.

Example 4D: Code numbers

A code has 5 digits.

Each position can contain any digit from 0 through 9.

Repetition is allowed.

Choices per position:

10, 10, 10, 10, 10.

Total:

$$10^5 = \boxed{100000}.$$

Example 4E: First digit restricted

A seven-digit number is formed.

The first digit can be:

$2, 3, \dots, 9,$

so there are 8 choices.

Each remaining digit has 10 choices.

Example 4E: First digit restricted

A seven-digit number is formed.

The first digit can be:

$$2, 3, \dots, 9,$$

so there are 8 choices.

Each remaining digit has 10 choices.

Total:

$$8 \cdot 10^6.$$

Example 4E: First digit restricted

A seven-digit number is formed.

The first digit can be:

$$2, 3, \dots, 9,$$

so there are 8 choices.

Each remaining digit has 10 choices.

Total:

$$8 \cdot 10^6.$$

$$\boxed{8,000,000}.$$

Skill 5: With replacement

A selected item is returned before the next draw.

Skill 5: With replacement

A selected item is returned before the next draw.
So the number of available choices is restored.

Skill 5: With replacement

A selected item is returned before the next draw.

So the number of available choices is restored.

Example: draw from 10 numbered balls three times, replacing the ball each time.

Choices:

10, 10, 10.

Skill 5: With replacement

A selected item is returned before the next draw.

So the number of available choices is restored.

Example: draw from 10 numbered balls three times, replacing the ball each time.

Choices:

10, 10, 10.

Total:

$$10^3 = 1000.$$

Skill 5: Without replacement

A selected item is **not** returned.

Skill 5: Without replacement

A selected item is **not** returned.

So the number of choices decreases.

Skill 5: Without replacement

A selected item is **not** returned.

So the number of choices decreases.

For 10 numbered balls drawn three times without replacement:

10

choices first,

9

choices second,

8

choices third.

Skill 5: Without replacement

A selected item is **not** returned.

So the number of choices decreases.

For 10 numbered balls drawn three times without replacement:

10

choices first,

9

choices second,

8

choices third.

Total:

$$10 \cdot 9 \cdot 8 = 720.$$

Example 5A: With versus without replacement

A box contains 5 distinct balls.
Two balls are selected in order.

Example 5A: With versus without replacement

A box contains 5 distinct balls.

Two balls are selected in order.

With replacement:

$$5 \cdot 5 = \boxed{25}.$$

Example 5A: With versus without replacement

A box contains 5 distinct balls.

Two balls are selected in order.

With replacement:

$$5 \cdot 5 = \boxed{25}.$$

Without replacement:

$$5 \cdot 4 = \boxed{20}.$$

Example 5B: Three cards from a small deck

A deck contains 8 distinct cards.

Three are drawn in order without replacement.

Example 5B: Three cards from a small deck

A deck contains 8 distinct cards.

Three are drawn in order without replacement.

8

choices for the first card.

7

for the second.

6

for the third.

Example 5B: Three cards from a small deck

A deck contains 8 distinct cards.

Three are drawn in order without replacement.

8

choices for the first card.

7

for the second.

6

for the third.

$$8 \cdot 7 \cdot 6 = 336.$$

Skill 6: Equal-likelihood principle

Suppose there are:

 n

equally likely elementary outcomes.

If an event E contains

 k

of them, then:

$$\mathbb{P}(E) = \frac{k}{n}.$$

Skill 6: Equal-likelihood principle

Suppose there are:

$$n$$

equally likely elementary outcomes.

If an event E contains

$$k$$

of them, then:

$$\mathbb{P}(E) = \frac{k}{n}.$$

In words:

$$\frac{\text{favorable outcomes}}{\text{total equally likely outcomes}}.$$

The basic probability workflow

For an equally likely sample space:

- 1 Count the total outcomes.

The basic probability workflow

For an equally likely sample space:

- 1 Count the total outcomes.
- 2 Count the favorable outcomes.

The basic probability workflow

For an equally likely sample space:

- 1 Count the total outcomes.
- 2 Count the favorable outcomes.
- 3 Divide:

$$\mathbb{P}(E) = \frac{\#E}{\#S}.$$

The basic probability workflow

For an equally likely sample space:

- 1 Count the total outcomes.
- 2 Count the favorable outcomes.
- 3 Divide:

$$\mathbb{P}(E) = \frac{\#E}{\#S}.$$

- 4 Check that the answer lies between 0 and 1.

Example 6A: Even result on one die

Event:

$$E = \{\text{even result}\}.$$

Example 6A: Even result on one die

Event:

$$E = \{\text{even result}\}.$$

Favorable outcomes:

$$\{2, 4, 6\}.$$

So:

$$k = 3.$$

Example 6A: Even result on one die

Event:

$$E = \{\text{even result}\}.$$

Favorable outcomes:

$$\{2, 4, 6\}.$$

So:

$$k = 3.$$

Total outcomes:

$$n = 6.$$

Example 6A: Even result on one die

Event:

$$E = \{\text{even result}\}.$$

Favorable outcomes:

$$\{2, 4, 6\}.$$

So:

$$k = 3.$$

Total outcomes:

$$n = 6.$$

$$\mathbb{P}(E) = \frac{3}{6} = \boxed{\frac{1}{2}}.$$

Example 6B: Sum of five with two dice

The favorable ordered pairs are:

$$(1, 4), (2, 3), (3, 2), (4, 1).$$

Example 6B: Sum of five with two dice

The favorable ordered pairs are:

$$(1, 4), (2, 3), (3, 2), (4, 1).$$

So:

$$k = 4.$$

Total outcomes:

$$n = 36.$$

Example 6B: Sum of five with two dice

The favorable ordered pairs are:

$$(1, 4), (2, 3), (3, 2), (4, 1).$$

So:

$$k = 4.$$

Total outcomes:

$$n = 36.$$

$$\mathbb{P}(\text{sum } 5) = \frac{4}{36} = \boxed{\frac{1}{9}}.$$

Example 6C: Sum of seven with two dice

Favorable outcomes:

$(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1).$

Example 6C: Sum of seven with two dice

Favorable outcomes:

$(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1).$

There are:

6

favorable outcomes.

Example 6C: Sum of seven with two dice

Favorable outcomes:

$(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1).$

There are:

6

favorable outcomes.

Therefore:

$$\mathbb{P}(\text{sum } 7) = \frac{6}{36} = \boxed{\frac{1}{6}}.$$

Example 6D: Exactly two heads in four tosses

Four tosses give:

$$2^4 = 16$$

equally likely sequences.

Example 6D: Exactly two heads in four tosses

Four tosses give:

$$2^4 = 16$$

equally likely sequences.

Exactly two heads occurs in:

HHTT, HTHT, HTTH,
THHT, THTH, TTHH.

Example 6D: Exactly two heads in four tosses

Four tosses give:

$$2^4 = 16$$

equally likely sequences.

Exactly two heads occurs in:

HHTT, HTHT, HTTH,
THHT, THTH, TTHH.

There are 6 favorable sequences.

Example 6D: Exactly two heads in four tosses

Four tosses give:

$$2^4 = 16$$

equally likely sequences.

Exactly two heads occurs in:

$$HHTT, HTHT, HTTH, \\ THHT, THTH, TTHH.$$

There are 6 favorable sequences.

$$\mathbb{P}(\text{exactly two heads}) = \frac{6}{16} = \boxed{\frac{3}{8}}.$$

Skill 7: Zero-one principle

Every probability satisfies:

$$0 \leq \mathbb{P}(E) \leq 1.$$

Skill 7: Zero-one principle

Every probability satisfies:

$$0 \leq \mathbb{P}(E) \leq 1.$$

Impossible event:

$$\mathbb{P}(E) = 0.$$

Certain event:

$$\mathbb{P}(E) = 1.$$

Skill 7: Zero-one principle

Every probability satisfies:

$$0 \leq \mathbb{P}(E) \leq 1.$$

Impossible event:

$$\mathbb{P}(E) = 0.$$

Certain event:

$$\mathbb{P}(E) = 1.$$

The probabilities of all elementary outcomes in the sample space add to:

$$1.$$

Example 7A: Spot impossible answers

Which could be a probability?

-0.2 , 0.35 , 1.4 , $\frac{7}{8}$.

Example 7A: Spot impossible answers

Which could be a probability?

-0.2 , 0.35 , 1.4 , $\frac{7}{8}$.

Valid:

$0.35, \frac{7}{8}$.

Example 7A: Spot impossible answers

Which could be a probability?

$$-0.2, \quad 0.35, \quad 1.4, \quad \frac{7}{8}.$$

Valid:

$$\boxed{0.35, \frac{7}{8}}.$$

Invalid:

$$-0.2 < 0, \quad 1.4 > 1.$$

Example 7B: Certain and impossible events

Roll one standard die.

Event A:

result is between 1 and 6.

Example 7B: Certain and impossible events

Roll one standard die.

Event A:

result is between 1 and 6.

This is certain:

$$\mathbb{P}(A) = 1.$$

Example 7B: Certain and impossible events

Roll one standard die.

Event A:

result is between 1 and 6.

This is certain:

$$\mathbb{P}(A) = 1.$$

Event B:

result is 9.

This is impossible:

$$\mathbb{P}(B) = 0.$$

Skill 8: Frequency rule

If:

$$\mathbb{P}(E) = p$$

and the experiment is repeated:

$$n$$

times,

then we expect E to occur approximately:

$$\boxed{np}$$

times.

Skill 8: Frequency rule

If:

$$\mathbb{P}(E) = p$$

and the experiment is repeated:

$$n$$

times,

then we expect E to occur approximately:

$$\boxed{np}$$

times.

This is an approximate long-run count, not a guarantee.

Example 8A: Expected number of sixes

A fair die is rolled:

600

times.

Example 8A: Expected number of sixes

A fair die is rolled:

600

times.

$$\mathbb{P}(6) = \frac{1}{6}.$$

Expected number of sixes:

$$600 \left(\frac{1}{6} \right).$$

Example 8A: Expected number of sixes

A fair die is rolled:

600

times.

$$\mathbb{P}(6) = \frac{1}{6}.$$

Expected number of sixes:

$$600 \left(\frac{1}{6} \right).$$

$$\boxed{100}.$$

Example 8B: Rare event

Suppose an event has probability:

0.024.

In a population of:

1000,

the expected number is:

$1000(0.024)$.

Example 8B: Rare event

Suppose an event has probability:

0.024.

In a population of:

1000,

the expected number is:

$1000(0.024).$

24.

Example 8B: Rare event

Suppose an event has probability:

0.024.

In a population of:

1000,

the expected number is:

$1000(0.024)$.

24.

For:

10000

trials:

$10000(0.024) =$ 240.

Expected does not mean exact

If we expect:

100

sixes in 600 die rolls,

Expected does not mean exact

If we expect:

100

sixes in 600 die rolls,
the actual count might be:

94, 103, 108, ...

Expected does not mean exact

If we expect:

100

sixes in 600 die rolls,
the actual count might be:

94, 103, 108, ...

The value 100 is a long-run benchmark, not a promise for every experiment.

Skill 9: Complement of an event

The complement of E , written

$$E^c,$$

is the event that E does **not** occur.

Skill 9: Complement of an event

The complement of E , written

$$E^c,$$

is the event that E does **not** occur.

The two events:

$$E \quad \text{and} \quad E^c$$

cover all possibilities and cannot both happen.

Skill 9: Complement of an event

The complement of E , written

$$E^c,$$

is the event that E does **not** occur.

The two events:

$$E \quad \text{and} \quad E^c$$

cover all possibilities and cannot both happen.

Therefore:

$$\boxed{\mathbb{P}(E) + \mathbb{P}(E^c) = 1}.$$

Complement rule

From:

$$\mathbb{P}(E) + \mathbb{P}(E^c) = 1,$$

we get:

$$\boxed{\mathbb{P}(E) = 1 - \mathbb{P}(E^c)}.$$

Complement rule

From:

$$\mathbb{P}(E) + \mathbb{P}(E^c) = 1,$$

we get:

$$\boxed{\mathbb{P}(E) = 1 - \mathbb{P}(E^c)}.$$

This is especially useful for events described by:

“at least one.”

Example 9A: At least one head in three tosses

Let:

$$E = \{\text{at least one head}\}.$$

Example 9A: At least one head in three tosses

Let:

$$E = \{\text{at least one head}\}.$$

The complement is:

$$E^c = \{\text{no heads}\}.$$

Example 9A: At least one head in three tosses

Let:

$$E = \{\text{at least one head}\}.$$

The complement is:

$$E^c = \{\text{no heads}\}.$$

“No heads” means:

$$TTT.$$

Example 9A: At least one head in three tosses

Let:

$$E = \{\text{at least one head}\}.$$

The complement is:

$$E^c = \{\text{no heads}\}.$$

“No heads” means:

$$TTT.$$

$$\mathbb{P}(E^c) = \frac{1}{8}.$$

Therefore:

$$\mathbb{P}(E) = 1 - \frac{1}{8} = \boxed{\frac{7}{8}}.$$

Example 9B: At least one six in two rolls

Let:

$$E = \{\text{at least one six}\}.$$

Example 9B: At least one six in two rolls

Let:

$$E = \{\text{at least one six}\}.$$

The complement is:

$$E^c = \{\text{no sixes}\}.$$

For each roll:

$$\mathbb{P}(\text{not six}) = \frac{5}{6}.$$

Example 9B: At least one six in two rolls

Let:

$$E = \{\text{at least one six}\}.$$

The complement is:

$$E^c = \{\text{no sixes}\}.$$

For each roll:

$$\mathbb{P}(\text{not six}) = \frac{5}{6}.$$

Two rolls:

$$\mathbb{P}(E^c) = \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36}.$$

Example 9B: At least one six in two rolls

Let:

$$E = \{\text{at least one six}\}.$$

The complement is:

$$E^c = \{\text{no sixes}\}.$$

For each roll:

$$\mathbb{P}(\text{not six}) = \frac{5}{6}.$$

Two rolls:

$$\mathbb{P}(E^c) = \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36}.$$

Therefore:

$$\mathbb{P}(E) = 1 - \frac{25}{36} = \boxed{\frac{11}{36}}.$$

Example 9C: At least one tail in five tosses

Let:

$$E = \{\text{at least one tail}\}.$$

Example 9C: At least one tail in five tosses

Let:

$$E = \{\text{at least one tail}\}.$$

The complement is:

$$E^c = \{\text{all heads}\}.$$

Example 9C: At least one tail in five tosses

Let:

$$E = \{\text{at least one tail}\}.$$

The complement is:

$$E^c = \{\text{all heads}\}.$$

$$\mathbb{P}(E^c) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

Example 9C: At least one tail in five tosses

Let:

$$E = \{\text{at least one tail}\}.$$

The complement is:

$$E^c = \{\text{all heads}\}.$$

$$\mathbb{P}(E^c) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

Thus:

$$\mathbb{P}(E) = 1 - \frac{1}{32} = \boxed{\frac{31}{32}}.$$

When complements save work

Direct method:

count every case with 1, 2, 3, 4, or 5 tails.

When complements save work

Direct method:

count every case with 1, 2, 3, 4, or 5 tails.

Complement method:

$$1 - \mathbb{P}(\text{no tails}).$$

When complements save work

Direct method:

count every case with 1, 2, 3, 4, or 5 tails.

Complement method:

$$1 - \mathbb{P}(\text{no tails}).$$

For “at least one,” the complement is often much simpler.

Skill 10: Birthday problem

Question:

What is the probability that at least two people in a group share a birthday?

Skill 10: Birthday problem

Question:

What is the probability that at least two people in a group share a birthday?

Directly counting all ways that some birthday is repeated is complicated.

Skill 10: Birthday problem

Question:

What is the probability that at least two people in a group share a birthday?

Directly counting all ways that some birthday is repeated is complicated.

The complement is much simpler:

everyone has a different birthday.

Example 10A: Ten people, total outcomes

Ignore February 29.

Each of 10 people can have any of:

365

birthdays.

Example 10A: Ten people, total outcomes

Ignore February 29.

Each of 10 people can have any of:

365

birthdays.

Total elementary outcomes:

$$365^{10}.$$

Example 10A: Count all-different birthdays

For everyone to have different birthdays:

Person 1:

365

choices.

Person 2:

364.

Person 3:

363.

Example 10A: Count all-different birthdays

For everyone to have different birthdays:

Person 1:

365

choices.

Person 2:

364.

Person 3:

363.

Continue down to Person 10:

356.

Example 10A: Count all-different birthdays

For everyone to have different birthdays:

Person 1:

365

choices.

Person 2:

364.

Person 3:

363.

Continue down to Person 10:

356.

Favorable outcomes for “all different”:

$$365 \cdot 364 \cdot 363 \cdots 356.$$

Example 10A: Probability of a shared birthday

$$\mathbb{P}(\text{all different}) = \frac{365 \cdot 364 \cdot 363 \cdots 356}{365^{10}}.$$

Example 10A: Probability of a shared birthday

$$\mathbb{P}(\text{all different}) = \frac{365 \cdot 364 \cdot 363 \cdots 356}{365^{10}}.$$

Numerically:

$$\mathbb{P}(\text{all different}) \approx 0.883.$$

Example 10A: Probability of a shared birthday

$$\mathbb{P}(\text{all different}) = \frac{365 \cdot 364 \cdot 363 \cdots 356}{365^{10}}.$$

Numerically:

$$\mathbb{P}(\text{all different}) \approx 0.883.$$

Therefore:

$$\mathbb{P}(\text{at least one shared birthday}) = 1 - 0.883.$$

Example 10A: Probability of a shared birthday

$$\mathbb{P}(\text{all different}) = \frac{365 \cdot 364 \cdot 363 \cdots 356}{365^{10}}.$$

Numerically:

$$\mathbb{P}(\text{all different}) \approx 0.883.$$

Therefore:

$$\mathbb{P}(\text{at least one shared birthday}) = 1 - 0.883.$$

$$\boxed{\approx 0.117}.$$

So the chance is about 12%.

Birthday problem: the important idea

The important skill is not memorizing the answer.

Birthday problem: the important idea

The important skill is not memorizing the answer.

The structure is:

- 1 “At least one match” is difficult to count directly.

Birthday problem: the important idea

The important skill is not memorizing the answer.

The structure is:

- ① “At least one match” is difficult to count directly.
- ② “No matches” is easy to count stage by stage.

Birthday problem: the important idea

The important skill is not memorizing the answer.

The structure is:

- ① “At least one match” is difficult to count directly.
- ② “No matches” is easy to count stage by stage.
- ③ Use:

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c).$$

Skill 11: Do not assume equal likelihood automatically

The formula:

$$\mathbb{P}(E) = \frac{\text{favorable outcomes}}{\text{total outcomes}}$$

works only when the elementary outcomes being counted are equally likely.

Skill 11: Do not assume equal likelihood automatically

The formula:

$$\mathbb{P}(E) = \frac{\text{favorable outcomes}}{\text{total outcomes}}$$

works only when the elementary outcomes being counted are equally likely.

Having two named possibilities does **not** mean each has probability $\frac{1}{2}$.

Example 11A: Passing or failing

A student either:

passes

or

fails.

Example 11A: Passing or failing

A student either:

passes

or

fails.

There are two possible labels.

But we cannot conclude:

$$\mathbb{P}(\text{pass}) = \frac{1}{2}.$$

Example 11A: Passing or failing

A student either:

passes

or

fails.

There are two possible labels.

But we cannot conclude:

$$\mathbb{P}(\text{pass}) = \frac{1}{2}.$$

The outcomes are not automatically equally likely.

We would need data or a probability model.

Example 11B: Blood type

Possible blood types include:

$O, A, B, AB.$

Example 11B: Blood type

Possible blood types include:

$O, A, B, AB.$

There are four categories.

But it would be wrong to assign:

$$\frac{1}{4}$$

to each without evidence.

Example 11B: Blood type

Possible blood types include:

$O, A, B, AB.$

There are four categories.

But it would be wrong to assign:

$$\frac{1}{4}$$

to each without evidence.

Observed population frequencies differ across the categories.

Example 11C: Unequal spinner sectors

A spinner has three numbered sectors, but the sectors have different sizes.

Example 11C: Unequal spinner sectors

A spinner has three numbered sectors, but the sectors have different sizes.
The three labels:

1, 2, 3

are not equally likely unless the sectors have equal area.

Example 11C: Unequal spinner sectors

A spinner has three numbered sectors, but the sectors have different sizes.

The three labels:

1, 2, 3

are not equally likely unless the sectors have equal area.

Probability depends on the physical model, not merely the number of labels.

Mixed Example 1

A fair coin is tossed 5 times.

How many exact sequences are possible?

Mixed Example 1

A fair coin is tossed 5 times.

How many exact sequences are possible?

There are 2 choices on each toss.

Mixed Example 1

A fair coin is tossed 5 times.

How many exact sequences are possible?

There are 2 choices on each toss.

$$2^5 = \boxed{32}.$$

Mixed Example 2

Two fair dice are rolled.

Find:

$$\mathbb{P}(\text{sum} = 2).$$

Mixed Example 2

Two fair dice are rolled.

Find:

$$\mathbb{P}(\text{sum} = 2).$$

Only one ordered pair works:

$$(1, 1).$$

Mixed Example 2

Two fair dice are rolled.

Find:

$$\mathbb{P}(\text{sum} = 2).$$

Only one ordered pair works:

$$(1, 1).$$

Total outcomes:

$$36.$$

Mixed Example 2

Two fair dice are rolled.

Find:

$$\mathbb{P}(\text{sum} = 2).$$

Only one ordered pair works:

$$(1, 1).$$

Total outcomes:

$$36.$$

$$\boxed{\mathbb{P}(\text{sum} = 2) = \frac{1}{36}}.$$

Mixed Example 3

Two fair dice are rolled.

Find:

$$\mathbb{P}(\text{sum} = 12).$$

Mixed Example 3

Two fair dice are rolled.

Find:

$$\mathbb{P}(\text{sum} = 12).$$

Only:

$$(6, 6)$$

works.

Mixed Example 3

Two fair dice are rolled.

Find:

$$\mathbb{P}(\text{sum} = 12).$$

Only:

$$(6, 6)$$

works.

Thus:

$$\boxed{\frac{1}{36}}.$$

Mixed Example 3

Two fair dice are rolled.

Find:

$$\mathbb{P}(\text{sum} = 12).$$

Only:

$$(6, 6)$$

works.

Thus:

$$\boxed{\frac{1}{36}}.$$

So sums 2 and 12 are equally likely.

Mixed Example 4

Two fair dice are rolled.

Which is more likely:

sum 7

or

sum 12?

Mixed Example 4

Two fair dice are rolled.

Which is more likely:

sum 7

or

sum 12?

Sum 7:

6

favorable outcomes.

Sum 12:

1

favorable outcome.

Mixed Example 4

Two fair dice are rolled.

Which is more likely:

sum 7

or

sum 12?

Sum 7:

6

favorable outcomes.

Sum 12:

1

favorable outcome.

So:

sum 7 is six times as likely.

Mixed Example 5

A bowl contains 8 distinct balls.

Three are selected in order without replacement.

How many outcomes?

Mixed Example 5

A bowl contains 8 distinct balls.

Three are selected in order without replacement.

How many outcomes?

$$8 \cdot 7 \cdot 6.$$

Mixed Example 5

A bowl contains 8 distinct balls.

Three are selected in order without replacement.

How many outcomes?

$$8 \cdot 7 \cdot 6.$$

$$\boxed{336}.$$

Mixed Example 6

A fair coin is tossed 4 times.
Find the probability of:

HHHH.

Mixed Example 6

A fair coin is tossed 4 times.
Find the probability of:

HHHH.

There are:

16

equally likely exact sequences.
Only one is *HHHH.*

Mixed Example 6

A fair coin is tossed 4 times.
Find the probability of:

HHHH.

There are:

16

equally likely exact sequences.
Only one is *HHHH*.

$$\boxed{\frac{1}{16}}.$$

Mixed Example 7

A fair coin is tossed 4 times.
Find the probability of:

exactly two heads.

Mixed Example 7

A fair coin is tossed 4 times.

Find the probability of:

exactly two heads.

There are 6 favorable sequences.

Total:

16.

Mixed Example 7

A fair coin is tossed 4 times.

Find the probability of:

exactly two heads.

There are 6 favorable sequences.

Total:

16.

$$\boxed{\frac{6}{16} = \frac{3}{8}}.$$

Mixed Example 8

A fair die is rolled 900 times.

Approximately how many times should we see a result greater than 4?

Mixed Example 8

A fair die is rolled 900 times.

Approximately how many times should we see a result greater than 4?

Favorable results:

5, 6.

Mixed Example 8

A fair die is rolled 900 times.

Approximately how many times should we see a result greater than 4?

Favorable results:

5, 6.

$$\mathbb{P}(\text{greater than 4}) = \frac{2}{6} = \frac{1}{3}.$$

Mixed Example 8

A fair die is rolled 900 times.

Approximately how many times should we see a result greater than 4?

Favorable results:

5, 6.

$$\mathbb{P}(\text{greater than 4}) = \frac{2}{6} = \frac{1}{3}.$$

Expected count:

$$900 \left(\frac{1}{3} \right) = \boxed{300}.$$

Mixed Example 9

A fair coin is tossed 8 times.

Find:

$$\mathbb{P}(\text{at least one head}).$$

Mixed Example 9

A fair coin is tossed 8 times.

Find:

$\mathbb{P}(\text{at least one head}).$

Complement:

all tails.

Mixed Example 9

A fair coin is tossed 8 times.

Find:

$$\mathbb{P}(\text{at least one head}).$$

Complement:

all tails.

$$\mathbb{P}(\text{all tails}) = \left(\frac{1}{2}\right)^8 = \frac{1}{256}.$$

Mixed Example 9

A fair coin is tossed 8 times.

Find:

$$\mathbb{P}(\text{at least one head}).$$

Complement:

all tails.

$$\mathbb{P}(\text{all tails}) = \left(\frac{1}{2}\right)^8 = \frac{1}{256}.$$

Therefore:

$$\boxed{1 - \frac{1}{256} = \frac{255}{256}}.$$

Mixed Example 10

A student says:

“There are two possibilities: it rains or it does not rain. Therefore the probability of rain is $\frac{1}{2}$.”

Mixed Example 10

A student says:

“There are two possibilities: it rains or it does not rain. Therefore the probability of rain is $\frac{1}{2}$.”

What is wrong?

Mixed Example 10

A student says:

“There are two possibilities: it rains or it does not rain. Therefore the probability of rain is $\frac{1}{2}$.”

What is wrong?

The two outcomes are not known to be equally likely.

Mixed Example 10

A student says:

“There are two possibilities: it rains or it does not rain. Therefore the probability of rain is $\frac{1}{2}$.”

What is wrong?

The two outcomes are not known to be equally likely.

The equal-likelihood method cannot be applied without justification.

The chapter in one slide

Relative frequency:

$$\mathbb{P}(E) \approx \frac{\# \text{ occurrences of } E}{\# \text{ trials}}$$

Equal-likelihood method:

$$\mathbb{P}(E) = \frac{\# \text{ favorable outcomes}}{\# \text{ total equally likely outcomes}}$$

Multiplication rule:

$$\# \text{ outcomes} = (\# \text{ choices at stage 1})(\# \text{ choices at stage 2}) \cdots$$

Complement rule:

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c)$$

Frequency rule:

$$\text{expected count} \approx n\mathbb{P}(E)$$

A practical probability workflow

When solving a basic probability problem:

- 1 Identify the experiment.

A practical probability workflow

When solving a basic probability problem:

- ① Identify the experiment.
- ② Describe the elementary outcomes.

A practical probability workflow

When solving a basic probability problem:

- ① Identify the experiment.
- ② Describe the elementary outcomes.
- ③ Decide whether they are equally likely.

A practical probability workflow

When solving a basic probability problem:

- ① Identify the experiment.
- ② Describe the elementary outcomes.
- ③ Decide whether they are equally likely.
- ④ Count the total outcomes.

A practical probability workflow

When solving a basic probability problem:

- ① Identify the experiment.
- ② Describe the elementary outcomes.
- ③ Decide whether they are equally likely.
- ④ Count the total outcomes.
- ⑤ Count the favorable outcomes.

A practical probability workflow

When solving a basic probability problem:

- ① Identify the experiment.
- ② Describe the elementary outcomes.
- ③ Decide whether they are equally likely.
- ④ Count the total outcomes.
- ⑤ Count the favorable outcomes.
- ⑥ Use a complement if the direct event is complicated.

A practical probability workflow

When solving a basic probability problem:

- ① Identify the experiment.
- ② Describe the elementary outcomes.
- ③ Decide whether they are equally likely.
- ④ Count the total outcomes.
- ⑤ Count the favorable outcomes.
- ⑥ Use a complement if the direct event is complicated.
- ⑦ Check that the final probability lies between 0 and 1.

Common errors checklist

Before submitting an answer, check:

- ▶ Did I distinguish an event from an elementary outcome?

Common errors checklist

Before submitting an answer, check:

- ▶ Did I distinguish an event from an elementary outcome?
- ▶ Did I keep ordered outcomes separate when order matters?

Common errors checklist

Before submitting an answer, check:

- ▶ Did I distinguish an event from an elementary outcome?
- ▶ Did I keep ordered outcomes separate when order matters?
- ▶ Did I multiply the choices across stages?

Common errors checklist

Before submitting an answer, check:

- ▶ Did I distinguish an event from an elementary outcome?
- ▶ Did I keep ordered outcomes separate when order matters?
- ▶ Did I multiply the choices across stages?
- ▶ Did I account for replacement or lack of replacement?

Common errors checklist

Before submitting an answer, check:

- ▶ Did I distinguish an event from an elementary outcome?
- ▶ Did I keep ordered outcomes separate when order matters?
- ▶ Did I multiply the choices across stages?
- ▶ Did I account for replacement or lack of replacement?
- ▶ Are the outcomes really equally likely?

Common errors checklist

Before submitting an answer, check:

- ▶ Did I distinguish an event from an elementary outcome?
- ▶ Did I keep ordered outcomes separate when order matters?
- ▶ Did I multiply the choices across stages?
- ▶ Did I account for replacement or lack of replacement?
- ▶ Are the outcomes really equally likely?
- ▶ Did I count favorable outcomes correctly?

Common errors checklist

Before submitting an answer, check:

- ▶ Did I distinguish an event from an elementary outcome?
- ▶ Did I keep ordered outcomes separate when order matters?
- ▶ Did I multiply the choices across stages?
- ▶ Did I account for replacement or lack of replacement?
- ▶ Are the outcomes really equally likely?
- ▶ Did I count favorable outcomes correctly?
- ▶ Would a complement make the problem easier?

Common errors checklist

Before submitting an answer, check:

- ▶ Did I distinguish an event from an elementary outcome?
- ▶ Did I keep ordered outcomes separate when order matters?
- ▶ Did I multiply the choices across stages?
- ▶ Did I account for replacement or lack of replacement?
- ▶ Are the outcomes really equally likely?
- ▶ Did I count favorable outcomes correctly?
- ▶ Would a complement make the problem easier?
- ▶ Is my answer between 0 and 1?

A fair coin is tossed 3 times.

Find:

- 1 the number of elementary outcomes;
- 2 the probability of exactly two heads;
- 3 the probability of at least one head.

Final practice: Part 1

Three tosses give:

$$2 \cdot 2 \cdot 2.$$

Final practice: Part 1

Three tosses give:

$$2 \cdot 2 \cdot 2.$$

8

elementary outcomes.

Exactly two heads:

HHT, HTH, THH.

Final practice: Part 2

Exactly two heads:

HHT, HTH, THH.

There are:

3

favorable outcomes out of 8.

Final practice: Part 2

Exactly two heads:

HHT, HTH, THH.

There are:

3

favorable outcomes out of 8.

$$\mathbb{P}(\text{exactly two heads}) = \frac{3}{8}.$$

Final practice: Part 3

At least one head is easiest by complement.

Final practice: Part 3

At least one head is easiest by complement.

Complement:

TTT.

Final practice: Part 3

At least one head is easiest by complement.

Complement:

TTT .

$$\mathbb{P}(TTT) = \frac{1}{8}.$$

Final practice: Part 3

At least one head is easiest by complement.

Complement:

TTT .

$$\mathbb{P}(TTT) = \frac{1}{8}.$$

Therefore:

$$\mathbb{P}(\text{at least one head}) = 1 - \frac{1}{8} = \frac{7}{8}.$$