Read the entire page 795 for proof of the theorem below.

$\boxed{5}$ **Theorem**

If f has a power series representation (expansion) at a, that is, if

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a|<R$$

then its coefficients are given by the formula

$$c_n=\frac{f^{(n)}(a)}{n!}$$

$\boxed{6}$
$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

Taylor series of the function f **at** a (or **about** a or **centered at** a).

For the special case $a=0$ the Taylor series becomes

$\boxed{7}$
$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

The name is **Maclaurin series**.

$\boxed{8}$ **Theorem**

If $f(x)=T_n(x)+R_n(x)$, where $T_n(x)$ is the nth-degree Taylor polynomial of f at a, and if

$$\lim_{n\to\infty} R_n(x)=0$$

For $|x-a|<R$, then f is equal to the sum of its Taylor series on the interval $|x-a|<R$.

$\boxed{9}$ **Taylor's Inequality**

If $\left|f^{(n+1)}(x)\right|\le M$ for $|x-a|\le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a|\le d$$

$\boxed{10}$
$$\lim_{n\to\infty}\frac{x^n}{n!}=0 \qquad \text{for every real number } x$$

$\boxed{11}$
$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \text{for all } x$$

$\boxed{12}$
$$e=\sum_{n=0}^{\infty}\frac{1}{n!}=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$$

$\boxed{17}$ **The Binomial Series**

If k is any real number and $|x|<1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Table 1 Important Maclaurin Series and Their Radii of Convergence

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R=1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R=\infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R=\infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R=\infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R=1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \cdots$	$R=1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$R=1$

Starting below are definitions given in Thomas' Calculus textbook

Definition

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by** f **at** $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series of** f is the Taylor series generated by f at $x=0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

Definition

Let f be a function with derivatives of order k for $k=1,2,\ldots,N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order** n generated by f at $x=a$ is the polynomial

$$P_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(k)}(a)}{k!}(x-a)^k+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem 23 - Taylor's Theorem

If f and its first n derivatives $f',f'',\ldots,f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b)=f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(b-a)^n+\frac{f^{(n+1)}(a)}{(n+1)!}(b-a)^{n+1}.$$

Taylor's Formula

Let f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_n(x), \tag{1}$$

where

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \tag{2}$$

If $R_n(x)\to 0$ as $n\to\infty$ for all $x\in I$, we say that the Taylor series generated by f at $x=a$ **converges** to f on I, and we write

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Theorem 24 - The Remainder Estimation Theorem

If there is a positive constant M such that $\left|f^{(n+1)}(t)\right|\le M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)|\le M\frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality hold for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

The Taylor series generated by $f(x)=(1+x)^m$, where m is constant, is

$$1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\cdots+\frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}x^k+\cdots. \qquad (1)$$

The Binomial Series

For $-1<x<1$,

$$(1+x)^m=1+\sum_{k=1}^{\infty}\binom{m}{k}x^k ,$$

where we define

$$\binom{m}{1}=m, \qquad \binom{m}{2}=\frac{m(m-1)}{2!},$$

and

$$\binom{m}{k}=\frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k\ge 3.$$

$$\frac{1}{1+t^2}=1-t^2+t^4-t^6+\cdots+(-1)^n t^{2n}+\frac{(-1)^{n+1}t^{2(n+1)}}{1+t^2} \qquad (2)$$

$$\tan^{-1}x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2x+1}}{2n+1} \qquad |x|\le 1$$

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \qquad |x|\le 1 \qquad (3)$$

Definition For any real number θ, $e^{i\theta}=\cos\theta+i\sin\theta$. (4)

Table 10.1 Frequently Used Taylor Series (from section 10.10)

$$\frac{1}{1-x}=1+x+x^2+\cdots+x^n+\cdots=\sum_{n=0}^{\infty}x^n \qquad |x|<1$$

$$\frac{1}{1+x}=1-x+x^2-\cdots+(-x)^n+\cdots=\sum_{n=0}^{\infty}(-1)^n x^n \qquad |x|<1$$

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad |x|<\infty$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad |x|<\infty$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \qquad |x|<\infty$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} \qquad -1<x\le 1$$

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \qquad |x|\le 1$$

6) $f(x) = \dfrac{1}{1+x} = (1+x)^{-1}$ $\quad a = 2$

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$(1+x)^{-1} = \dfrac{1}{1+x}$	$\dfrac{1}{1+(2)} = \dfrac{1}{3}$
1	$-1(1+x)^{-2} = \dfrac{-1}{(1+x)^2}$	$\dfrac{-1}{(1+(2))^2} = \dfrac{-1}{9}$
2	$2(1+x)^{-3} = \dfrac{2}{(1+x)^3}$	$\dfrac{2}{(1+(2))^3} = \dfrac{2}{27}$
3	$-6(1+x)^{-4} = \dfrac{-6}{(1+x)^4}$	$\dfrac{-6}{(1+(2))^4} = \dfrac{-6}{81}$

$$\sum_{n=0}^{3} \frac{f^{(n)}(2)}{n!}(x-2)^n = \frac{\frac{1}{3}}{0!}(x-2)^0 - \frac{\frac{1}{9}}{1!}(x-2)^1$$

$$+ \frac{\frac{2}{27}}{2!}(x-2)^2 - \frac{\frac{6}{81}}{3!}(x-2)^3$$

$$= \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3$$

---

8) $f(x) = \ln x$ $\quad a = 1$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	$\ln(1) = 0$
1	$\dfrac{1}{x} = x^{-1}$	$\dfrac{1}{(1)} = 1$
2	$-1x^{-2} = \dfrac{-1}{x^2}$	$\dfrac{-1}{(1)^2} = -1$
3	$2x^{-3} = \dfrac{2}{x^3}$	$\dfrac{2}{(1)^3} = 2$
4	$-6x^{-4} = \dfrac{-6}{x^4}$	$\dfrac{-6}{(1)^4} = -6$

$$\sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1$$

$$- \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

---

10) $f(x) = \cos^2 x$ $\quad a = 0$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos^2 x$	$\cos^2(0) = (1)^2 = 1$
1	$-2\cos x \sin x = -\sin 2x$	$-\sin 2(0) = 0$
2	$-2\cos(2x)$	$-2\cos(2(0)) = -2$
3	$4\sin(2x)$	$4\sin(2(0)) = 0$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
4	$8\cos(2x)$	$8\cos(2(0)) = 8$
5	$-16\sin(2x)$	$-16\sin(2(0)) = 0$
6	$-32\cos(2x)$	$-32\cos(2(0)) = -32$

10) continued

$$\sum_{n=0}^{6} \frac{f^{(n)}(0)}{n!}(x-0)^n = \frac{1}{0!}x^0 + \frac{0}{1!}x^1 - \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{8}{4!}x^4 - \frac{0}{5!}x^5 - \frac{32}{6!}x^6$$

$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6$$

---

22) $f(x) = x^6 - x^4 + 2$ $\quad a = -2$

n	$f^{(n)}(x)$	$f^{(n)}(-2)$
0	$x^6 - x^4 + 2$	50
1	$6x^5 - 4x^3$	-160
2	$30x^4 - 12x^2$	432
3	$120x^3 - 24x$	-912
4	$360x^2 - 24$	1416
5	$720x$	-1440
6	720	720
7	0	0
$\vdots$	$\vdots$	$\vdots$

$$f(x) = x^6 - x^4 + 2 = \sum_{n=0}^{6} \frac{f^{(n)}(-2)}{n!}(x-(-2))^n$$

$$= \frac{50}{0!}(x+2)^0 - \frac{160}{1!}(x+2)^1 + \frac{432}{2!}(x+2)^2$$

$$- \frac{912}{3!}(x+2)^3 + \frac{1416}{4!}(x+2)^4 - \frac{1440}{5!}(x+2)^5$$

$$+ \frac{720}{6!}(x+2)^6$$

$$= 50 - 160(x+2) + 216(x+2)^2 - 152(x+2)^3$$

$$+ 59(x+2)^4 - 12(x+2)^5 + (x+2)^6$$

a finite series converges for all x $\quad R = \infty$

---

24) $f(x) = \frac{1}{x} = x^{-1}$ $\quad a = -3$

n	$f^{(n)}(x)$	$f^{(n)}(-3)$
0	$\frac{1}{x}$	$\frac{1}{(-3)} = -\frac{1}{3}$
1	$-1x^{-2} = \frac{-1}{x^2}$	$\frac{-1}{(-3)^2} = \frac{-1}{9} = \frac{-1}{3^2}$
2	$2x^{-3} = \frac{2}{x^3}$	$\frac{2}{(-3)^3} = \frac{-2}{3^3} = \frac{-2}{27}$

n	$f^{(n)}(x)$	$f^{(n)}(-3)$
3	$-6x^{-4} = \frac{-6}{x^4}$	$\frac{-6}{(-3)^4} = \frac{-6}{3^4}$
4	$24x^{-5} = \frac{24}{x^5}$	$\frac{24}{(-3)^5} = \frac{-24}{3^5}$

24) continued...

$$f(x) = \frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x-(-3))^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n$$

$$= \frac{\left(\frac{-1}{3}\right)}{0!}(x+3)^0 + \frac{\left(\frac{-1}{3^2}\right)}{1!}(x+3)^1 + \frac{\left(\frac{-2}{3^3}\right)}{2!}(x+3)^2 + \frac{\left(\frac{-6}{3^4}\right)}{3!}(x+3)^3 + \frac{\left(\frac{-24}{3^5}\right)}{4!}(x+3)^4 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{\frac{-(n!)}{3^{n+1}}}{n!}(x+3)^n = -\sum_{n=0}^{\infty} \frac{(x+3)^n}{3^{n+1}} \rightarrow a_n = \frac{(x+3)^n}{3^{n+1}} \quad a_{n+1} = \frac{(x+3)^{n+1}}{3^{n+2}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \ldots = \lim_{n\to\infty}\left|\left(\frac{(x+3)^{n+1}}{3^{n+2}}\right)\left(\frac{3^{n+1}}{(x+3)^n}\right)\right| = \lim_{n\to\infty}\left|\left(\frac{(x+3)^n(x+3)^1}{(3^{n+1})(3^1)}\right)\left(\frac{3^{n+1}}{(x+3)^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{x+3}{3}\right| = \left|\frac{x+3}{3}\right| < 1 \text{ for convergence} \rightarrow |x+3| < 3 \rightarrow R = 3$$

---

26) $f(x) = \frac{1}{x^2} = x^{-2}$ $\qquad a = 1$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\frac{1}{x^2}$	$\frac{1}{(1)^2} = 1$
1	$-2x^{-3} = \frac{-2}{x^3}$	$\frac{-2}{(1)^3} = -2$
2	$6x^{-4} = \frac{6}{x^4}$	$\frac{6}{(1)^4} = 6$
3	$-24x^{-5} = \frac{-24}{x^5}$	$\frac{-24}{(1)^5} = -24$
4	$120x^{-6} = \frac{120}{x^6}$	$\frac{120}{(1)^6} = 120$
$\vdots$	$\vdots$	$\vdots$

$$f(x) = \frac{1}{x^2} = \frac{1}{0!}(x-1)^0 - \frac{2}{1!}(x-1)^1 + \frac{6}{2!}(x-1)^2 - \frac{24}{3!}(x-1)^3 + \frac{120}{4!}(x-1)^4 - \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$$

$$a_n = (-1)^n(n+1)(x-1)^n \quad a_{n+1} = (-1)^{n+1}(n+2)(x-1)^{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \ldots = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)(x-1)^{n+1}}{(-1)^n(n+1)(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)(x-1)^n(x-1)^1}{(-1)^n(n+1)(x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n+2}{n+1}\right)(x-1)\right| = \ldots = \lim_{n\to\infty}\left|\left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)(x-1)\right| = \left|\left(\frac{1+0}{1+0}\right)(x-1)\right| = |x-1| < 1$$

for convergence $|x-1| < 1 \rightarrow R = 1$

28) $f(x) = \cos x \qquad a = \frac{\pi}{2}$

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{2}\right)$
0	$\cos x$	$\cos\left(\frac{\pi}{2}\right) = 0$
1	$-\sin x$	$-\sin\left(\frac{\pi}{2}\right) = -1$
2	$-\cos x$	$-\cos\left(\frac{\pi}{2}\right) = 0$
3	$\sin x$	$\sin\left(\frac{\pi}{2}\right) = 1$
4	$\cos x$	$\cos\left(\frac{\pi}{2}\right) = 0$
5	$-\sin x$	$-\sin\left(\frac{\pi}{2}\right) = -1$
6	$-\cos x$	$-\cos\left(\frac{\pi}{2}\right) = 0$
7	$\sin x$	$\sin\left(\frac{\pi}{2}\right) = 1$
$\vdots$	$\vdots$	$\vdots$

$$f(x) = \cos\left(\frac{\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(\frac{\pi}{2}\right)}{n!}\left(x - \frac{\pi}{2}\right)^n$$

$$= \frac{-1}{1!}\left(x - \frac{\pi}{2}\right)^1 + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 + \frac{-1}{5!}\left(x - \frac{\pi}{2}\right)^5$$

$$+ \frac{1}{7!}\left(x - \frac{\pi}{2}\right)^7 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x - \frac{\pi}{2}\right)^{2n+1}$$

$$a_n = \frac{(-1)^{n+1}}{(2n+1)!}\left(x - \frac{\pi}{2}\right)^{2n+1} \qquad a_{n+1} = \frac{(-1)^{n+2}}{(2n+3)!}\left(x - \frac{\pi}{2}\right)^{2n+3}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \cdots = \lim_{n\to\infty}\left|\left(\frac{(-1)^{n+2}\left(x - \frac{\pi}{2}\right)^{2n+3}}{(2n+3)!}\right)\left(\frac{(2n+1)!}{(-1)^{n+1}\left(x - \frac{\pi}{2}\right)^{2n+1}}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(-1)^{n+2}\left(x - \frac{\pi}{2}\right)^{2n+1}\left(x - \frac{\pi}{2}\right)^2}{(2n+3)(2n+2)(2n+1)!}\right)\left(\frac{(2n+1)!}{(-1)^{n+1}\left(x - \frac{\pi}{2}\right)^{2n+1}}\right)\right| = \cdots = \lim_{n\to\infty}\left|\frac{\left(x - \frac{\pi}{2}\right)^2}{(2n+3)(2n+2)}\right| = 0 < 1$$

true for all x $\rightarrow$ $R = \infty$

---

40) $f(x) = \sin\left(\frac{\pi x}{4}\right) \qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = \infty$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi x}{4}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}\, x^{2n+1} \qquad R = \infty$$

---

42) $f(x) = e^{3x} - e^{2x} \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty$

$$= \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} - \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{3^n x^n}{n!} - \frac{2^n x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{3^n - 2^n}{n!}\, x^n \qquad R = \infty$$

44) $f(x) = x^2 \ln(1+x^3)$ $\qquad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad R=1$

$$= x^2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x^3)^n}{n} = x^2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n}}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n+2}}{n} \quad R=1$$

---

56) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$\frac{1}{\sqrt[10]{e}} = e^{-\frac{1}{10}} = 1 + \frac{\left(\frac{-1}{10}\right)}{1!} + \frac{\left(\frac{-1}{10}\right)^2}{2!} + \frac{\left(\frac{-1}{10}\right)^3}{3!} + \frac{\left(\frac{-1}{10}\right)^4}{4!} + \frac{\left(\frac{-1}{10}\right)^5}{5!} + \dots$$

$$\frac{1}{\sqrt[10]{e}} \approx 1 - \frac{1}{10} + \frac{1}{2!}\left(\frac{1}{10}\right)^2 - \frac{1}{3!}\left(\frac{1}{10}\right)^3 + \frac{1}{4!}\left(\frac{1}{10}\right)^4 \approx 0.90484$$

because $\frac{1}{5!}\left(\frac{1}{10}\right)^5 \approx 8.3 \times 10^{-8}$ which does not affect fifth decimal place.

---

60) $\int x^2 \sin(x^2)\,dx$ $\qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad R=\infty$

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

$$x^2 \sin(x^2) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{(2n+1)!}$$

$$\int x^2 \sin(x^2)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{(2n+1)!}\,dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+5}}{(4n+5)(2n+1)!} \quad R=\infty$$

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62) $\int \arctan(x^2)\,dx$ $\qquad \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad R=1$

$$\arctan(x^2) = \tan^{-1}(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$\int \arctan(x^2)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)} \quad R=1$$

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64) $\int_0^1 \sin(x^4)\,dx$ (4 decimal places)

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\quad \sin(x^4) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+4}}{(2n+1)!}$

$$\int \sin(x^4)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+4}}{(2n+1)!}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+5}}{(8n+5)(2n+1)!}$$

$$\int_0^1 \sin(x^4)\,dx = \left[C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+5}}{(8n+5)(2n+1)!}\right]_0^1 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(8n+5)(2n+1)!}$$

$$= \frac{1}{5(1!)} - \frac{1}{13(3!)} + \frac{1}{21(5!)} - \frac{1}{29(7!)} + \ldots$$

now $\frac{1}{29(7!)} \simeq 6.64 \times 10^{-6}$ which does not affect 4th decimal place

so $\int_0^1 \sin(x^4)\,dx \simeq \frac{1}{5(1!)} - \frac{1}{13(3!)} + \frac{1}{21(5!)} \simeq 0.1876$

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66) $\int_0^{0.5} x^2 e^{-x^2}\,dx$ $\quad |\text{error}| < 0.001$ $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad R = \infty$

$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$ $\quad x^2 e^{-x^2} = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!}$

$$\int_0^{0.5} x^2 e^{-x^2}\,dx = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!}\,dx = \left[C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+3)n!}\right]_0^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)2^{2n+3}}$$

$$= \underbrace{\frac{(-1)^0}{0!(2(0)+3)2^{2(0)+3}}}_{= \frac{1}{24}} + \underbrace{\frac{(-1)^1}{1!(2(1)+3)2^{2(1)+3}}}_{= \frac{-1}{160}} + \underbrace{\frac{(-1)^2}{2!(2(2)+3)2^{(2(2)+3)}}}_{= \frac{1}{1792}} + \ldots$$

$$\frac{1}{1792} < \frac{1}{1000} = 0.001$$

we only need the 1st 2 terms to have $|\text{error}| < 0.001$

$$\int_0^{0.5} x^2 e^{-x^2}\,dx \simeq \frac{1}{24} - \frac{1}{160} \simeq 0.0354$$