$\boxed{1}$ $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

$\boxed{2}$ **Theorem**

If the power series $\sum c_n (x-a)^n$ has radius of convergence $R > 0$, then the function $f$ defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$$

(ii) $$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both $R$.

More generally, a series of the form

$\boxed{3}$ $$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

is called a **power series in** $(x-a)$ or a **power series centered at** $a$ or a **power series about** $a$.

**Pages 2 and 3 lists descriptions from Thomas' Calculus textbook (also repeated in section 8).**

**Definition**

**A power series about** $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \tag{1}$$

**A power series about** $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the **center** $a$ and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

**Theorem 18 - The Convergence Theorem for Power Series**

If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

converges at $x = c \neq 0$, then it converges absolutely for all $x$ with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all $x$ with $|x| > |d|$.

**Corollary to Theorem 18**

The convergence of the series $\sum c_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number $R$ such that the series diverges for $x$ with $|x-a| > R$ but converges absolutely for $x$ with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.

2. The series converges absolutely for every $x$ ($R = \infty$).

3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

**How to Test a Power Series for Convergence**

1. Use the Ratio Test (or Root Test) to find the largest open interval where the series converges absolutely.
   $|x-a| < R$ or $a - R < x < a + R$.

2. If $R$ is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b (see pages 626 to 628). Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. If $R$ is finite, the series diverges for $|x-a| > R$ (it does not even converge conditionally) because the $n$th term does not approach zero for those values of $x$.

**Theorem 19 - Series Multiplication for Power Series**

If $A(x)=\sum_{n=0}^{\infty} a_n x^n$ and $B(x)=\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x|<R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x|<R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n .$$

**Theorem 20**

If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<R$ and $f$ is a continuous function, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely on the set of points $x$ where $|f(x)|<R$.

**Theorem 21 - Term-by-Term Differentiation**

If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$, it defines a function

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{on the interval} \qquad a-R<x<a+R .$$

This function $f$ has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x)=\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

and so on. Each of these derived series converges at every point of the interval $a-R<x<a+R$.

**Theorem 22 - Term-by-Term Integration**

Suppose that

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a-R<x<a+R$ ($R>0$). Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a-R<x<a+R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a-R<x<a+R$.

4) $f(x) = \frac{x}{1+x} = x\left(\frac{1}{1-(-x)}\right) = x\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^{n+1}$

the series converges when $|-x|<1 \rightarrow |x|<1 \rightarrow R=1$ and $I=(-1,1)$

---

6) $f(x) = \frac{5}{1-4x^2} = 5\left(\frac{1}{1-(4x^2)}\right) = 5\sum_{n=0}^{\infty}(4x^2)^n = 5\sum_{n=0}^{\infty}4^n x^{2n}$

the series converges when $|4x^2|<1 \rightarrow |x^2|<\frac{1}{4} \rightarrow |x|<\frac{1}{2}$

$R=\frac{1}{2}$ and $I=(\frac{-1}{2},\frac{1}{2})$

---

8) $f(x) = \frac{4}{2x+3} = \frac{4}{3+2x} = \frac{4}{3(1+\frac{2}{3}x)} = \frac{4}{3}\left(\frac{1}{1-(\frac{-2}{3}x)}\right) = \frac{4}{3}\sum_{n=0}^{\infty}\left(\frac{-2}{3}x\right)^n$

the series converges when $|\frac{-2}{3}x|<1 \rightarrow \frac{2}{3}|x|<1 \rightarrow |x|<\frac{3}{2}$

$R=\frac{3}{2}$ and $I=(\frac{-3}{2},\frac{3}{2})$

---

10) $f(x) = \frac{x}{2x^2+1} = x\left(\frac{1}{1-(-2x^2)}\right) = x\sum_{n=0}^{\infty}(-2x^2)^n = \sum_{n=0}^{\infty}(-2)^n x^{2n+1}$

the series converges when $|-2x^2|<1 \rightarrow |x^2|<\frac{1}{2} \rightarrow |x|<\frac{1}{\sqrt{2}}$

$R=\frac{1}{\sqrt{2}}$ and $I=(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

---

12) $f(x) = \frac{x+a}{x^2+a^2}$ $[a>0]$ $\quad f(x) = \frac{x+a}{a^2+x^2} = \frac{x}{a^2+x^2} + \frac{a}{a^2+x^2}$

$f(x) = \frac{x}{a^2(1+\frac{x^2}{a^2})} + \frac{a}{a^2(1+\frac{x^2}{a^2})} = \frac{x}{a^2}\left(\frac{1}{1-(\frac{-x^2}{a^2})}\right) + \frac{1}{a}\left(\frac{1}{1-(\frac{-x^2}{a^2})}\right)$

$= \frac{x}{a^2}\sum_{n=0}^{\infty}\left(\frac{-x^2}{a^2}\right)^n + \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{-x^2}{a^2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{a^{2n+2}} + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{a^{2n+1}}$

The geometric series $\sum_{n=0}^{\infty}\left(\frac{-x^2}{a^2}\right)^n$ converges when $|\frac{-x^2}{a^2}|<1 \rightarrow |x^2|<a^2 \rightarrow |x|<a$

$R=a$ and $I=(-a,a)$

11.9/5

14) $f(x) = \dfrac{2x+3}{x^2+3x+2} = \dfrac{2x+3}{(x+2)^1(x+1)^1}$

$\dfrac{2x+3}{(x+2)^1(x+1)^1} = \dfrac{A}{(x+2)^1} + \dfrac{B}{(x+1)^1}$

$2x+3 = A(x+1) + B(x+2)$

constant term

$3 = A + 2B$

$3 = (2-B) + 2B$

$3 = 2 + B$

$1 = B$

x-term

$2 = A + B$

$(2-B) = A$

$A = 2-(1) = 1$

$f(x) = \dfrac{2x+3}{x^2+3x+2} = \dfrac{(+1)}{(x+2)^1} + \dfrac{(+1)}{(x+1)^1} = \dfrac{1}{2+x} + \dfrac{1}{1+x} = \dfrac{1}{2(1+\frac{x}{2})} + \dfrac{1}{1+x}$

$= \dfrac{1}{2}\left(\dfrac{1}{1-(\frac{-x}{2})}\right) + \dfrac{1}{1-(-x)} = \dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{-x}{2}\right)^n + \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}\left((-1)^n\left(\dfrac{1}{2^{n+1}}+1\right)\right)x^n$

$\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n$ converges when $\left|\frac{-x}{2}\right| < 1 \rightarrow |x| < 2$

$\sum_{n=0}^{\infty}(-x)^n$ converges when $|-x| < 1 \rightarrow |x| < 1$

therefore the sum of these 2 mini sums converges when $|x| < 1$ $R = 1$ and $I = (-1, 1)$

---

18) $f(x) = \left(\dfrac{x}{2-x}\right)^3$

start with $\dfrac{1}{2-x} = \dfrac{1}{2(1-\frac{x}{2})} = \dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n = \sum_{n=0}^{\infty}\dfrac{1}{2^{n+1}}x^n$

$\dfrac{d}{dx}\left(\dfrac{1}{2-x}\right) = \dfrac{d}{dx}\left(\sum_{n=0}^{\infty}\dfrac{1}{2^{n+1}}x^n\right) \rightarrow \dfrac{1}{(2-x)^2} = \sum_{n=1}^{\infty}\dfrac{1}{2^{n+1}}n\,x^{n-1}$

$\dfrac{d}{dx}\left(\dfrac{1}{(2-x)^2}\right) = \dfrac{d}{dx}\left(\sum_{n=1}^{\infty}\dfrac{1}{2^{n+1}}n\,x^{n-1}\right) \rightarrow \dfrac{2}{(2-x)^3} = \sum_{n=2}^{\infty}\dfrac{1}{2^{n+1}}n(n-1)x^{n-2}$

$\sum_{n=2}^{\infty}\dfrac{1}{2^{n+1}}n(n-1)x^{n-2} = \sum_{n=0}^{\infty}\dfrac{1}{2^{n+3}}(n+2)(n+1)x^n$

18) continued...

$$f(x) = \left(\frac{x}{2-x}\right)^3 = \frac{x^3}{(2-x)^3} = \left(\frac{x^3}{(2-x)^3}\right)\left(\frac{2}{2}\right) = \frac{x^3}{2}\left(\frac{2}{(2-x)^3}\right)$$

$$= \frac{x^3}{2}\sum_{n=0}^{\infty}\frac{1}{2^{n+3}}(n+2)(n+1)x^n = \sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2^{n+4}}x^{n+3}$$

and it converges when $\left|\frac{x}{2}\right| < 1 \rightarrow |x| < 2 \rightarrow R = 2$

---

20) $f(x) = \dfrac{x^2+x}{(1-x)^3}$

see Example 4: $\dfrac{1}{(1-x)^2} = \displaystyle\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$

$$\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(n+1)x^n\right) \rightarrow \frac{2}{(1-x)^3} = \sum_{n=1}^{\infty}(n+1)n\,x^{n-1}$$

$$f(x) = \frac{x^2+x}{(1-x)^3} = \frac{x^2}{(1-x)^3} + \frac{x}{(1-x)^3} = \frac{x^2}{2}\left(\frac{2}{(1-x)^3}\right) + \frac{x}{2}\left(\frac{2}{(1-x)^3}\right)$$

$$= \frac{x^2}{2}\sum_{n=1}^{\infty}(n+1)n\,x^{n-1} + \frac{x}{2}\sum_{n=1}^{\infty}(n+1)n\,x^{n-1}$$

$$= \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^{n+1} + \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^n$$

$$= \sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^n + \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^n$$ changed the 1st sum to match the exponents of $x$

$$= \sum_{n=2}^{\infty}\frac{n^2-n}{2}x^n + \frac{((1)+1)(1)}{2}x^1 + \sum_{n=2}^{\infty}\frac{n^2+n}{2}x^n$$ match starting value of sum

$$= x + \underbrace{\sum_{n=2}^{\infty}\frac{n^2-n}{2}x^n + \sum_{n=2}^{\infty}\frac{n^2+n}{2}x^n}$$

$$= x + \sum_{n=2}^{\infty}n^2x^n = \sum_{n=1}^{\infty}n^2x^2$$ converges when $|x|<1 \rightarrow R=1$

11.9/7

22) $f(x) = x^2 \tan^{-1}(x^3)$

see example 6: $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$$f(x) = x^2 \tan^{-1}(x^3) = x^2 \sum_{n=0}^{\infty} (-1) \frac{(x^3)^{2n+1}}{2n+1} = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+5}}{2n+1}$$

converges for $|x^3| < 1 \rightarrow |x| < 1 \rightarrow R = 1$

---

24) $f(x) = \ln(1+x^4)$

see example 5: $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $|x| < 1$

$$f(x) = \ln(1+x^4) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x^4)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{4n}}{n}$$

$|x^4| < 1 \rightarrow |x| < 1 \rightarrow R = 1$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{4n}}{n} = (-1)^{(1)-1} \frac{x^{4(1)}}{(1)} + (-1)^{(2)-1} \frac{x^{4(2)}}{(2)} + (-1)^{(3)-1} \frac{x^{4(3)}}{(3)} + (-1)^{(4)-1} \frac{x^{4(4)}}{(4)} + (-1)^{(5)-1} \frac{x^{4(5)}}{(5)} + \dots$$

$$= x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} - \frac{x^{16}}{4} + \frac{x^{20}}{5} - \dots$$

$s_1 = x^4$

$s_2 = s_1 - \frac{1}{2} x^8$

$s_3 = s_2 + \frac{1}{3} x^{12}$

$s_4 = s_3 - \frac{1}{4} x^{16}$

$s_5 = s_4 + \frac{1}{5} x^{20}$

$\vdots$

28) $\int \frac{t}{1+t^3}\,dt$

$$\frac{t}{1+t^3} = t\left(\frac{1}{1-(-t^3)}\right) = t\sum_{n=0}^{\infty}(-t^3)^n = t\sum_{n=0}^{\infty}(-1)^n t^{3n} = \sum_{n=0}^{\infty}(-1)^n t^{3n+1}$$

$$\int \frac{t}{1+t^3}\,dt = \int \sum_{n=0}^{\infty}(-1)^n t^{3n+1}\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{t^{3n+2}}{3n+2} + C$$

Since $\sum_{n=0}^{\infty} \frac{1}{1+t^3}$ converges when $|t^3|<1 \to |t|<1 \to R=1$,
our integral $\int \frac{t}{1+t^3}\,dt$ will also have $R=1$ (Thm 2)

---

30) $\int \frac{\tan^{-1}x}{x}\,dx$ see example 6 or exercise 22

$$\frac{\tan^{-1}x}{x} = \frac{\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}}{x} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2n+1}$$

$$\int \frac{\tan^{-1}x}{x}\,dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2n+1}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)^2} + C$$

Since $\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ has $R=1$,
our integral $\int \frac{\tan^{-1}x}{x}\,dx$ will also have $R=1$ (Thm 2)

32) $\int_0^{\frac{1}{2}} \arctan \frac{x}{2}\, dx = \int_0^{\frac{1}{2}} \tan^{-1}\left(\frac{x}{2}\right) dx$ $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

11.9/9

$$\int \tan^{-1}\left(\frac{x}{2}\right) dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{2}\right)^{2n+1}}{2n+1}\, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^{2n+1}(2n+1)}\, dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2^{2n+1}(2n+1)(2n+2)}$$

$$I = \int_0^{\frac{1}{2}} \tan^{-1}\left(\frac{x}{2}\right) dx = \left[C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2^{2n+1}(2n+1)(2n+2)}\right]_0^{\frac{1}{2}}$$

$$= \left[\frac{x^2}{2(1)(2)} - \frac{x^4}{2^3(3)(4)} + \frac{x^6}{2^5(5)(6)} - \frac{x^8}{2^7(7)(8)} + \frac{x^{10}}{2^9(9)(10)} - \cdots\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{2^3(1)(2)} - \frac{1}{2^7(3)(4)} + \frac{1}{2^{11}(5)(6)} - \frac{1}{2^{15}(7)(8)} + \frac{1}{2^{19}(9)(10)} - \cdots$$

The series is alternating; using first four terms, the error is at most $\frac{1}{2^{19}(9)(10)} \simeq 2.1 \times 10^{-8}$.

So $I \approx \frac{1}{2^3(1)(2)} - \frac{1}{2^7(3)(4)} + \frac{1}{2^{11}(5)(6)} - \frac{1}{2^{15}(7)(8)}$

$\approx \frac{1}{16} - \frac{1}{1536} + \frac{1}{61440} - \frac{1}{1835008} \simeq 0.061865$ to 6 decimal places

34) $\displaystyle\int_0^{0.3} \frac{x^2}{1+x^4}\,dx$

11.9/10

$$\frac{x^2}{1+x^4} = x^2\left(\frac{1}{1-(-x^4)}\right) = x^2\sum_{n=0}^{\infty}(-x^4)^n = x^2\sum_{n=0}^{\infty}(-1)^n x^{4n} = \sum_{n=0}^{\infty}(-1)^n x^{4n+2}$$

$$\int \frac{x^2}{1+x^4}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{4n+2}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{4n+3}$$

$$\int_0^{0.3} \frac{x^2}{1+x^4}\,dx = \left[C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{4n+3}\right]_0^{\frac{3}{10}} = \sum_{n=0}^{\infty}\frac{(-1)^n (3)^{4n+3}}{(4n+3)10^{4n+3}}$$

$$= \frac{3^3}{(3)(10^3)} - \frac{3^7}{(7)(10^7)} + \frac{3^{11}}{(11)(10^{11})} - \cdots$$

the series is alternating; if we use only 2 terms, then the error will be at most $\frac{3^{11}}{11\times 10^{11}} \approx 0.00000016$

to six decimal places

$$\int_0^{0.3} \frac{x^2}{1+x^4}\,dx \approx \frac{3^3}{3\times 10^3} - \frac{3^7}{7\times 10^7} \approx 0.008969$$