A **power series** is a series of the form

$\boxed{1}$ $$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric sereis

$\boxed{2}$ $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

More generally, a series of the form

$\boxed{3}$ $$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

is called a **power series in** $(x-a)$ or a **power series centered at** a or a **power series about** a.

$\boxed{4}$ **Theorem**

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. (The number R is called the **radius of convergence** of the power series.)

The table below summarizes the radius of convergence for each examples in section 11.8 (recopy the examples to help understand better).

	Series	Radius of Convergence	Interval of Convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1,1)$
Example 1	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2,4)$
Example 2	$\sum_{n=0}^{\infty} n!x^n$	$R = 0$	$\{0\}$
Example 3	$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$	$R = \infty$	$(-\infty,\infty)$

Pages 2 and 3 lists descriptions from Thomas' Calculus textbook.

Definition

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \qquad (1)$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \qquad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Theorem 18 - The Convergence Theorem for Power Series

If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

Corollary to Theorem 18

The convergence of the series $\sum c_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.

2. The series converges absolutely for every x ($R = \infty$).

3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the largest open interval where the series converges absolutely.
   $|x-a| < R$ or $a - R < x < a + R$.

2. If R is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b (see pages 626 to 628). Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. If R is finite, the series diverges for $|x-a| > R$ (it does not even converge conditionally) because the nth term does not approach zero for those values of x.

Theorem 19 - Series Multiplication for Power Series

If $A(x)=\sum_{n=0}^{\infty} a_n x^n$ and $B(x)=\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x|<R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x|<R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n .$$

Theorem 20

If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<R$ and f is a continuous function, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely on the set of points x where $|f(x)|<R$.

Theorem 21 - Term-by-Term Differentiation

If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$, it defines a function

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{on the interval} \qquad a-R<x<a+R .$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x)=\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

and so on. Each of these derived series converges at every point of the interval $a-R<x<a+R$.

Theorem 22 - Term-by-Term Integration

Suppose that

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a-R<x<a+R$ ($R>0$). Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a-R<x<a+R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a-R<x<a+R$.

4) $\sum_{n=1}^{\infty} (-1)^n n x^n \qquad a_n = (-1)^n n x^n \qquad a_{n+1} = (-1)^{n+1}(n+1)x^{n+1}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{(-1)^n n x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(-1)^1(n+1)(x^n)(x^1)}{(-1)^n n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)(n+1)x}{n}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)(-x)\right| = \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)(-x)\right|$$

$$= \left|(1+0)(-x)\right| = |-x| = |x| \rightarrow |x| < 1 \rightarrow -1 < x < 1$$

radius of convergence: $R = 1$

endpoints when $x = -1$, $\sum_{n=1}^{\infty} (-1)^n n (-1)^n = \sum_{n=1}^{\infty} n \rightarrow c_n = n$

when $x = 1$, $\sum_{n=1}^{\infty} (-1)^n n (1)^n = \sum_{n=1}^{\infty} (-1)^n n \rightarrow d_n = (-1)^n n \rightarrow |d_n| = c_n = n$

$$\lim_{n\to\infty}|d_n| = \lim_{n\to\infty} c_n = \lim_{n\to\infty} n = +\infty$$

by the Test for Divergence, diverges when $x = -1$ and $x = 1$

Interval of Convergence: $I = (-1, 1)$

---

6) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}} \qquad a_n = \frac{(-1)^n x^n}{\sqrt[3]{n}} \qquad a_{n+1} = \frac{(-1)^{n+1} x^{n+1}}{\sqrt[3]{n+1}}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{\sqrt[3]{n+1}}}{\frac{(-1)^n x^n}{\sqrt[3]{n}}}\right| = \lim_{n\to\infty}\left|\left(\frac{(-1)^{n+1}x^{n+1}}{\sqrt[3]{n+1}}\right)\left(\frac{\sqrt[3]{n}}{(-1)^n x^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(-1)^n(-1)^1(x^n)(x^1)}{\sqrt[3]{n+1}}\right)\left(\frac{\sqrt[3]{n}}{(-1)^n x^n}\right)\right| = \lim_{n\to\infty}\left|\frac{(-1)(x)\sqrt[3]{n}}{\sqrt[3]{n+1}}\right|$$

6) continued...

$$= \lim_{n\to\infty} \left| (x) \sqrt[3]{\frac{n}{n+1}} \right| = \lim_{n\to\infty} \left| x \sqrt[3]{\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}} \right| = \lim_{n\to\infty} \left| x \sqrt[3]{\frac{1}{1+\frac{1}{n}}} \right|$$

$$= \left| x \sqrt[3]{\frac{1}{1+0}} \right| = |x(1)| = |x| \to |x| < 1 \to -1 < x < 1$$

radius of convergence: $R = 1$

when $x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$ is a p-series with $p = \frac{1}{3} \leq 1$ which diverges

when $x = 1$: $\sum_{n=1}^{\infty} \frac{(-1)^n(1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{3}}}$ $\quad c_n = \frac{(-1)^n}{n^{\frac{1}{3}}}$

$b_n = |c_n| = \frac{1}{n^{\frac{1}{3}}}$ "not absolutely convergent (see above)"

① for $n \geq 1$, $b_n = \frac{1}{n^{\frac{1}{3}}} > 0$, $\frac{1}{n^{\frac{1}{3}}} = b_n > b_{n+1} = \frac{1}{(n+1)^{\frac{1}{3}}}$,

$\{b_n\}$ is decreasing ② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$

by the Alternating Series test, convergent when $x = 1$

Interval of convergence: $I = (-1, 1]$

---

8) $\sum_{n=2}^{\infty} \frac{5^n}{n} x^n \qquad a_n = \frac{5^n}{n} x^n \qquad a_{n+1} = \frac{5^{n+1}}{n+1} x^{n+1}$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{5^{n+1} x^{n+1}}{n+1}}{\frac{5^n x^n}{n}} \right| = \lim_{n\to\infty} \left| \left(\frac{5^{n+1} x^{n+1}}{n+1} \right) \left(\frac{n}{5^n x^n} \right) \right|$$

8) continued...

$$= \lim_{n\to\infty}\left|\left(\frac{(5^n)(5^1)(x^n)(x^1)}{n+1}\right)\left(\frac{n}{5^n x^n}\right)\right| = \lim_{n\to\infty}\left|\frac{5n}{n+1}x\right| = \lim_{n\to\infty}\left|\frac{\frac{5n}{n}}{\frac{n}{n}+\frac{1}{n}}x\right|$$

$$= \lim_{n\to\infty}\left|\frac{5}{1+\frac{1}{n}}x\right| = \left|\frac{5}{1+0}x\right| = |5x| = 5|x| \to 5|x|<1 \to |x|<\frac{1}{5}$$

$$\to \frac{-1}{5} < x < \frac{1}{5}$$

radius of convergence: $R=\frac{1}{5}$

when $x=\frac{1}{5}$, $\sum_{n=2}^{\infty}\frac{5^n}{n}\left(\frac{1}{5}\right)^n = \sum_{n=2}^{\infty}\frac{1}{n}$ is a p-series (partial) with

$p=1\le 1$ which diverges

when $x=\frac{-1}{5}$, $\sum_{n=2}^{\infty}\frac{5^n}{n}\left(\frac{-1}{5}\right)^n = \sum_{n=2}^{\infty}\frac{(-1)^n}{n}$ $\quad c_n=\frac{(-1)^n}{n}$ $\quad b_n=|c_n|=\frac{1}{n}$

for $n\ge 2$, $b_n=\frac{1}{n}>0$, $\frac{1}{n}=b_n>b_{n+1}=\frac{1}{n+1}$ $\{b_n\}$ is decreasing

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n}=0$

by the Alternating Series Test, convergent when $x=\frac{-1}{5}$

Interval of convergence: $I=\left[\frac{-1}{5},\frac{1}{5}\right)$

---

10) $\sum_{n=1}^{\infty}\frac{n}{n+1}x^n$ $\qquad a_n=\frac{n}{n+1}x^n$ $\qquad a_{n+1}=\frac{(n+1)}{(n+1)+1}x^{n+1}=\frac{(n+1)}{n+2}x^{n+1}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)x^{n+1}}{n+2}}{\frac{nx^n}{n+1}}\right| = \lim_{n\to\infty}\left|\left(\frac{(n+1)x^{n+1}}{n+2}\right)\left(\frac{n+1}{nx^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(n+1)(x^n)(x^1)}{n+2}\right)\left(\frac{n+1}{nx^n}\right)\right| = \lim_{n\to\infty}\left|\frac{(n+1)(n+1)}{n(n+2)}x\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^2+2n+1}{n^2+2n}x\right| = \lim_{n\to\infty}\left|\frac{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}+\frac{2n}{n^2}}x\right| = \lim_{n\to\infty}\left|\frac{1+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{2}{n}}x\right|$$

10) continued...

$$= \left|\frac{1+0+0}{1+0}x\right| = |x| \rightarrow |x|<1 \rightarrow -1<x<1$$

radius of convergence: $R=1$

when $x=1$, $\sum_{n=1}^{\infty} \frac{n}{n+1}(1)^n = \sum_{n=1}^{\infty} \frac{n}{n+1}$ $\quad c_n = \frac{n}{n+1}$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{n}{n+1} \overset{L}{=} \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0$$

when $x=-1$, $\sum_{n=1}^{\infty} \frac{n}{n+1}(-1)^n$ $\quad d_n = \frac{n}{n+1}(-1)^n$ $\quad b_n = |d_n| = \frac{n}{n+1}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \Rightarrow$ Alternating Series Test does not apply

$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{n}{n+1}(-1)^n$ D.N.E.

by the Test for Divergence, diverges for when $x=1$ and $x=-1$

Interval of convergence: $I=(-1,1)$

---

12) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ $\quad a_n = \frac{(-1)^n x^n}{n^2}$ $\quad a_{n+1} = \frac{(-1)^{n+1} x^{n+1}}{(n+1)^2}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{(n+1)^2}}{\frac{(-1)^n x^n}{n^2}}\right| = \lim_{n\to\infty}\left|\left(\frac{(-1)^{n+1}x^{n+1}}{(n+1)^2}\right)\left(\frac{n^2}{(-1)^n x^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(-1)^n(-1)^1(x^n)(x^1)}{(n+1)^2}\right)\left(\frac{n^2}{(-1)^n x^n}\right)\right| = \lim_{n\to\infty}\left|\frac{(-1)(x)n^2}{(n+1)^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}x\right| = \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^2 x\right| = \lim_{n\to\infty}\left|\left(\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)^2 x\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{1}{1+\frac{1}{n}}\right)^2 x\right| = \left|\left(\frac{1}{1+0}\right)^2 x\right| = |x| \rightarrow |x|<1 \rightarrow -1<x<1$$

12) continued... radius of convergence: $R=1$

when $x=-1$, $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p=2>1$ which converges.

when $x=1$, $\sum_{n=1}^{\infty} \frac{(-1)^n(1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ $\quad c_n = \frac{(-1)^n}{n^2} \quad b_n = |c_n| = \frac{1}{n^2}$

it is absolutelly convergent and convergent by the Alternating Series Test.

Interval of convergence: $I=[-1,1]$

---

14) $\sum_{n=1}^{\infty} n^n x^n \qquad a_n = n^n x^n = (nx)^n$ "use Root Test"

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|(nx)^n|} = \lim_{n\to\infty} |nx| = \lim_{n\to\infty} n|x| = +\infty$$

if $x \neq 0$

if $x=0$ then $\lim_{n\to\infty} n|x| = 0$

radius of convergence: $R=0$ no endpoints

Interval of convergence: $I=\{0\}$

---

16) $\sum_{n=1}^{\infty} 2^n n^2 x^n \qquad a_n = 2^n n^2 x^n \qquad a_{n+1} = 2^{n+1}(n+1)^2 x^{n+1}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(n+1)^2 x^{n+1}}{2^n n^2 x^n}\right| = \lim_{n\to\infty}\left|\frac{(2^n)(2^1)(n+1)^2(x^n)(x^1)}{2^n n^2 x^n}\right|$$

$$= \lim_{n\to\infty}\left|(2)\left(\frac{n+1}{n}\right)^2 x\right| = \lim_{n\to\infty}\left|2\left(\frac{\frac{n}{n}+\frac{1}{n}}{\frac{n}{n}}\right)^2 x\right| = \lim_{n\to\infty}\left|2\left(\frac{1+\frac{1}{n}}{1}\right)^2 x\right|$$

$$= \left|2\left(\frac{1+0}{1}\right)^2 x\right| = |2x| = 2|x| \to 2|x|<1 \to |x|<\frac{1}{2} \to -\frac{1}{2}<x<\frac{1}{2}$$

16) continued...

radius of convergence: $R = \frac{1}{2}$

when $x = \pm\frac{1}{2}$, $\sum_{n=1}^{\infty} (2)^n n^2 \left(\pm\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n n^2 \quad c_n = (-1)^n n^2$

$b_n = |c_n| = n^2 \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} n^2 = +\infty$

by the Test for Divergence, diverges for both $x = \pm\frac{1}{2}$

Interval of convergence: $I = \left(\frac{-1}{2}, \frac{1}{2}\right)$

---

18) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n} x^n \quad a_n = \frac{(-1)^{n-1}}{n 5^n} x^n \quad a_{n+1} = \frac{(-1)^{(n+1)-1}}{(n+1) 5^{n+1}} x^{n+1}$

Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{(n+1)-1} x^{n+1}}{(n+1) 5^{n+1}}}{\frac{(-1)^{n-1} x^n}{n 5^n}}\right| = \lim_{n\to\infty} \left|\left(\frac{(-1)^{(n+1)-1} x^{n+1}}{(n+1) 5^{n+1}}\right)\left(\frac{n 5^n}{(-1)^{n-1} x^n}\right)\right|$$

$$= \lim_{n\to\infty} \left|\left(\frac{(-1)^{n-1}(-1)^1 (x^n)(x^1)}{(n+1)(5^n)(5^1)}\right)\left(\frac{n 5^n}{(-1)^{n-1} x^n}\right)\right| = \lim_{n\to\infty} \left|\frac{(-1)\, x\, n}{5\,(n+1)}\right|$$

$$= \lim_{n\to\infty} \left|\frac{n}{n+1} \frac{x}{5}\right| = \lim_{n\to\infty} \left|\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} \frac{x}{5}\right| = \lim_{n\to\infty} \left|\frac{1}{1+\frac{1}{n}} \frac{x}{5}\right|$$

$$= \left|\frac{1}{1+0} \frac{x}{5}\right| = \left|\frac{x}{5}\right| = \frac{|x|}{5} \rightarrow \frac{|x|}{5} < 1 \rightarrow |x| < 5 \rightarrow -5 < x < 5$$

radius of convergence: $R = 5$

when $x = -5$, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n} (-5)^n = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series

with $p = 1 \leq 1$ which diverges

when $x = 5$, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n} (5)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by the Alternate Series Test. Interval of convergence: $(-5, 5]$

20) $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$ $a_n = \frac{x^{2n}}{n!}$ $a_{n+1} = \frac{x^{2(n+1)}}{(n+1)!}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}}\right| = \lim_{n\to\infty}\left|\left(\frac{x^{2n+2}}{(n+1)!}\right)\left(\frac{n!}{x^{2n}}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(x^{2n})(x^2)}{(n+1)(n!)}\right)\left(\frac{n!}{x^{2n}}\right)\right| = \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0 < 1$$

true statement for all x

radius of convergence: $R = \infty$

Interval of convergence: $I = (-\infty, \infty)$

---

22) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n$ $a_n = \frac{(-1)^n}{(2n-1)2^n}(x-1)^n = \frac{(-1)^n(x-1)^n}{(2n-1)2^n}$

$$a_{n+1} = \frac{(-1)^{n+1}}{(2(n+1)-1)2^{n+1}}(x-1)^{n+1} = \frac{(-1)^n(-1)^1(x-1)^n(x-1)^1}{(2n+1)(2^n)(2^1)}$$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^n(-1)^1(x-1)^n(x-1)^1}{(2n+1)(2^n)(2^1)}}{\frac{(-1)^n(x-1)^n}{(2n-1)2^n}}\right| = \lim_{n\to\infty}\left|\left(\frac{(-1)^n(-1)^1(x-1)^n(x-1)^1}{(2n+1)(2^n)(2^1)}\right)\left(\frac{(2n-1)2^n}{(-1)^n(x-1)^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)(2n-1)\ (x-1)}{2(2n+1)}\right| = \lim_{n\to\infty}\left|\frac{2n-1}{2n+1}\ \frac{x-1}{2}\right| \overset{L}{=} \lim_{n\to\infty}\left|\frac{2}{2}\ \frac{x-1}{2}\right|$$

$$= \lim_{n\to\infty}\left|1\ \frac{x-1}{2}\right| = \left|\frac{x-1}{2}\right| = \frac{|x-1|}{2} \rightarrow \frac{|x-1|}{2} < 1 \rightarrow |x-1| < 2$$

radius of convergence: $R = 2$

$-2 < x-1 < 2$

$-1 < x < 3$

when $x=-1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}((-1)-1)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)2^n}(-2)^n = \sum_{n=1}^{\infty}\frac{1}{2n-1}$

22) continued...

11.8/11

let $f(x) = \frac{1}{2x-1}$ for $x \geq 1$

① $f(x)$ is continuous
② $f(x)$ is positive
③ $f(x)$ is decreasing as x increases

$$\int_1^\infty \frac{1}{2x-1}\,dx = \lim_{v\to\infty}\int_1^v \frac{1}{2x-1}\,dx = \lim_{v\to\infty}\left[\frac{1}{2}\ln|2x-1|+C\right]_1^v$$

$$= \lim_{v\to\infty}\left\{\left[\frac{1}{2}\ln|2v-1|+C\right]-\left[\frac{1}{2}\ln|2(1)-1|+C\right]\right\} = +\infty$$

by the Integral Test, diverges when $x = -1$

when $x = 3$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)2^n}((3)-1)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)2^n}(2)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}$

$b_n = \left|\frac{(-1)^n}{2n-1}\right| = \frac{1}{2n-1}$ for $n \geq 1$ $\quad b_n = \frac{1}{2n-1} > 0$

$\frac{1}{2n-1} = b_n > b_{n+1} = \frac{1}{2(n+1)-1}$ $\quad \{b_n\}$ is decreasing

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{2n-1} = 0$

by the Alternate Series Test, converges when $x = 3$

Interval of convergence: $I = (-1, 3]$

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24) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{8^n}(x+6)^n \qquad a_n = \frac{\sqrt{n}}{8^n}(x+6)^n = \frac{\sqrt{n}\,(x+6)^n}{8^n}$

$$a_{n+1} = \frac{\sqrt{n+1}}{8^{n+1}}(x+6)^{n+1} = \frac{\sqrt{n+1}\,(x+6)^n(x+6)^1}{(8^n)(8^1)}$$

Ratio Test

24) continued...

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\sqrt{n+1}\ (x+6)^n\ (x+6)^1}{(8^n)(8^1)}}{\dfrac{\sqrt{n}\ (x+6)^n}{8^n}}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{\sqrt{n+1}\ (x+6)^n (x+6)^1}{(8^n)(8^1)}\right)\left(\frac{8^n}{\sqrt{n}\ (x+6)^n}\right)\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}\ (x+6)}{8\sqrt{n}}\right|$$

$$= \lim_{n\to\infty}\left|\sqrt{\frac{n+1}{n}}\ \frac{x+6}{8}\right| = \lim_{n\to\infty}\left|\sqrt{\frac{\frac{n}{n}+\frac{1}{n}}{\frac{n}{n}}}\ \frac{x+6}{8}\right| = \lim_{n\to\infty}\left|\sqrt{\frac{1+\frac{1}{n}}{1}}\ \frac{x+6}{8}\right|$$

$$= \left|\sqrt{\frac{1+0}{1}}\ \frac{x+6}{8}\right| = \left|\frac{x+6}{8}\right| = \frac{|x+6|}{8} \rightarrow \frac{|x+6|}{8} < 1 \rightarrow |x+6| < 8$$

radius of convergence: $R = 8$

$$-8 < x+6 < 8$$
$$-14 < x < 2$$

when $x = 2$, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}((2)+6)^n = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{8^n}(8^n) = \sum_{n=1}^{\infty}\sqrt{n}$ $\quad c_n = \sqrt{n}$

$\lim_{n\to\infty} c_n = \lim_{n\to\infty}\sqrt{n} = +\infty$

when $x = -14$, $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{8^n}((-14)+6)^n = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{8^n}(-8)^n = \sum_{n=1}^{\infty}(-1)^n\sqrt{n}$ $\quad d_n = (-1)^n\sqrt{n}$

$\lim_{n\to\infty} d_n = \lim_{n\to\infty}(-1)^n\sqrt{n}$ D.N.E.

by the Test for Divergence, diverges for when $x=2$ and $x=-14$

Interval of convergence: $I = (-14, 2)$

11.8/13

26) $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$ $\quad a_n = \frac{(2x-1)^n}{5^n\sqrt{n}}$ $\quad a_{n+1} = \frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}}{\frac{(2x-1)^n}{5^n\sqrt{n}}}\right| = \lim_{n\to\infty}\left|\left(\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}\right)\left(\frac{5^n\sqrt{n}}{(2x-1)^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(2x-1)^n(2x-1)^1}{(5^n)(5^1)\sqrt{n+1}}\right)\left(\frac{5^n\sqrt{n}}{(2x-1)^n}\right)\right| = \lim_{n\to\infty}\left|\frac{2x-1}{5}\sqrt{\frac{n}{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2x-1}{5}\sqrt{\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}}\right| = \lim_{n\to\infty}\left|\frac{2x-1}{5}\sqrt{\frac{1}{1+\frac{1}{n}}}\right| = \left|\frac{2x-1}{5}\sqrt{\frac{1}{1+0}}\right|$$

$$= \left|\frac{2x-1}{5}\right| = \frac{|2x-1|}{5} \rightarrow \frac{|2x-1|}{5} < 1 \rightarrow |2x-1| < 5 \rightarrow$$

$$-5 < 2x-1 < 5$$
$$-4 < 2x < 6$$
$$-2 < x < 3$$

$$\Downarrow$$
$$2|x-\tfrac{1}{2}| < 5$$
$$|x-\tfrac{1}{2}| < \tfrac{5}{2}$$

radius of convergence: $R = \frac{5}{2}$

when $x = 3$, $\sum_{n=1}^{\infty}\frac{(2(3)-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(5)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$

is a p-series with $p = \frac{1}{2} \leq 1$ which diverges

when $x = -2$, $\sum_{n=1}^{\infty}\frac{(2(-2)-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-5)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ $\quad b_n = \left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}}$

for $n \geq 1$, $b_n = \frac{1}{\sqrt{n}} > 0$, $\quad \frac{1}{\sqrt{n}} = b_n > b_{n+1} = \frac{1}{\sqrt{n+1}}$ $\{b_n\}$ is decreasing

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

by the Alternative Series Test, converges when $x = -2$

Interval of convergence: $I = [-2, 3)$

11/8/14

28) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} x^n \qquad a_n = \frac{(-1)^n}{n \ln n} x^n \qquad a_{n+1} = \frac{(-1)^{n+1}}{(n+1)\ln(n+1)} x^{n+1}$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{(n+1)\ln(n+1)}}{\frac{(-1)^n x^n}{n \ln n}} \right| = \lim_{n\to\infty} \left| \left(\frac{(-1)^{n+1} x^{n+1}}{(n+1)\ln(n+1)} \right) \left(\frac{n \ln n}{(-1)^n x^n} \right) \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{(-1)^n (-1)^1 (x^n)(x^1)}{(n+1)\ln(n+1)} \right) \left(\frac{n \ln n}{(-1)^n x^n} \right) \right| = \lim_{n\to\infty} \left| \frac{(-1)(x) n \ln n}{(n+1)\ln(n+1)} \right|$$

$$= \lim_{n\to\infty} \left| (-1)(x) \left(\frac{n}{n+1} \right) \left(\frac{\ln n}{\ln(n+1)} \right) \right| = \left| \lim_{n\to\infty} x \right| \left| \lim_{n\to\infty} \frac{n}{n+1} \right| \left| \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \right|$$

$$= |x| |1| |1| = |x| \to |x| < 1 \longrightarrow -1 < x < 1$$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n}} = \frac{1}{1+0} = 1$$

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \stackrel{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} \stackrel{L}{=} \lim_{n\to\infty} \frac{1}{1} = 1$$

when $x = -1$, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} (-1)^n = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ let $f(x) = \frac{1}{x \ln x}$

for $x \geq 2$
1. $f(x)$ is continuous
2. $f(x)$ is positive
3. $f(x)$ is decreasing as x increases

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{U\to\infty} \int_2^{U} \frac{1}{x \ln x} dx = \lim_{U\to\infty} \Big[\ln|\ln x| + C \Big]_2^U$$

$$= \lim_{U\to\infty} \Big\{ \big[\ln|\ln U| + C \big] - \big[\ln|\ln(2)| + C \big] \Big\} = +\infty$$

by Integral Test, diverges when $x = -1$

28) continued...

when $x=1$, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}(1)^n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$ $\quad c_n = \frac{(-1)^n}{n\ln n}$ $\quad b_n = |c_n| = \frac{1}{n\ln n}$

for $n \geq 2$, $b_n = \frac{1}{n\ln n} > 0$ $\quad \frac{1}{n\ln n} = b_n > b_{n+1} = \frac{1}{(n+1)\ln(n+1)}$

$\{b_n\}$ is decreasing $\quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n\ln n} = 0$

by the Alternating Series Test, converges when $x=1$

Interval of convergence: $I = (-1, 1]$

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30) $\sum_{n=2}^{\infty} \frac{b^n}{\ln n}(x-a)^n$, $b>0$ $\qquad a_n = \frac{b^n(x-a)^n}{\ln n}$ $\quad a_{n+1} = \frac{b^{n+1}(x-a)^{n+1}}{\ln(n+1)}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{b^{n+1}(x-a)^{n+1}}{\ln(n+1)}}{\frac{b^n(x-a)^n}{\ln n}}\right| = \lim_{n\to\infty}\left|\left(\frac{b^{n+1}(x-a)^{n+1}}{\ln(n+1)}\right)\left(\frac{\ln n}{b^n(x-a)^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(b^n)(b^1)(x-a)^n(x-a)^1}{\ln(n+1)}\right)\left(\frac{\ln n}{b^n(x-a)^n}\right)\right| = \lim_{n\to\infty}\left|b(x-a)\left(\frac{\ln n}{\ln(n+1)}\right)\right|$$

see ex. 28

$$= |b(x-a)(1)| = b|x-a| \rightarrow b|x-a| < 1 \rightarrow |x-a| < \frac{1}{b}$$

radius of convergence: $R = \frac{1}{b}$

$$\frac{-1}{b} < x-a < \frac{1}{b}$$

$$a-\frac{1}{b} < x < a+\frac{1}{b}$$

when $x = a+\frac{1}{b}$, $\sum_{n=2}^{\infty} \frac{b^n}{\ln n}\left(\left(a+\frac{1}{b}\right)-a\right)^n = \sum_{n=2}^{\infty} \frac{b^n}{\ln n}\left(\frac{1}{b}\right)^n = \sum_{n=2}^{\infty} \frac{1}{\ln n}$

for $n \geq 2$, $\ln n < n$ so $\frac{1}{\ln n} > \frac{1}{n} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$

30) continued...

$\sum_{n=2}^{\infty} \frac{1}{n}$ is a p-series (partial) with $p=1\le 1$ which diverges

by the Direct Comparison Test, diverges when $x = a + \frac{1}{b}$

when $x = a - \frac{1}{b}$, $\sum_{n=2}^{\infty} \frac{b^n}{\ln n}\left(\left(a-\frac{1}{b}\right)-a\right)^n = \sum_{n=2}^{\infty} \frac{b^n}{\ln n}\left(\frac{-1}{b}\right)^n = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

let $c_n = \left|\frac{(-1)^n}{\ln n}\right| = \frac{1}{\ln n}$ for $n \ge 2$ $\quad c_n = \frac{1}{\ln n} > 0$

$\frac{1}{\ln n} = c_n > c_{n+1} = \frac{1}{\ln(n+1)}$ $\{c_n\}$ is decreasing

$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0$

by the Alternating Series Test, converges when $x = a - \frac{1}{b}$

Interval of convergence: $I = \left[a - \frac{1}{b}, a + \frac{1}{b}\right)$

---

32) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{(2)(4)(6)\cdots(2n)}$

$a_n = \frac{n^2 x^n}{(2)(4)(6)\cdots(2n)} = \frac{n^2 x^n}{(2(1))(2(2))(2(3))\cdots(2n)} = \frac{n^2 x^n}{2^n n!} = \frac{n^2 x^n}{2^n (n)(n-1)!}$

$a_n = \frac{n x^n}{2^n (n-1)!}$ $\qquad a_{n+1} = \frac{(n+1)x^{n+1}}{2^{n+1}((n+1)-1)!} = \frac{(n+1)x^{n+1}}{2^{n+1} n!}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)x^{n+1}}{2^{n+1} n!}}{\frac{n x^n}{2^n (n-1)!}}\right| = \lim_{n\to\infty}\left|\left(\frac{(n+1)x^{n+1}}{2^{n+1} n!}\right)\left(\frac{2^n (n-1)!}{n x^n}\right)\right|$$

32) continued...

$$= \lim_{n\to\infty}\left|\left(\frac{(n+1)(x^n)(x^1)}{(2^n)(2^1)(n)(n-1)!}\right)\left(\frac{2^n(n-1)!}{n\,x^n}\right)\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x)}{2(n)(n)}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n+1}{n^2}\right)\frac{x}{2}\right| = \lim_{n\to\infty}\left|\left(\frac{\frac{n}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}}\right)\frac{x}{2}\right| = \lim_{n\to\infty}\left|\left(\frac{\frac{1}{n}+\frac{1}{n^2}}{1}\right)\frac{x}{2}\right|$$

$$= \left|\left(\frac{0+0}{1}\right)\frac{x}{2}\right| = 0 < 1$$ "true statement for all x"

radius of convergence: $R = \infty$

Interval of convergence: $I = (-\infty, \infty)$

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34) $\displaystyle\sum_{n=2}^{\infty}\frac{x^{2n}}{n(\ln n)^2}$ $\quad a_n = \dfrac{x^{2n}}{n(\ln n)^2}$ $\quad a_{n+1} = \dfrac{x^{2(n+1)}}{(n+1)(\ln(n+1))^2}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{2(n+1)}}{(n+1)(\ln(n+1))^2}}{\frac{x^{2n}}{n(\ln n)^2}}\right| = \lim_{n\to\infty}\left|\left(\frac{x^{2n+2}}{(n+1)(\ln(n+1))^2}\right)\left(\frac{n(\ln n)^2}{x^{2n}}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(x^{2n})(x^2)}{(n+1)(\ln(n+1))^2}\right)\left(\frac{n(\ln n)^2}{x^{2n}}\right)\right| = \lim_{n\to\infty}\left|\frac{x^2\,n(\ln n)^2}{(n+1)(\ln(n+1))^2}\right|$$

$$= \lim_{n\to\infty}\left|x^2\left(\frac{n}{n+1}\right)\left(\frac{\ln n}{\ln(n+1)}\right)^2\right| = \left|\lim_{n\to\infty}x^2\right|\left|\lim_{n\to\infty}\frac{n}{n+1}\right|\left(\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}\right)$$

see ex 28 see ex 28

$$= x^2|1|(1) = x^2 \to x^2 < 1 \implies |x| < 1 \to -1 < x < 1$$

34) continued...

radius of convergence: $R=1$

when $x=1$, $x^{2n}=(1)^{2n}=1$ and $x=-1$, $x^{2n}=(-1)^{2n}=((-1)^2)^n=(1)^n=1$

$$\sum_{n=2}^{\infty}\frac{(\pm 1)^{2n}}{n(\ln n)^2}=\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2} \qquad \text{let } f(x)=\frac{1}{x(\ln x)^2}$$

for $x \geq 2$ ① $f(x)$ is continuous

② $f(x)$ is positive

③ $f(x)$ is decreasing as x increases

$$\int \frac{1}{x(\ln x)^2}\,dx=\int\frac{1}{(\ln x)^2}\left(\frac{1}{x}dx\right)=\int\frac{1}{p^2}\,dp=\int p^{-2}\,dp$$

$p=\ln x$

$dp=\frac{1}{x}dx$

$$=\left[\frac{p^{-1}}{-1}\right]+C=\frac{-1}{p}+C=\frac{-1}{\ln x}+C$$

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}\,dx=\lim_{U\to\infty}\int_2^{U}\frac{1}{x(\ln x)^2}\,dx=\lim_{U\to\infty}\left[\frac{-1}{\ln x}+C\right]_2^U$$

$$=\lim_{U\to\infty}\left\{\left[\frac{-1}{\ln U}+C\right]-\left[\frac{-1}{\ln(2)}+C\right]\right\}=[0]-\left[\frac{-1}{\ln 2}\right]=\frac{1}{\ln 2}$$

by the Integral Test, converges when $x=\pm 1$

Interval of convergence: $I=[-1,1]$

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36) $\displaystyle\sum_{n=1}^{\infty}\frac{n!\,x^n}{(1)(3)(5)\cdots(2n-1)}$ $\qquad a_n=\dfrac{n!\,x^n}{(1)(3)(5)\cdots(2n-1)}$

$$a_{n+1}=\frac{(n+1)!\,x^{n+1}}{(1)(3)(5)\cdots(2n-1)(2(n+1)-1)}=\frac{(n+1)!\,x^{n+1}}{(1)(3)(5)\cdots(2n-1)(2n+1)}$$

36) continued...

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!\,x^{n+1}}{(1)(3)(5)\cdots(2n-1)(2n+1)}}{\frac{n!\,x^n}{(1)(3)(5)\cdots(2n-1)}}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(n+1)!\,x^{n+1}}{(1)(3)(5)\cdots(2n-1)(2n+1)}\right)\left(\frac{(1)(3)(5)\cdots(2n-1)}{n!\,x^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(n+1)(n!)\,(x^n)(x^1)}{(1)(3)(5)\cdots(2n-1)(2n+1)}\right)\left(\frac{(1)(3)(5)\cdots(2n-1)}{n!\,x^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\,(x)}{2n+1}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{2n+1}\right)x\right| \overset{L}{=} \lim_{n\to\infty}\left|\left(\frac{1}{2}\right)x\right|$$

$$= \left|\frac{1}{2}x\right| = \frac{1}{2}|x| \rightarrow \quad \frac{1}{2}|x| < 1 \rightarrow |x| < 2 \rightarrow -2 < x < 2$$

radius of convergence: $R = 2$

when $x = 2$, $\sum_{n=1}^{\infty} \frac{n!\,(2)^n}{(1)(3)(5)\cdots(2n-1)}$ $\qquad b_n = \frac{n!\,(2)^n}{(1)(3)(5)\cdots(2n-1)}$

when $x = -2$, $\sum_{n=1}^{\infty} \frac{n!\,(-2)^n}{(1)(3)(5)\cdots(2n-1)}$ $\qquad a_n = \frac{n!\,(-2)^n}{(1)(3)(5)\cdots(2n-1)}$

$$b_n = |a_n| = \frac{n!\,(2)^n}{(1)(3)(5)\cdots(2n-1)} = \frac{\overbrace{\{(1)(2)(3)\cdots(n)\}}^{n\text{-values}}(2)^n}{\underbrace{(1)(3)(5)\cdots(2n-1)}_{n\text{-values}}}$$

$$= \frac{(2)\ (4)\ (6)\ \cdots\ (2n)}{(1)\ (3)\ (5)\ \cdots\ (2n-1)} > 1$$

each values of numerator is larger than corresponding values on denominator

$\lim_{n\to\infty} b_n = +\infty$

by the Test for Divergence, diverges when $x=-2$ and $x=2$

Interval of convergence: $I = (-2, 2)$