Strategy for Testing Series

1. Test for Divergence If you can see that $\lim_{n\to\infty} a_n$ may be different from 0, then apply the Test for Divergence.

2. *p*-Series If the series id of the form $\sum \frac{1}{n^p}$, then it is a *p*-series, which we know to be convergent if $p>1$ and divergent if $p \le 1$.

3. Geometric Series If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, then it is a geometric series, which converges if $|r|<1$ and diverges if $|r|\ge 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

4. Comparison Tests If the series has a form that is similar to a *p*-series or a geometric series, then one of the comparison test should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a *p*-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply a comparison test to $\sum |a_n|$ and test for absolute convergence.

5. Alternating Series Test If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test is an obvious possibility. Note that if $\sum b_n$ converges, then the given series is absolutely convergent and therefore convergent.

6. Ratio Test Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ration Test. Bear in mind that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$ for all *p*-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

7. Root Test If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8. Integral Test If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

Page 2 are descriptions from Thomas's Calculus textbook including description for Alternating Series Test.

Theorem 15 - The Alternating Series Test

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. The u_n's are all positive.
2. The u_n's are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Theorem 16 - The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$.

$$s_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n+1}u_n.$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

Definition

A series that is convergent but not absolutely convergent is called **conditionally convergent**.

Summary of Tests to Determine Convergence or Divergence

1. **The *n*th-Term Test for Divergence**: Unless $a_n \to 0$, the series diverges.

2. **Geometric Series**: $\sum ar^n$ converges if $|r| < 1$; otherwise diverges.

3. ***p*-series**: $\sum \frac{1}{n^p}$ converges if $p > 1$; otherwise diverges.

4. **Series with nonnegative terms**: Try the Integral Test or try comparing to a known series with the Direct Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.

5. **Series with some negative terms**: Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.

6 **Alternating series**: $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

2-a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ $\quad a_n = \frac{(-1)^n}{n^{3/2}}$ $\quad b_n = |a_n| = \frac{1}{n^{3/2}}$

for $n \geq 1$ ① $b_n = \frac{1}{n^{3/2}} > 0$ and $\frac{1}{n^{3/2}} = b_n \geq b_{n+1} = \frac{1}{(n+1)^{3/2}}$

$\{b_n\}$ is decreasing ② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^{3/2}} = 0$

by the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ converges

2-b) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a p-series with $p = \frac{3}{2} > 1$ which converges.

4-a) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ $\quad a_n = \frac{n+1}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{\frac{n}{n} + \frac{1}{n}}{\frac{n}{n}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = \frac{1+0}{1} = 1 \neq 0$

by the Test for Divergence, $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges

4-b) $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n}$ $\quad a_n = (-1)^n \frac{n+1}{n}$ $\quad b_n = |a_n| = \frac{n+1}{n}$

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$ (see 4-a)

now check Alternating Series Test

for $n \geq 1$ ① $b_n = \frac{n+1}{n} > 0$

$\frac{n+1}{n} = b_n \quad b_{n+1} = \frac{(n+1)+1}{(n+1)} = \frac{n+2}{n+1}$ can't be certain that

$\{b_n\}$ is decreasing.

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$ so Alternating Series Test

does not apply. $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n}$ diverges.

6-a) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $\ln e = 1$ so for $n \geq 3$

$$\frac{\ln n}{n} > \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n} > \sum_{n=1}^{\infty} \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1 \leq 1$ which diverges

by the Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

6-b) $\sum_{n=10}^{\infty} \frac{1}{n \ln n}$ let $f(x) = \frac{1}{x \ln x} =$

for interval $[10, \infty)$ ① $f(x)$ is continuous

② $f(x)$ is positive

$$\frac{df}{dx} = \frac{(x \ln x)[0] - (1)\left[(x)\left[\frac{1}{x}(1)\right] + (\ln x)[1]\right]}{(x \ln x)^2} = \frac{-1 - \ln x}{(x \ln x)^2} < 0 \text{ for } x > \frac{1}{e}$$

③ $f(x)$ is decreasing (for $x > \frac{1}{e}$)

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x}\left(\frac{1}{x} dx\right) = \int \frac{1}{p} dp = \ln|p| + C$$

$p = \ln x$

$dp = \frac{1}{x} dx$

$$= \ln|\ln x| + C$$

$$\int_{10}^{\infty} \frac{1}{x \ln x} dx = \lim_{U \to \infty} \int_{10}^{U} \frac{1}{x \ln x} dx = \lim_{U \to \infty} \Big[\ln|\ln x| + C\Big]_{10}^{U}$$

$$= \lim_{U \to \infty} \Big\{ \big[\ln|\ln U| + C\big] - \big[\ln|\ln(10)| + C\big] \Big\} = +\infty$$

by the Integral Test, $\sum_{n=10}^{\infty} \frac{1}{n \ln n}$ diverges.

8-a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ $\quad a_n = \frac{1}{\sqrt{n^2+1}}$

for $n \geq 1$ $\quad \frac{1}{\sqrt{n^2+1}} < \frac{1}{\sqrt{n^2}} = \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}} < \sum_{n=1}^{\infty} \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1 \leq 1$ which diverges. Let $b_n = \frac{1}{n}$

$$\frac{a_n}{b_n} = \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \frac{n}{\sqrt{n^2+1}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^2}}}{\frac{\sqrt{n^2+1}}{\sqrt{n^2}}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\sqrt{\frac{n^2+1}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{1+0}} = \frac{1}{1} = 1 > 0$$

by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges

8-b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

for $n \geq 1$ $\quad 0 \leq \frac{1}{n\sqrt{n^2+1}} < \frac{1}{n\sqrt{n^2}} = \frac{1}{n(n)} = \frac{1}{n^2}$

Since $\quad \sum_{n=1}^{\infty} 0 \leq \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ which converges

by the Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ converges.

10) $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

for $n \geq 1$ $\quad 0 \leq \frac{n-1}{n^3+1} < \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} 0 \leq \sum_{n=1}^{\infty} \frac{n-1}{n^3+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ which converges

by the Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ converges

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12) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$ $\qquad a_n = (-1)^n \frac{n^2-1}{n^2+1}$ $\qquad b_n = |a_n| = \frac{n^2-1}{n^2+1}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2-1}{n^2+1} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2} - \frac{1}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}} = \lim_{n\to\infty} \frac{1 - \frac{1}{n^2}}{1 + \frac{1}{n^2}} = \frac{1-0}{1+0} = 1 \neq 0$$

note: this is also same as testing if $\lim_{n\to\infty} b_n = 0$ for the Alternating Series Test

by the Test for Divergence, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$ diverges

---

14) $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$ $\qquad a_n = \frac{n^{2n}}{(1+n)^{3n}} = \frac{(n^2)^n}{((1+n)^3)^n} = \left(\frac{n^2}{(1+n)^3}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n^2}{(1+n)^3}\right)^n\right|} = \lim_{n\to\infty} \left|\frac{n^2}{(1+n)^3}\right| = \lim_{n\to\infty} \left|\frac{\frac{n^2}{n^3}}{\frac{(1+n)^3}{n^3}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{\frac{1}{n}}{\left(\frac{1+n}{n}\right)^3}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n}}{\left(\frac{1}{n}+1\right)^3}\right| = \frac{0}{(0+1)^3} = 0 < 1$$

by the Root Test, $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$ absolutely converges

11.7/7

16) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$ $\quad a_n = (-1)^{n-1} \frac{n^4}{4^n}$ $\quad a_{n+1} = (-1)^{(n+1)-1} \frac{(n+1)^4}{4^{n+1}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{(n+1)-1} \frac{(n+1)^4}{4^{n+1}}}{(-1)^{n-1} \frac{n^4}{4^n}} \right| = \lim_{n\to\infty} \left| \left((-1)^n \frac{(n+1)^4}{4^{n+1}}\right)\left(\frac{1}{(-1)^{n-1}} \frac{4^n}{n^4}\right) \right|$$

$$= \lim_{n\to\infty} \left| \left((-1)^{n-1}(-1)^1 \frac{(n+1)^4}{(4^n)(4^1)}\right)\left(\frac{1}{(-1)^{n-1}} \frac{4^n}{n^4}\right) \right| = \lim_{n\to\infty} \left| \frac{(-1)(n+1)^4}{4\, n^4} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-1}{4} \left(\frac{n+1}{n}\right)^4 \right| = \lim_{n\to\infty} \left| \frac{-1}{4} \left(\frac{\frac{n}{n} + \frac{1}{n}}{\frac{n}{n}}\right)^4 \right| = \lim_{n\to\infty} \left| \frac{-1}{4} \left(\frac{1+\frac{1}{n}}{1}\right)^4 \right|$$

$$= \left| \frac{-1}{4} \left(\frac{1+0}{1}\right)^4 \right| = \frac{1}{4} < 1$$

by the Ratio Test, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$ absolutely converges

---

18) $\sum_{n=1}^{\infty} n^2 e^{-n^3} = \sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$ $\quad$ let $f(x) = \frac{x^2}{e^{x^3}} = x^2 e^{-x^3}$

for interval $[1, \infty)$ $\quad$ 1 $f(x)$ is continuous

2 $f(x)$ is positive

$$\frac{df}{dx} = \frac{(e^{x^3})[2x] - (x^2)[e^{x^3}(3x^2)]}{(e^{x^3})^2} = \frac{e^{x^3}\{2x - 3x^4\}}{(e^{x^3})^2} = \frac{x(2-3x^3)}{e^{x^3}}$$

$$0 = \frac{df}{dx} = \frac{x(2-3x^3)}{e^{x^3}}$$

$$0 = x(2-3x^3)$$

$x = 0$ discard $\quad \Big| \quad 0 = 2 - 3x^3$

$3x^3 - 2 = 0$

$$(\sqrt[3]{3}x)^3 - (\sqrt[3]{2})^3 = 0$$

$$(\sqrt[3]{3}x) - (\sqrt[3]{2})\left((\sqrt[3]{3}x)^2 + (\sqrt[3]{3}x)(\sqrt[3]{2}) + (\sqrt[3]{2})^2\right) = 0$$

$$\sqrt[3]{3}x - \sqrt[3]{2} = 0$$

$$\sqrt[3]{3}x = \sqrt[3]{2}$$

$$x = \frac{\sqrt[3]{2}}{\sqrt[3]{3}} = \sqrt[3]{\frac{2}{3}} < 1$$

18) continued...

so for $x \geq 1$ ③ $\frac{df}{dx} = \frac{x(2-3x^3)}{e^{x^3}} < 0$, $f(x)$ is decreasing

$$\int x^2 e^{-x^3} dx = \int e^{-x^3}(x^2 dx) = \int e^{p}\left(\frac{-1}{3}dp\right) = \frac{-1}{3}e^{p} + C$$

$p = -x^3$

$dp = -3x^2 dx \Rightarrow \frac{-1}{3}dp = x^2 dx$

$$= \frac{-1}{3}e^{-x^3} + C = \frac{-1}{3e^{x^3}} + C$$

$$\int_1^{\infty} \frac{x^2}{e^{x^3}} dx = \lim_{v \to \infty} \int_1^{v} \frac{x^2}{e^{x^3}} dx = \lim_{v \to \infty} \left[\frac{-1}{3e^{x^3}} + C\right]_1^{v}$$

$$= \lim_{v \to \infty} \left\{\left[\frac{-1}{3e^{v^3}} + C\right] - \left[\frac{-1}{3e^{(1)^3}} + C\right]\right\} = [0] - \left[\frac{-1}{3e}\right] = \frac{1}{3e}$$

by the Integral Test, $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges

---

20) $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$

for $k \geq 1$ $\quad 0 \leq \frac{1}{k\sqrt{k^2+1}} < \frac{1}{k\sqrt{k^2}} = \frac{1}{k(k)} = \frac{1}{k^2}$

Since $\sum_{k=1}^{\infty} 0 \leq \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}} < \sum_{k=1}^{\infty} \frac{1}{k^2}$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p-series with $p = 2 > 1$ which converges

by the Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$ converges.

11.7/9

22) $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$ $\quad a_n = \frac{\sin 2n}{1+2^n}$ $\quad |a_n| = \left|\frac{\sin 2n}{1+2^n}\right|$

for $n \geq 1$ $\quad 0 \leq \left|\frac{\sin(2n)}{1+2^n}\right| \leq \frac{1}{1+2^n} < \frac{1}{2^n}$

Since $\sum_{n=1}^{\infty} 0 \leq \sum_{n=1}^{\infty} \left|\frac{\sin(2n)}{1+2^n}\right| < \sum_{n=1}^{\infty} \frac{1}{2^n}$

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1}$ is a geometric series with

$a = \frac{1}{2}$ and $|r| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$ which converges.

by the Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$ absolutely converges.

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24) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ $\quad a_n = \frac{\sqrt{n^4+1}}{n^3+n}$

for $n \geq 1$ $\quad 0 \leq \frac{\sqrt{n^4+1}}{n^3+n} \quad \frac{\sqrt{n^4}}{n^3+n} < \frac{\sqrt{n^4}}{n^3} = \frac{n^2}{n^3} = \frac{1}{n}$

not completely confident that the sign is <

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1 \leq 1$ which diverges. Let $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n^4+1}}{n^3+n}}{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{\sqrt{n^4+1}}{n(n^2+1)}\right)\left(\frac{n}{1}\right) = \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{\frac{\sqrt{n^4+1}}{\sqrt{n^4}}}{\frac{n^2+1}{\sqrt{n^4}}} = \lim_{n\to\infty} \frac{\sqrt{\frac{n^4+1}{n^4}}}{\frac{n^2+1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^4}}}{1+\frac{1}{n^2}} = \frac{\sqrt{1+0}}{1+0} = 1 > 0$$

by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ diverges.

11.7/10

26) $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ $\quad a_n = \frac{(-1)^{n-1}}{\sqrt{n}-1}$ $\quad b_n = |a_n| = \frac{1}{\sqrt{n}-1}$

for $n \geq 2$ $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} > \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a p-series (partial) with $p = \frac{1}{2} \leq 1$

which diverges. So $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ does not absolutely converge

for $n \geq 2$ ① $b_n = \frac{1}{\sqrt{n}-1} > 0$ and $\frac{1}{\sqrt{n}-1} = b_n > b_{n+1} = \frac{1}{\sqrt{n+1}-1}$

$\{b_n\}$ is decreasing ② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$

By the Alternating Series Test, $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ converges (conditionally)

---

28) $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$

for $k \geq 1$ $\quad 0 \leq \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)} < \frac{\sqrt[3]{k}}{k(\sqrt{k}+1)} < \frac{\sqrt[3]{k}}{k(\sqrt{k})} = \frac{k^{\frac{1}{3}}}{k^{\frac{3}{2}}} = \frac{k^{\frac{2}{6}}}{k^{\frac{9}{6}}} = \frac{1}{k^{7/6}}$

Since $\sum_{k=1}^{\infty} 0 \leq \sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)} \leq \sum_{k=1}^{\infty} \frac{1}{k^{7/6}}$

$\sum_{k=1}^{\infty} \frac{1}{k^{7/6}}$ is a p-series with $p = \frac{7}{6} > 1$ which converges

By the Direct Comparison Test, $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$ converges.

30) $\sum_{k=1}^{\infty} \frac{1}{2+\sin k}$ $\quad a_k = \frac{1}{2+\sin k}$

$\lim_{n\to\infty} |a_k| = \lim_{n\to\infty} \left|\frac{1}{2+\sin k}\right| = \lim_{n\to\infty} \frac{1}{2+\sin k}$ D.N.E.

because the terms will occilate betwen $\frac{1}{3}$ and 1.

by the Test for Divergence, $\sum_{k=1}^{\infty} \frac{1}{2+\sin k}$ diverges

---

32) $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$ $\quad a_n = n \sin\left(\frac{1}{n}\right)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{n^{-1}} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$

$= \lim_{p\to 0^+} \frac{\sin p}{p} = 1 \neq 0$

by the Test for Divergence, $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$ diverges

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34) $\sum_{n=1}^{\infty} \frac{8+(-1)^n n}{n}$ $\quad a_n = \frac{8+(-1)^n n}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{8+(-1)^n n}{n} = \lim_{n\to\infty} \left(\frac{8}{n} + \frac{(-1)^n n}{n}\right) = \lim_{n\to\infty} \left(\frac{8}{n} + (-1)^n\right)$

$= \lim_{n\to\infty} \left(\frac{8}{n}\right) + \lim_{n\to\infty} (-1)^n = 0 +$ D.N.E $=$ D.N.E.

because $\lim_{n\to\infty} (-1)^n$ alternates between -1 and 1

by the Test for Divergence, $\sum_{n=1}^{\infty} \frac{8+(-1)^n n}{n}$ diverges

11.7/12

36) $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$ $\quad a_n = \frac{n^2+1}{5^n}$ $\quad a_{n+1} = \frac{(n+1)^2+1}{5^{n+1}} = \frac{(n^2+2n+1)+1}{(5^n)(5^1)} = \frac{n^2+2n+2}{(5^n)(5)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{n^2+2n+2}{(5^n)(5)}}{\frac{n^2+1}{5^n}}\right| = \lim_{n\to\infty}\left|\left(\frac{n^2+2n+2}{(5^n)(5)}\right)\left(\frac{5^n}{n^2+1}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^2+2n+2}{5n^2+5}\right| = \lim_{n\to\infty}\left|\frac{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{2}{n^2}}{\frac{5n^2}{n^2}+\frac{5}{n^2}}\right| = \lim_{n\to\infty}\left|\frac{1+\frac{2}{n}+\frac{2}{n^2}}{5+\frac{5}{n^2}}\right|$$

$$= \left|\frac{1+0+0}{5+0}\right| = \left|\frac{1}{5}\right| = \frac{1}{5} < 1$$

by the Ratio Test, $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$ converges

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38) $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$

option 1:

for $n \geq 1$ $\; e = e^{\frac{1}{1}} > e^{\frac{1}{2}} > \dots e^{\frac{1}{n}}$ so $\; 0 \leq \frac{e^{\frac{1}{n}}}{n^2} \leq \frac{e}{n^2}$

Since $\sum_{n=1}^{\infty} 0 \leq \sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2} \leq \sum_{n=1}^{\infty} \frac{e}{n^2}$

$\sum_{n=1}^{\infty} \frac{e}{n^2} = e \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ which converges

by the Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ converges

option 2:

let $f(x) = \frac{e^{\frac{1}{x}}}{x^2}$ for $x \geq 1$ $\quad$ ① $f(x)$ is continuous

② $f(x)$ is positive

③ $f(x)$ is decreasing

$$\int \frac{e^{\frac{1}{x}}}{x^2}\,dx = \int e^{\frac{1}{x}}\left(\frac{1}{x^2}\,dx\right) = \int e^{p}(-1\,dp) = -1e^{p} + C = -1\,e^{\frac{1}{x}} + C$$

$p = \frac{1}{x} \quad dp = \frac{-1}{x^2}\,dx \rightarrow -1\,dp = \frac{1}{x^2}\,dx$

38) continued...

$$\int_1^\infty \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{u\to\infty}\int_1^u \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{u\to\infty}\left[-e^{\frac{1}{x}}+C\right]_1^u$$

$$= \lim_{u\to\infty}\left\{\left[-e^{\frac{1}{u}}+C\right]-\left[-e^{\frac{1}{(1)}}+C\right]\right\} = \left[-e^0\right]-\left[-e^1\right] = e-1$$

by the Integral Test, $\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n^2}$ converges

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40) $\sum_{j=1}^{\infty}(-1)^j\frac{\sqrt{j}}{j+5}$ $\quad a_j = (-1)^j\frac{\sqrt{j}}{j+5}$ $\quad b_j = |a_j| = \frac{\sqrt{j}}{j+5}$

let $f(x) = \frac{\sqrt{x}}{x+5}$ for $x \geq 1$ ① $f(x)$ is continuous

② $f(x)$ is positive

$$\frac{df}{dx} = \frac{(x+5)\left[\frac{1}{2\sqrt{x}}\right]-(\sqrt{x})[1]}{(x+5)^2} = \frac{\frac{x+5}{2\sqrt{x}} - \sqrt{x}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)}{(x+5)^2} = \frac{\frac{x+5-2x}{2\sqrt{x}}}{(x+5)^2} = \frac{5-x}{2\sqrt{x}(x+5)^2}$$

$$0 = \frac{df}{dx} = \frac{5-x}{2\sqrt{x}(x+5)^2}$$

$0 = 5-x$

$x = 5$

for $x > 5$ $\quad \frac{df}{dx} = \frac{5-x}{2\sqrt{x}(x+5)^2} < 0$

③ $f(x)$ is decreasing (for $x > 5$)

$$\int \frac{\sqrt{x}}{x+5}\,dx = \int \frac{p}{(p^2)+5}(2p\,dp) = \int \frac{2p^2}{p^2+5}\,dp = \int\left(2+\frac{(-10)}{p^2+5}\right)dp$$

$p = \sqrt{x}$

$p^2 = x$

$2p\,dp = dx$

$$\begin{array}{r} 2 \\ p^2+0p+5\overline{)\,2p^2+0p+0} \\ -(2p^2+0p+10) \\ \hline -10 \end{array}$$

$$= \int\left(2-\frac{10}{p^2+(\sqrt{5})^2}\right)dp$$

$$= 2[p] - 10\left[\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{p}{\sqrt{5}}\right)\right]+C$$

$$= 2\sqrt{x} - \frac{10}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{x}}{\sqrt{5}}\right)+C$$

40) continued...

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$$\int_1^{\infty} \frac{\sqrt{x}}{x+5}\,dx = \lim_{v\to\infty}\int_1^{v} \frac{\sqrt{x}}{x+5}\,dx = \lim_{v\to\infty}\left[2\sqrt{x} - \frac{10}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{x}}{\sqrt{5}}\right)+C\right]_1^{v}$$

$$= \lim_{v\to\infty}\left\{\left[2\sqrt{v} - \frac{10}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{v}}{\sqrt{5}}\right)+C\right] - \left[2\sqrt{(1)} - \frac{10}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{(1)}}{\sqrt{5}}\right)+C\right]\right\} = +\infty$$

This means that $\sum_{j=1}^{\infty}(-1)^j\frac{\sqrt{j}}{j+5}$ is not absolutelly convergent.

for $j \geq 1$ "using criterias of the Integral Test above"

1. $b_j > 0$ and for $j > 5$ $\{b_j\}$ is decreasing
2. $\lim_{j\to\infty} b_j = \lim_{j\to\infty}\frac{\sqrt{j}}{j+5} = \lim_{j\to\infty}\frac{\frac{\sqrt{j}}{\sqrt{j}}}{\frac{j}{\sqrt{j}}+\frac{5}{\sqrt{j}}} = \lim_{j\to\infty}\frac{1}{\sqrt{j}+\frac{5}{j}} = 0$

by the Alternate Series Test, $\sum_{j=1}^{\infty}(-1)^j\frac{\sqrt{j}}{j+5}$ converges (conditionally)

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42) $\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{4n}}$ $\quad a_n = \frac{(n!)^n}{n^{4n}} = \frac{(n!)^n}{(n^4)^n} = \left(\frac{n!}{n^4}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n!}{n^4}\right)^n\right|} = \lim_{n\to\infty}\left|\frac{n!}{n^4}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\left(\frac{n-3}{n}\right)(n-4)!\right|$$

$$= \lim_{n\to\infty}\left|(1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)(n-4)!\right| = +\infty$$

by the Root Test, $\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{4n}}$ diverges

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44) $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$

for $n \geq 1$ $\quad 0 \leq n+n\cos^2 n \leq n+n = 2n$

so $\quad \frac{1}{n+n\cos^2 n} \geq \frac{1}{n+n} = \frac{1}{2n}$

Since $\quad \sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n} \geq \sum_{n=1}^{\infty} \frac{1}{2n}$

$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p=1 \leq 1$ which diverges

by the Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ diverges

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46) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ $\quad$ note: $x = e^{\ln x}$

colors used here to make it easier to follow

$(\ln n)^{\ln n} = \left(e^{\ln \ln n}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln \ln n} = n^{\ln \ln n}$

and $\ln \ln n \to \infty$ as $n \to \infty$

so $\ln \ln n > 2$ for sufficiently large n

$\ln \ln n = 2$
$\Downarrow$
$\ln n = e^2$
$\Downarrow$
$n = e^{e^2}$

$n = e^{e^2} \approx 1618.177992\ldots$ so for $n \geq 1619$

we get $\quad (\ln n)^{\ln n} > n^2$

and $\quad 0 \leq \frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}$

Since $\sum_{n=2}^{\infty} 0 \leq \sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}} < \sum_{n=2}^{\infty} \frac{1}{n^2}$

46) continued...

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a p-series with $p=2>1$ which converges

By the Direct Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converges

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48) $\sum_{n=1}^{\infty} \left(\sqrt[n]{2}-1\right)$ $a_n = \sqrt[n]{2}-1 = 2^{\frac{1}{n}}-1$

this one has $\frac{1}{n}$ as exponent with a constant base, so the best option is to use Limit Comparison Test with $b_n = \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p=1\leq 1$ which diverges

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^{\frac{1}{n}}-1}{\frac{1}{n}} \overset{L}{=} \lim_{n\to\infty} \frac{2^{\frac{1}{n}}(\ln 2)\left(\frac{-1}{n^2}\right)}{\frac{-1}{n^2}} = \lim_{n\to\infty} 2^{\frac{1}{n}}(\ln 2)$$

$$= 2^0(\ln 2) = 1(\ln 2) = \ln 2 > 0$$

By the Limit Comparison Test, $\sum_{n=1}^{\infty} \left(\sqrt[n]{2}-1\right)$ diverges