The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

The Root Test

(i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Below are descriptions from Thomas's Calculus textbook.

Theorem 13 - The Ratio Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho .$$

Then **(a)** the series *converges absolutely* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

Theorem 14 - The Root Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \rho .$$

Then **(a)** the series *converges absolutely* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

2) given Σa_n with $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 2$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{\frac{a_n}{a_{n+1}}}\right| = \frac{\lim_{n\to\infty}|1|}{\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|} = \frac{|1|}{|2|} = \frac{1}{2} < 1$$

By the Ratio Test, the series Σa_n is absolutely convergent.

---

4) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ $\quad a_n = \frac{(-2)^n}{n^2} \quad a_{n+1} = \frac{(-2)^{n+1}}{(n+1)^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-2)^{n+1}}{(n+1)^2}}{\frac{(-2)^n}{n^2}}\right| = \lim_{n\to\infty}\left|\left(\frac{(-2)^{n+1}}{(n+1)^2}\right)\left(\frac{n^2}{(-2)^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(-2)^n(-2)^1}{(n+1)^2}\right)\left(\frac{n^2}{(-2)^n}\right)\right| = \lim_{n\to\infty}\left|\frac{(-2)\,n^2}{(n+1)^2}\right|$$

$$= \lim_{n\to\infty}|-2|\left|\frac{n^2}{n^2+2n+1}\right| = |-2|\lim_{n\to\infty}\left|\frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}}\right|$$

$$= 2\lim_{n\to\infty}\left|\frac{1}{1+\frac{2}{n}+\frac{1}{n^2}}\right| = 2\left|\frac{1}{1+0+0}\right| = 2 > 1$$

By the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$ diverges

6) $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ $\quad a_n = \frac{(-3)^n}{(2n+1)!}$ $\quad a_{n+1} = \frac{(-3)^{n+1}}{(2(n+1)+1)!} = \frac{(-3)^{n+1}}{(2n+3)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-3)^{n+1}}{(2n+3)!}}{\frac{(-3)^n}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\left(\frac{(-3)^{n+1}}{(2n+3)!}\right)\left(\frac{(2n+1)!}{(-3)^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(-3)^n(-3)^1}{(2n+3)(2n+2)(2n+1)!}\right)\left(\frac{(2n+1)!}{(-3)^n}\right)\right| = \lim_{n\to\infty}\left|\frac{-3}{(2n+3)(2n+2)}\right| = 0 < 1$$

by the Ratio Test, $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ absolutely converges.

---

8) $\sum_{k=1}^{\infty} ke^{-k} = \sum_{k=1}^{\infty} \frac{k}{e^k}$ $\quad a_n = \frac{n}{e^n}$ $\quad a_{n+1} = \frac{(n+1)}{e^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)}{e^{n+1}}}{\frac{n}{e^n}}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{e^{n+1}}\right)\left(\frac{e^n}{n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n+1}{(e^n)(e^1)}\right)\left(\frac{e^n}{n}\right)\right| = \lim_{n\to\infty}\left|\frac{n+1}{en}\right| \overset{L}{=} \lim_{n\to\infty}\left|\frac{1}{e}\right| = \frac{1}{e} < 1$$

by the Ratio Test, $\sum_{k=1}^{\infty} ke^{-k}$ absolutely converges.

---

10) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ $\quad a_n = \frac{n!}{100^n}$ $\quad a_{n+1} = \frac{(n+1)!}{100^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}}\right| = \lim_{n\to\infty}\left|\left(\frac{(n+1)!}{100^{n+1}}\right)\left(\frac{100^n}{n!}\right)\right|$$

10) continued...

$$= \lim_{n\to\infty}\left|\left(\frac{(n+1)(n!)}{(100^n)(100^1)}\right)\left(\frac{100^n}{n!}\right)\right| = \lim_{n\to\infty}\left|\frac{n+1}{100}\right| = +\infty$$

by the Ratio Test, $\sum_{n=1}^{\infty}\frac{n!}{100^n}$ diverges

---

12) $\sum_{n=1}^{\infty}\frac{n^{10}}{(-10)^{n+1}}$ $\quad a_n = \frac{n^{10}}{(-10)^{n+1}}$ $\quad a_{n+1} = \frac{(n+1)^{10}}{(-10)^{(n+1)+1}} = \frac{(n+1)^{10}}{(-10)^{n+2}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{10}}{(-10)^{n+2}}}{\frac{n^{10}}{(-10)^{n+1}}}\right| = \lim_{n\to\infty}\left|\left(\frac{(n+1)^{10}}{(-10)^{n+2}}\right)\left(\frac{(-10)^{n+1}}{n^{10}}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(n+1)^{10}}{((-10)^{n+1})(-10^1)}\right)\left(\frac{(-10)^{n+1}}{n^{10}}\right)\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{10}}{-10n^{10}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{10}}{n^{10}}}{\frac{-10n^{10}}{n^{10}}}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{n+1}{n}\right)^{10}}{-10}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{n}{n}+\frac{1}{n}\right)^{10}}{-10}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)^{10}}{-10}\right| = \left|\frac{(1+0)^{10}}{-10}\right| = \frac{1}{10} < 1$$

by the Ratio Test, $\sum_{n=1}^{\infty}\frac{n^{10}}{(-10)^{n+1}}$ absolutely converges.

14) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\quad a_n = \frac{n!}{n^n}$ $\quad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| = \lim_{n\to\infty}\left|\left(\frac{(n+1)!}{(n+1)^{n+1}}\right)\left(\frac{n^n}{n!}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(n+1)(n!)}{((n+1)^n)((n+1)^1)}\right)\left(\frac{n^n}{n!}\right)\right| = \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n}\right| = \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^n\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)^n\right| = \lim_{n\to\infty}\left|\left(\frac{1}{1+\frac{1}{n}}\right)^n\right| = \lim_{n\to\infty}\left|\frac{(1)^n}{(1+\frac{1}{n})^n}\right|$$

$$= \left|\frac{1}{\underbrace{\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n}}\right| = \left|\frac{1}{e}\right| = \frac{1}{e} < 1$$

see pg 4 of my 11.1 examples

by the Ratio Test, $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ absolutely converges.

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16) $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ $\quad a_n = \frac{(2n)!}{(n!)^2}$ $\quad a_{n+1} = \frac{(2(n+1))!}{((n+1)!)^2} = \frac{(2n+2)!}{((n+1)!)^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}}\right| = \lim_{n\to\infty}\left|\left(\frac{(2n+2)!}{((n+1)!)^2}\right)\left(\frac{(n!)^2}{(2n)!}\right)\right|$$

16) continued...

$$= \lim_{n\to\infty}\left|\left(\frac{(2n+2)(2n+1)(2n)!}{((n+1)(n!))^2}\right)\left(\frac{(n!)^2}{(2n)!}\right)\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)}{(n+1)^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{4n^2+6n+2}{n^2+2n+1}\right| = \lim_{n\to\infty}\left|\frac{\frac{4n^2}{n^2}+\frac{6n}{n^2}+\frac{2}{n^2}}{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{4+\frac{6}{n}+\frac{2}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}}\right| = \left|\frac{4+0+0}{1+0+0}\right| = 4 > 1$$

by the Ratio Test, $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ diverges

---

18) $\frac{2}{3} + \frac{2}{3}\cdot\frac{5}{5} + \frac{2}{3}\cdot\frac{5}{5}\cdot\frac{8}{7} + \frac{2}{3} + \frac{5}{5}\cdot\frac{8}{7}\cdot\frac{11}{9} + \ldots$

$+ \frac{2}{3}\cdot\frac{5}{5}\cdot\frac{8}{7}\cdot\frac{11}{9}\cdot\ldots\cdot\frac{3n-1}{2n+1} + \ldots = \sum_{n=1}^{\infty}\left(\prod_{i=1}^{n}\frac{3i-1}{2i+1}\right)$

(note $\prod_{i=1}^{4} i = (1)(2)(3)(4)$ "product notation")

$$a_n = \prod_{i=1}^{n}\frac{3i-1}{2i+1} \qquad a_{n+1} = \prod_{i=1}^{n+1}\frac{3i-1}{2i+1} = \left(\prod_{i=1}^{n}\frac{3i-1}{2i+1}\right)\left(\frac{3(n+1)-1}{2(n+1)+1}\right)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\prod_{i=1}^{n+1}\frac{3i-1}{2i+1}\right)}{\prod_{i=1}^{n}\frac{3i-1}{2i+1}}\right| = \lim_{n\to\infty}\left|\frac{\left(\prod_{i=1}^{n}\frac{3i-1}{2i+1}\right)\left(\frac{3(n+1)-1}{2(n+1)+1}\right)}{\prod_{i=1}^{n}\frac{3i-1}{2i+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3(n+1)-1}{2(n+1)+1}\right| = \lim_{n\to\infty}\left|\frac{3n+2}{2n+3}\right| \stackrel{L}{=} \lim_{n\to\infty}\left|\frac{3}{2}\right| = \frac{3}{2} > 1$$

18) continued...

by the Ratio Test, $\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n}\frac{3i-1}{2i+1}\right)$ diverges.

---

20) $\sum_{n=1}^{\infty}(-1)^n\frac{2^n n!}{(5)(8)(11)\cdots(3n+2)}$ $\quad a_n=(-1)^n\frac{2^n n!}{(5)(8)(11)\cdots(3n+2)}$

$$a_{n+1}=(-1)^{n+1}\frac{2^{n+1}(n+1)!}{(5)(8)(11)\cdots(3n+2)(3(n+1)+2)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\dfrac{2^{(n+1)}(n+1)!}{(5)(8)(11)(3n+2)(3(n+1)+2)}}{\dfrac{2^n n!}{(5)(8)(11)(3n+2)}}\right|$$

$$=\lim_{n\to\infty}\left|\left(\frac{2^{n+1}(n+1)!}{(5)(8)(11)(3n+2)(3(n+1)+2)}\right)\left(\frac{(5)(8)(11)(3n+2)}{2^n n!}\right)\right|$$

$$=\lim_{n\to\infty}\left|\left(\frac{(2^n)(2^1)(n+1)(n!)}{(5)(8)(11)(3n+2)(3n+5)}\right)\left(\frac{(5)(8)(11)(3n+2)}{2^n n!}\right)\right|$$

$$=\lim_{n\to\infty}\left|\frac{2(n+1)}{3n+5}\right|=\lim_{n\to\infty}\left|\frac{2n+2}{3n+5}\right|\leq\lim_{n\to\infty}\left|\frac{2}{3}\right|=\frac{2}{3}<1$$

by the Ratio Test, $\sum_{n=1}^{\infty}(-1)^n\frac{2^n n!}{(5)(8)(11)(3n+2)}$ absolutely converges.

22) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ $\quad a_n = \frac{(-2)^n}{n^n}$ $\quad |a_n| = \frac{2^n}{n^n} = \left(\frac{2}{n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2}{n}\right)^n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$$

by the Root Test, $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ absolutely converges.

---

24) $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ $\quad a_n = \left(\frac{-2n}{n+1}\right)^{5n}$ $\quad |a_n| = \left(\frac{2n}{n+1}\right)^{5n} = \left(\left(\frac{2n}{n+1}\right)^5\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\left(\frac{2n}{n+1}\right)^5\right)^n} = \lim_{n\to\infty} \left(\frac{2n}{n+1}\right)^5$$

$$= \lim_{n\to\infty} \left(\frac{\frac{2n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)^5 = \lim_{n\to\infty} \left(\frac{2}{1+\frac{1}{n}}\right)^5 = \left(\frac{2}{1+0}\right)^5 = (2)^5 > 1$$

by the Root Test, $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ diverges

---

26) $\sum_{n=0}^{\infty} (\arctan n)^n = \sum_{n=0}^{\infty} (\tan^{-1} n)^n$ $\quad a_n = (\tan^{-1} n)^n$ $\quad |a_n| = (\tan^{-1} n)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{(\tan^{-1} n)^n} = \lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2} > 1$$

by the Root Test, $\sum_{n=0}^{\infty} (\tan^{-1} n)^n$ diverges.

28) $\sum_{n=1}^{\infty}\left(\frac{1-n}{2+3n}\right)^n$ Root Test

$a_n=\left(\frac{1-n}{2+3n}\right)^n \qquad |a_n|=\left|\left(\frac{1-n}{2+3n}\right)^n\right|=\left(\left|\frac{1-n}{2+3n}\right|\right)^n$

$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left(\left|\frac{1-n}{2+3n}\right|\right)^n}=\lim_{n\to\infty}\left|\frac{1-n}{2+3n}\right|\overset{L}{=}\lim_{n\to\infty}\left|\frac{-1}{3}\right|=\frac{1}{3}<1$

By the Root Test $\sum_{n=1}^{\infty}\left(\frac{1-n}{2+3n}\right)^n$ absolutely converges.

---

30) $\sum_{n=1}^{\infty}\frac{n\,5^{2n}}{10^{n+1}}$ Ratio Test

$a_n=\frac{n\,5^{2n}}{10^{n+1}} \qquad a_{n+1}=\frac{(n+1)5^{2(n+1)}}{10^{(n+1)+1}}=\frac{(n+1)5^{2n+2}}{10^{n+2}}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{(n+1)5^{2n+2}}{10^{n+2}}}{\frac{n\,5^{2n}}{10^{n+1}}}\right|=\lim_{n\to\infty}\left|\left(\frac{(n+1)5^{2n+2}}{10^{n+2}}\right)\left(\frac{10^{n+1}}{n\,5^{2n}}\right)\right|$

$=\lim_{n\to\infty}\left|\left(\frac{(n+1)(5^{2n})(5^2)}{(10^{n+1})(10^1)}\right)\left(\frac{10^{n+1}}{n\,5^{2n}}\right)\right|=\lim_{n\to\infty}\left|\frac{(5^2)(n+1)}{10\,n}\right|$

$=\lim_{n\to\infty}\left|\frac{5n+5}{2n}\right|\overset{L}{=}\lim_{n\to\infty}\left|\frac{5}{2}\right|=\frac{5}{2}>1$

By the Ratio Test, $\sum_{n=1}^{\infty}\frac{n\,5^{2n}}{10^{n+1}}$ diverges

32) $\sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{6})}{1+n\sqrt{n}}$ Direct Comparison Test

for $n \geq 1$ $\quad 0 \leq \frac{\sin(n\frac{\pi}{6})}{1+n\sqrt{n}} \leq \frac{1}{1+n\sqrt{n}} < \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$

Since $\sum_{n=1}^{\infty} 0 \leq \sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{6})}{1+n\sqrt{n}} < \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ and

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a p-series with $p = \frac{3}{2} > 1$ which converges,

$\sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{6})}{1+n\sqrt{n}}$ converges.

---

34) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ [hint: $\ln n < \sqrt{x}$]

let $a_n = \frac{(-1)^n}{\sqrt{n}\ln n}$ $\qquad b_n = |a_n| = \frac{1}{\sqrt{n}\ln n}$

for $n \geq 2$, $\frac{1}{\sqrt{n}\ln n} > \frac{1}{\sqrt{n}(\sqrt{n})} = \frac{1}{n} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$

$\sum_{n=2}^{\infty} \frac{1}{n}$ is a p-series with $p = 1 \leq 1$ which diverges

so $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ (diverges) is not absolutely convergent.

for $n \geq 2$ ① $b_n = \frac{1}{\sqrt{n}\ln n} > 0$ and $\frac{1}{\sqrt{n}\ln n} = b_n > b_{n+1} = \frac{1}{\sqrt{n+1}\ln(n+1)}$

$\{b_n\}$ is decreasing.

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}\ln n} = 0$ by the Alternating Series Test

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ converges.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ is conditionally convergent.

11.6/11

36) Σa_n defined by $a_1 = 1$ $a_{n+1} = \frac{2+\cos n}{\sqrt{n}} a_n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2+\cos n}{\sqrt{n}}a_n}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2+\cos n}{\sqrt{n}}\right| \le \lim_{n\to\infty}\left|\frac{2+1}{\sqrt{n}}\right| = 0 < 1$$

by the Ratio Test, Σa_n absolutely converges

---

38) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n b_1 b_2 b_3 \cdots b_n}$ and $\sum_{n=1}^{\infty} b_n = \frac{1}{2}$

$$a_n = \frac{(-1)^n n!}{n^n b_1 b_2 b_3 \cdots b_n} \qquad a_{n+1} = \frac{(-1)^{n+1}(n+1)!}{(n+1)^{n+1} b_1 b_2 b_3 \cdots b_n b_{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)!}{(n+1)^{n+1} b_1 b_2 b_3 \cdots b_n b_{n+1}}}{\frac{(-1)^n n!}{n^n b_1 b_2 b_3 \cdots b_n}}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{(-1)^{n+1}(n+1)!}{(n+1)^{n+1} b_1 b_2 b_3 \cdots b_n b_{n+1}}\right)\left(\frac{n^n b_1 b_2 b_3 \cdots b_n}{(-1)^n n!}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{((-1)^n)((-1)^1)(n+1)(n!)}{((n+1)^n)((n+1)^1) b_1 b_2 b_3 \cdots b_n b_{n+1}}\right)\left(\frac{n^n b_1 b_2 b_3 \cdots b_n}{(-1)^n n!}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)\, n^n}{(n+1)^n b_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)}{b_{n+1}}\left(\frac{n}{n+1}\right)^n\right| = \lim_{n\to\infty}\left|\frac{(-1)}{b_{n+1}}\left(\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)^n\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)}{b_{n+1}}\left(\frac{1}{1+\frac{1}{n}}\right)^n\right| = \frac{|(-1)|}{\left|\lim_{n\to\infty}(b_{n+1})\left(1+\frac{1}{n}\right)^n\right|} = \frac{|-1|}{|(\frac{1}{2})(e)|} = \frac{1}{\frac{1}{2}e} = \frac{2}{e} < 1$$

38) continued...

by the Ratio Test, $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n b_1 b_2 b_3 \cdots b_n}$ absolutely converges.

---

40) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$ $\quad a_n = \frac{(n!)^2}{(kn)!}$ $\quad a_{n+1} = \frac{((n+1)!)^2}{(k(n+1))!}$

k is positive integer

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2}{(k(n+1))!}}{\frac{(n!)^2}{(kn)!}}\right| = \lim_{n\to\infty}\left|\left(\frac{((n+1)!)^2}{(k(n+1))!}\right)\left(\frac{(kn)!}{(n!)^2}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{((n+1)(n!))^2}{(k(n+1))(k(n+1)-1)(k(n+1)-2)\cdots(kn+1)(kn)!}\right)\left(\frac{(kn)!}{(n!)^2}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(k(n+1))(k(n+1)-1)(k(n+1)-2)\cdots(kn+1)}\right|$$

if $k=1$ then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{1(n+1)}\right| = \lim_{n\to\infty}\left|\frac{n^2+2n+1}{n+1}\right| \overset{L}{=} \lim_{n\to\infty}\left|\frac{2n+2}{1}\right| = \infty$$

diverges

if $k=2$ then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2(n+1))(2n+1)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^2+2n+1}{4n^2+6n+2}\right| = \lim_{n\to\infty}\left|\frac{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}}{\frac{4n^2}{n^2}+\frac{6n}{n^2}+\frac{2}{n^2}}\right| = \lim_{n\to\infty}\left|\frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+\frac{6}{n}+\frac{2}{n^2}}\right|$$

40) continued...

$$= \left| \frac{1+0+0}{4+0+0} \right| = \left| \frac{1}{4} \right| = \frac{1}{4} < 1 \text{ converges}$$

if $k=3$ then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(3(n+1))(3(n+1)-1)(3(n+1)-2)\cdots(3n+1)} \right| = 0 < 1$$

because the highest power of n in the denominator is larger than 2. This will be true for all $k \geq 3$.

converges.

Therefore $\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$ converges for $k \geq 2$.