Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad (b_n > 0)$$

satisfies the conditions

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Alternating Series Estimation Theorem

If $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

(i) $b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

then $|R_n| = |s - s_n| \le b_{n+1}$

$\boxed{1}$ **Definition**

A series $\sum a_n$ is called **absolutely convergent** is the series of absolute values $\sum |a_n|$ is convergent.

$\boxed{2}$ **Definition**

A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent; that is, if $\sum a_n$ converges but $\sum |a_n|$ diverges.

$\boxed{3}$ **Theorem**

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Below are descriptions from Thomas's Calculus textbook.

Definition

A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$ converges.

Theorem 12 - The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Description of Alternating Series Test from Thomas' Calculus textbook is located in notes of section 11.7.

2) $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \dots$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{2n+1} \longrightarrow b_n = \frac{2}{2n+1}$

① for $n \geq 1$, $0 < \frac{2}{2n+1} = b_n \geq b_{n+1} = \frac{2}{2(n+1)+1} = \frac{2}{2n+2}$, $\{b_n\}$ is decreasing

② $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{2}{2n+1} = 0$

by the Alternate Series Test, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{2n+1}$ converges.

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4) $\frac{1}{\ln(3)} - \frac{1}{\ln(4)} + \frac{1}{\ln(5)} - \frac{1}{\ln(6)} + \frac{1}{\ln(7)} - \dots$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+2)} \longrightarrow b_n = \frac{1}{\ln(n+2)}$

① for $n \geq 1$, $0 < \frac{1}{\ln(n+2)} = b_n \geq b_{n+1} = \frac{1}{\ln((n+1)+2)} = \frac{1}{\ln(n+3)}$, $\{b_n\}$ is decreasing

② $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{\ln(n+2)} = 0$

by the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+2)}$ converges.

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6) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}} = \sum_{n=0}^{\infty} a_n \longrightarrow \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \rightarrow b_n = \frac{1}{\sqrt{n+1}}$

① for $n \geq 0$, $0 < \frac{1}{\sqrt{n+1}} = b_n \geq b_{n+1} = \frac{1}{\sqrt{(n+1)+1}} = \frac{1}{\sqrt{n+2}}$, $\{b_n\}$ is decreasing

② $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = 0$

by the Alternating Series Test, $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$ converges.

8) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1} = \sum_{n=1}^{\infty} a_n \rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^2}{n^2+n+1} \rightarrow b_n = \frac{n^2}{n^2+n+1}$

① for $n \geq 1$, $0 < \frac{n^2}{n^2+n+1} = b_n \geq b_{n+1} = \frac{(n+1)^2}{(n+1)^2+(n+1)+1}$, $\{b_n\}$ is decreasing

② option 1:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^2+n+1} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2}+\frac{n}{n^2}+\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}+\frac{1}{n^2}} = \frac{1}{1+0+0} = 1 \neq 0$$

option 2:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^2+n+1} \stackrel{L}{=} \lim_{n\to\infty} \frac{2n}{2n+1} \stackrel{L}{=} \lim_{n\to\infty} \frac{2}{2} = \lim_{n\to\infty} 1 = 1 \neq 0$$

Since $\lim_{n\to\infty} b_n = 1 \neq 0 \Rightarrow \lim_{n\to\infty} a_n \neq 0$, by the Test for Divergence (sec. 11.2), $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}$ diverges.

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10) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3} = \sum_{n=1}^{\infty} a_n \rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n+3} \rightarrow b_n = \frac{\sqrt{n}}{2n+3}$

① for $n \geq 1$, $b_n = \frac{\sqrt{n}}{2n+3} > 0$ let $f(x) = \frac{\sqrt{x}}{2x+3}$

$$\frac{df}{dx} = \frac{(2x+3)\left[\frac{1}{2\sqrt{x}}\right] - (\sqrt{x})[2]}{(2x+3)^2} = \frac{\frac{2x+3}{2\sqrt{x}} - 2\sqrt{x}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)}{(2x+3)^2} = \frac{\frac{(2x+3)-(4x)}{2\sqrt{x}}}{(2x+3)^2}$$

$$= \frac{3-2x}{2\sqrt{x}(2x+3)^2}$$

$0 = \frac{df}{dx} = \frac{3-2x}{2\sqrt{x}(2x+3)^2}$

$0 = \frac{3-2x}{2\sqrt{x}(2x+3)^2}$

$0 = 3-2x$

$x = \frac{3}{2}$

for $x > \frac{3}{2}$ (test at $x=2$)

$\left.\frac{df}{dx}\right|_{x=2} = \frac{3-2(2)}{2\sqrt{(2)}\,(2(2)+3)^2} < 0$

so for $n \geq 2$, $\{b_n\}$ is decreasing

10) continued...

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+3} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n}}{\frac{n}{n} + \frac{3}{n}} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{1+\frac{3}{n}} = \frac{0}{1+0} = 0$

by the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$ converges.

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12) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n} = \sum_{n=1} a_n \rightarrow \sum_{n=1} b_n = \sum_{n=1} \frac{n}{2^n} \rightarrow b_n = \frac{n}{2^n}$

① for $n \geq 1$, $b_n = \frac{n}{2^n} > 0$ let $f(x) = \frac{x}{2^x}$

$\frac{df}{dx} = \frac{(2^x)[1] - (x)[2^x(\ln 2)]}{(2^x)^2} = \frac{2^x(1-(\ln 2)x)}{(2^x)^2} = \frac{1-(\ln 2)x}{2^x}$

$0 = \frac{df}{dx} = \frac{1-(\ln 2)x}{2^x}$

$0 = \frac{1-(\ln 2)x}{2^x}$

$0 = 1-(\ln 2)x$

$x = \frac{1}{\ln 2}$

for $x > \frac{1}{\ln 2}$ (test at $x = 3$)

over estimate without calculator

$\left.\frac{df}{dx}\right|_{x=3} = \frac{1-(\ln 2)(3)}{2^{(3)}} < 0$

so for $n \geq 3$, $\{b_n\}$ is decreasing

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{2^n} \stackrel{L}{=} \lim_{n\to\infty} \frac{1}{2^n(\ln 2)} = 0$

by the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$ converges.

14) $\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n = \sum_{n=1}^{\infty} (-1)^{n-1} \tan^{-1}(n) = \sum_{n=1}^{\infty} a_n \rightarrow$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \tan^{-1}(n) \rightarrow b_n = \tan^{-1}(n)$

① for $n \geq 1$, $0 < \tan^{-1}(n) = b_n \leq b_{n+1} = \tan^{-1}(n+1)$ "does not satisfy"

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2} \rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n-1} \tan^{-1}(n)$ does not exist

by the Test for Divergence (sec. 11.2),
$\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n = \sum_{n=1}^{\infty} (-1)^{n-1} \tan^{-1}(n)$ diverges.

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16) $\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n} = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (\cos \pi n)\left(\frac{n}{2^n}\right) = \sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{2^n}\right) \rightarrow$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{2^n} \rightarrow b_n = \frac{n}{2^n}$

① for $n \geq 1$, $b_n = \frac{n}{2^n} > 0$ and for $n \geq 3$, $\{b_n\}$ is decreasing

see ex 12 for estimation of n and proof of decreasing

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{2^n} \stackrel{L}{=} \lim_{n\to\infty} \frac{1}{2^n(\ln 2)} = 0$

by the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$ converges.

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18) $\sum_{n=1}^{\infty} (-1)^n \cos \frac{\pi}{n} = \sum_{n=1}^{\infty} a_n \rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \cos\left(\frac{\pi}{n}\right) \rightarrow b_n = \cos\left(\frac{\pi}{n}\right)$

① for $n \geq 1$, $b_n = \cos\left(\frac{\pi}{n}\right) > 0$ $\quad 0 < \cos\left(\frac{\pi}{n}\right) b_n \leq b_{n+1} = \cos\left(\frac{\pi}{n+1}\right)$

"does not satisfy"

18) continued...

(2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist

by the Test for Divergence (sec. 11.2),

$\sum_{n=1}^{\infty} (-1)^n \cos\frac{\pi}{n}$ diverges.

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20) $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right) = \sum_{n=1}^{\infty} a_n \to \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \to$

$b_n = \left(\sqrt{n+1} - \sqrt{n}\right)$

(1) for $n \geq 1$, $b_n = \left(\sqrt{n+1} - \sqrt{n}\right) = \left(\frac{\sqrt{n+1} - \sqrt{n}}{1}\right)\left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right)$

$$= \frac{\left(\sqrt{n+1}\right)^2 - \left(\sqrt{n}\right)^2}{\sqrt{n+1} + \sqrt{n}} = \frac{(n+1) - (n)}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} > 0$$

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} = b_n \geq b_{n+1} = \frac{1}{\sqrt{(n+1)+1} + \sqrt{(n+1)}} = \frac{1}{\sqrt{n+2} + \sqrt{n+1}}$$

$\{b_n\}$ is decreasing

(2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$

by the Alternating Series Test,

$\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$ converges.

22) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \sum_{n=1}^{\infty} a_n \rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^4}$

$\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a p-series with $p = 4 > 1$ which converges

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is Absolutely Convergent.

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24) $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+1} = \sum_{n=0}^{\infty} a_n \rightarrow \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} \frac{n^2}{n^2+1}$

① for $n \geq 1$ $\quad 0 < \frac{n^2}{n^2+1} = b_n \leq b_{n+1} = \frac{(n+1)^2}{(n+1)^2+1}$ "does not satisfy"

② option 1:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = \frac{1}{1+0} = 1 \neq 0$$

option 2:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^2+1} \stackrel{L}{=} \lim_{n\to\infty} \frac{2n}{2n} = \lim_{n\to\infty} 1 = 1 \neq 0$$

but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n+1}\left(\frac{n^2}{n^2+1}\right)$ does not exist because

$\lim_{n\to\infty} (-1)^{n+1}$ does not exist and $\lim_{n\to\infty} b_n \neq 0$

by the Test for Divergence, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+1}$ diverges.

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26) $\sum_{n=1}^{\infty} \frac{-n}{n^2+1} = -\sum_{n=1}^{\infty} \frac{n}{n^2+1} = -\sum_{n=1}^{\infty} a_n$ "using L.C."

let $a_n = \frac{n}{n^2+1}$ $\qquad b_n = \frac{1}{n} \rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1 \leq 1$ which diverges

26) continued...

$$\frac{a_n}{b_n} = \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \left(\frac{n}{n^2+1}\right)\left(\frac{n}{1}\right) = \frac{n^2}{n^2+1}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 > 0$$ see exercise 24 for limit computation

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges by the Limit Comparison Test $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

$\sum_{n=1}^{\infty} \frac{-n}{n^2+1}$ diverges because it is negative of $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

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28) $\sum_{n=1}^{\infty} \frac{\sin n}{2^n} = \sum_{n=1}^{\infty} a_n \rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{\sin n}{2^n}\right|$

for $n \geq 1$, $0 < \left|\frac{\sin n}{2^n}\right| < \frac{1}{2^n}$

$$\sum_{n=1}^{\infty} 0 < \sum_{n=1}^{\infty} \left|\frac{\sin n}{2^n}\right| < \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2} < 1$ which converges.

Since $0 < \sum_{n=1}^{\infty} \left|\frac{\sin n}{2^n}\right| < \sum_{n=1}^{\infty} \frac{1}{2^n}$ by Direct Comparison Test, $\sum_{n=1}^{\infty} \left|\frac{\sin n}{2^n}\right|$ converges.

$\therefore \sum_{n=1}^{\infty} \frac{\sin n}{2^n}$ is Absolutely Convergent.

$\boxed{11.5/9}$

30) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4} = \sum_{n=1}^{\infty} a_n \rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2+4}$

let $b_n = \dfrac{n}{n^2+4} \longrightarrow c_n = \dfrac{1}{n} \rightarrow \displaystyle\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n}$ "using L.C. to see Abs. Convergence"

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1 \leq 1$ which diverges

$$\frac{b_n}{c_n} = \frac{\frac{n}{n^2+4}}{\frac{1}{n}} = \left(\frac{n}{n^2+4}\right)\left(\frac{n}{1}\right) = \frac{n^2}{n^2+4}$$

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{n^2}{n^2+4} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2} + \frac{4}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{4}{n^2}} = \frac{1}{1+0} = 1 > 0$$

(see exercise 24 for alternate limit calculation)

Since $\displaystyle\sum_{n=1}^{\infty} c_n$ is divergent and $\displaystyle\lim_{n\to\infty} \frac{b_n}{c_n} = 1 > 0$ by Limit Comparison Test

$\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{n^2+4}$ is divergent "not Abs. Convergent"

① for $n \geq 1$ $b_n = \dfrac{n}{n^2+4} > 0$ and for $n \geq 2$ $\dfrac{n}{n^2+4} = b_n \geq b_{n+1} = \dfrac{(n+1)}{(n+1)^2+4}$

$\{b_n\}$ is decreasing

② $$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{n}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = \frac{0}{1+0} = 0$$

by the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4} = \sum_{n=1}^{\infty} a_n$ converges.

Since $\displaystyle\sum_{n=1}^{\infty} a_n$ converges and $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} |a_n|$ diverges,

$\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ is conditionally convergent.

11.5/10

32) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}} = \sum_{n=1} a_n \rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}}$

for $n \geq 1$ $\quad 0 < \frac{n}{\sqrt{n^3+2}} < \frac{n}{\sqrt{n^3}} = \frac{n}{n\sqrt{n}} = \frac{1}{\sqrt{n}}$

$$\sum_{n=1}^{\infty} 0 < \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

can't use D.C.

let $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}}$

so use L.C.

$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a p-series with $p = \frac{1}{2} \leq 1$ which diverges

$$\frac{b_n}{c_n} = \frac{\frac{n}{\sqrt{n^3+2}}}{\frac{1}{\sqrt{n}}} = \left(\frac{n}{\sqrt{n^3+2}}\right)\left(\frac{\sqrt{n}}{1}\right) = \frac{n\sqrt{n}}{\sqrt{n^3+2}} = \frac{\sqrt{n^3}}{\sqrt{n^3+2}} = \sqrt{\frac{n^3}{n^3+2}}$$

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3+2}} = \sqrt{\lim_{n\to\infty}\left(\frac{n^3}{n^3+2}\right)} = \sqrt{\lim_{n\to\infty}\left(\frac{\frac{n^3}{n^3}}{\frac{n^3}{n^3}+\frac{2}{n^3}}\right)}$$

$$= \sqrt{\lim_{n\to\infty}\left(\frac{1}{1+\frac{2}{n^3}}\right)} = \sqrt{\frac{1}{1+0}} = \sqrt{1} = 1 > 0$$

Since $\lim_{n\to\infty} \frac{b_n}{c_n} = 1 > 0$ and $\sum_{n=1}^{\infty} c_n$ is divergent by Limit Comparison Test $\sum_{n=1}^{\infty} b_n$ is divergent "not Abs. Convergent"

① for $n \geq 1$, $b_n = \frac{n}{\sqrt{n^3+2}} > 0$ $\quad$ for $n \geq 2$ $\frac{n}{\sqrt{n^3+2}} = b_n \geq b_{n+1} = \frac{(n+1)}{\sqrt{(n+1)^3+2}}$

$\{b_n\}$ is decreasing

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^3+2}} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^3}}}{\frac{\sqrt{n^3+2}}{\sqrt{n^3}}} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\sqrt{\frac{n^3+2}{n^3}}}$

$$= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\sqrt{\frac{n^3}{n^3}+\frac{2}{n^3}}} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\sqrt{1+\frac{2}{n^3}}} = \frac{0}{\sqrt{1+0}} = 0$$

32) continued...

by Alternating Series Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ converges.

Since $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} |a_n|$ diverges,

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ is conditionally convergent.

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34) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} = \sum_{n=2}^{\infty} a_n \rightarrow \sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$

let $f(x) = \frac{1}{x \ln x}$, for $x \geq 2$

① $f(x) > 0$

② $f(x)$ is continuous

③ $f(x)$ is decreasing as x increases

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x} \left(\frac{1}{x} dx\right)$$

$p = \ln x$

$dp = \frac{1}{x} dx$

$$= \int \frac{1}{p} (dp)$$
$$= \ln|p| + C$$
$$= \ln|\ln x| + C$$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{U \to \infty} \int_2^U \frac{1}{x \ln x} dx = \lim_{U \to \infty} \Big[\ln|\ln x| + C\Big]_2^U$$

$$= \lim_{U \to \infty} \Big\{ \big[\ln|\ln U| + C\big] - \big[\ln|\ln(2)| + C\big] \Big\} = +\infty$$

Since $\int_2^{\infty} \frac{1}{x \ln x} dx$ diverges by the Integral Test,

$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges "not Abs. Convergent"

① for $n \geq 2$ $b_n = \frac{1}{n \ln n} > 0$ and $\{b_n\}$ is decreasing

34) continued...

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n \ln n} = 0$

by Alternating Series Test, $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges.

Since $\sum_{n=2}^{\infty} a_n$ converges and $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} |a_n|$ diverges,

$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is conditionally convergent.

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46) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = \sum_{n=1}^{\infty} a_n \rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^p} \rightarrow b_n = \frac{1}{n^p}$

if $p \le 0$, then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^p} = +\infty$ and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ does not exist

by the Test for Divergence (sec. 11.2) the series diverges

if $p > 0$, then ① for $n \ge 1$ $b_n = \frac{1}{n^p} > 0$ and $\frac{1}{n^p} = b_n \ge b_{n+1} = \frac{1}{(n+1)^p}$

$\{b_n\}$ is decreasing

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^p} = 0$

by the Alternating Series Test, the series converges.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ is convergent if $p > 0$.

48) $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n} = \sum_{n=2}^{\infty} a_n \rightarrow \sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{(\ln n)^p}{n}$

let $f(x) = \frac{(\ln x)^p}{x}$, for $x \geq 2$

① $f(x) > 0$

② $f(x)$ is continuous

$$\frac{df}{dx} = \frac{(x)\left[p(\ln x)^{p-1}\left(\frac{1}{x}(1)\right)\right] - ((\ln x)^p)\,[1]}{(x)^2} = \frac{(\ln x)^{p-1}\{p - \ln x\}}{x^2}$$

$$0 = \frac{df}{dx} = \frac{(\ln x)^{p-1}\{p - \ln x\}}{x^2}$$

$$0 = \frac{(\ln x)^{p-1}\{p - \ln x\}}{x^2}$$

$$0 = (\ln x)^{p-1}\{p - \ln x\}$$

$(\ln x)^{p-1} = 0$

$\ln x = 0$

$\Downarrow$

$x = e^0 = 1$

discard "for $x \geq 2$"

$p - \ln x = 0$

$p = \ln x$

$\Downarrow$

$e^p = x$

for $x > e^p$ (test at $x = e^{p+1}$ if $p > 0$)

$$\left.\frac{df}{dx}\right|_{x=e^{p+1}} = \frac{(\ln(e^{p+1}))\,\{p - \ln(e^{p+1})\}}{(e^{p+1})^2} = \frac{(\ln(e^p \cdot e^1))\{p - \ln(e^p \cdot e^1)\}}{(e^{p+1})^2}$$

$$= \frac{(\ln(e^p) + \ln(e^1))\,\{p - (\ln(e^p) + \ln(e^1))\}}{(e^{p+1})^2} = \frac{(p+1)\{p - (p+1)\}}{(e^{p+1})^2}$$

$$= \frac{(p+1)\{-1\}}{(e^{p+1})^2} < 0$$

48) continued...

for $x > e^{p}$ (test at $x = e^{p-1}$ if $p < 0$)

$$\left.\frac{df}{dx}\right|_{x=e^{p-1}} = \frac{(\ln(e^{p-1}))\{p - \ln(e^{p-1})\}}{(e^{p-1})^2} = \frac{(\ln(e^{p}\cdot e^{-1}))\{p - \ln(e^{p}\cdot e^{-1})\}}{(e^{p-1})^2}$$

$$= \frac{(\ln(e^{p}) + \ln(e^{-1}))\{p - (\ln(e^{p}) + \ln(e^{-1}))\}}{(e^{p-1})^2} = \frac{(p+(-1))\{p-(p+(-1))\}}{(e^{p-1})^2}$$

$$= \frac{(p-1)\{1\}}{(e^{p-1})^2} < 0$$

③ $f(x)$ is decreasing as x increases and $(x > e^{p})$

the 3 conditions above also satisfies the 1st criteria of the Alternate Series Test.

① for $x \geq 2$ and for all p, $b_n = \frac{(\ln x)^{p}}{n} > 0$ and $\{b_n\}$ is decreasing (eventually)

② if $p \leq 0$, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{(\ln n)^{p}}{n} = \lim\limits_{n\to\infty} \frac{1}{n(\ln n)^{-p}} = 0$

if $p > 0$, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{(\ln n)^{p}}{n} =$ apply L'Hospital Rule p-times $= 0$

by the Alternating Series Test, the series converges.

$\therefore \sum\limits_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^{p}}{n}$ is convergent for all p.