The Direct Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Below are definitions from Thomas's Calculus textbook.

Theorem 10 - Direct Comparison Test

Let $\sum a_n$ and $\sum b_n$ be two series with $0 \le a_n \le b_n$ for all n. Then

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.

2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

Theorem 11 - Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

4-a) $\sum_{n=2}^{\infty} \frac{n^2+n}{n^3-2}$ $\qquad \sum_{n=2}^{\infty} \frac{1}{n}$ "Direct Comparison Test"

for $n \geq 2$, $\frac{n^2+n}{n^3-2} > \frac{n^2}{n^3-2} > \frac{n^2}{n^3} = \frac{1}{n} \rightarrow \frac{n^2+n}{n^3-2} > \frac{1}{n}$

$\sum_{n=2}^{\infty} \frac{1}{n}$ is a partial of $\sum_{n=1}^{\infty} \frac{1}{n}$ which is a p-series with $p = 1 \leq 1$ which diverges

Since $\frac{n^2+n}{n^3-2} > \frac{1}{n}$, $\sum_{n=2}^{\infty} \frac{n^2+n}{n^3-2} > \sum_{n=2}^{\infty} \frac{1}{n}$ and the Direct Comparison Test $\sum_{n=2}^{\infty} \frac{n^2+n}{n^3-2}$ diverges.

4-b) $\sum_{n=2}^{\infty} \frac{n^2-n}{n^3+2}$ $\qquad \sum_{n=2}^{\infty} \frac{1}{n}$ "Limit Comparison Test"

let $a_n = \frac{n^2-n}{n^3+2}$ $\qquad b_n = \frac{1}{n}$

$$\frac{a_n}{b_n} = \frac{\frac{n^2-n}{n^3+2}}{\frac{1}{n}} = \left(\frac{n^2-n}{n^3+2}\right)\left(\frac{n}{1}\right) = \frac{n^3-n^2}{n^3+2}$$

option 1:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2-n}{n^3+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-n^2}{n^3+2} = \lim_{n\to\infty} \frac{\frac{n^3}{n^3}-\frac{n^2}{n^3}}{\frac{n^3}{n^3}+\frac{2}{n^3}} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1+\frac{2}{n^3}}$$

$$= \frac{1-0}{1+0} = 1 > 0$$

option 2:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2-n}{n^3+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-n^2}{n^3+2} \overset{L}{=} \lim_{n\to\infty} \frac{3n^2-2n}{3n^2} \overset{L}{=} \lim_{n\to\infty} \frac{6n-2}{6n}$$

$$\overset{L}{=} \lim_{n\to\infty} \frac{6}{6} = \lim_{n\to\infty} 1 = 1 > 0$$

4-b) continued...

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ is a (partial) p-series with $p=1 \le 1$ which diverges and $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$, by the Limit Comparison Test $\sum_{n=2}^{\infty} \frac{n^2-n}{n^3+2}$ diverges.

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6) show $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges

a) $\frac{n}{n^2+1} \ge \frac{1}{n^2+1}$

$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ which converges, so by Direct Comparison Test, $\sum \frac{1}{n^2+1}$ converges

Since $\frac{1}{n^2+1}$ converges and $\frac{n}{n^2+1} \ge \frac{1}{n^2+1}$, we cannot determine that $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

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b) $\frac{n}{n^2+1} \le \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1 \le 1$ which diverges.

$$\frac{n}{n^2+1} \le \frac{1}{n} \implies \sum_{n=1}^{\infty} \frac{n}{n^2+1} \le \sum_{n=1}^{\infty} \frac{1}{n}$$

Since the divergent series is not "larger" in our inequality, we cannot determine that $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

6) continued...

c) $\frac{n}{n^2+1} \geq \frac{1}{2n}$ $\qquad \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with

$\Downarrow$ $\qquad p = 1 \leq 1$ which diverges

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \geq \sum_{n=1}^{\infty} \frac{1}{2n}$$

Since the divergent series is "lower" in our inequality, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

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8) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ "D. C."

for $n \geq 2$ $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} > \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a (partial) p-series with

$p = \frac{1}{2} \leq 1$ which diverges

Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} > \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$, by Direct Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges

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10) $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ "D. C."

for $n \geq 1$, $\frac{n-1}{n^3+1} < \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$

10) continued...

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p=2>1$ which converges

Since $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$ and by Direct Comparison Test,

$\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ converges.

"D.C."

12) $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$

for $n \geq 1$, $\frac{6^n}{5^n-1} > \frac{6^n}{5^n} = \left(\frac{6}{5}\right)^n \Rightarrow \sum_{n=1}^{\infty} \frac{6^n}{5^n-1} > \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$

$\sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n = \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)\left(\frac{6}{5}\right)^{n-1}$ is a geometric series with $a=\frac{6}{5}$

and $|r|=\frac{6}{5}>1$ which diverges.

Since $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1} > \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ and by Direct Comparison Test,

$\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$ diverges.

"D.C."

14) $\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3}$

for $k \geq 1$, $\frac{k \sin^2 k}{1+k^3} \leq \frac{k}{1+k^3} < \frac{k}{k^3} = \frac{1}{k^2} \Rightarrow \sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3} < \sum_{k=1}^{\infty} \frac{1}{k^2}$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p-series with $p=2>1$ which converges.

Since $\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3} < \sum_{k=1}^{\infty} \frac{1}{k^2}$ and by Direct Comparison Test,

$\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3}$ converges.

16) $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ "D.C."

for $k \geq 1$, $\frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} < \frac{(2k)(k^2)}{(k)(k^2)^2} = \frac{2k^3}{k^5} = \frac{2}{k^2} \Rightarrow$

$$\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} < \sum_{k=1}^{\infty} \frac{2}{k^2}$$

$\sum_{k=1}^{\infty} \frac{2}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p-series with $p = 2 > 1$ which converges.

Since $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} < \sum_{k=1}^{\infty} \frac{2}{k^2}$ and by Direct Comparison Test, $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ converges.

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18) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$ "D.C."

for $n \geq 1$, $\frac{1}{\sqrt[3]{3n^4+1}} < \frac{1}{\sqrt[3]{3n^4}} < \frac{1}{\sqrt[3]{n^4}} = \frac{1}{n^{\frac{4}{3}}} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}} < \sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$ is a p-series with $p = \frac{4}{3} > 1$ which converges

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}} < \sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$ and by Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$ converges.

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20) $\sum_{n=1}^{\infty} \frac{1}{n^n}$ "D.C."

for $n \geq 1$, $\frac{1}{n^n} \leq \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$

20) continued...

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ which converges.

Since $\sum_{n=1}^{\infty} \frac{1}{n^n} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$ and by Direct Comparison Test,

$\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges.

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22) $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$ "L.C."

let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2} \rightarrow a_n = \frac{2}{\sqrt{n}+2}$ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \rightarrow b_n = \frac{1}{\sqrt{n}}$

$$\frac{a_n}{b_n} = \frac{\frac{2}{\sqrt{n}+2}}{\frac{1}{\sqrt{n}}} = \left(\frac{2}{\sqrt{n}+2}\right)\left(\frac{\sqrt{n}}{1}\right) = \frac{2\sqrt{n}}{\sqrt{n}+2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2}{\sqrt{n}+2}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{\sqrt{n}+2} = \lim_{n\to\infty} \frac{\frac{2\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n}}{\sqrt{n}} + \frac{2}{\sqrt{n}}} = \lim_{n\to\infty} \frac{2}{1+\frac{2}{\sqrt{n}}}$$

$$= \frac{2}{1+0} = 2 > 0$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a p-series with $p = \frac{1}{2} \le 1$ which diverges.

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 2 > 0$ and by Limit Comparison Test,

$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$ diverges.

24) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$ "L.C." 11.4/8

let $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2} \rightarrow a_n = \frac{n^2+n+1}{n^4+n^2}$ $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{\frac{n^2+n+1}{n^4+n^2}}{\frac{1}{n^2}} = \left(\frac{n^2+n+1}{n^4+n^2}\right)\left(\frac{n^2}{1}\right) = \frac{n^4+n^3+n^2}{n^4+n^2}$$

option 1:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+n+1}{n^4+n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4+n^3+n^2}{n^4+n^2} = \lim_{n\to\infty} \frac{\frac{n^4}{n^4}+\frac{n^3}{n^4}+\frac{n^2}{n^4}}{\frac{n^4}{n^4}+\frac{n^2}{n^4}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^2}} = \frac{1+0+0}{1+0} = 1 > 0$$

option 2:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+n+1}{n^4+n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4+n^3+n^2}{n^4+n^2} \stackrel{L}{=} \lim_{n\to\infty} \frac{4n^3+3n^2+2n}{4n^3+2n}$$

$$\stackrel{L}{=} \lim_{n\to\infty} \frac{12n^2+6n+2}{12n^2+2} \stackrel{L}{=} \lim_{n\to\infty} \frac{24n+6}{24n} \stackrel{L}{=} \lim_{n\to\infty} \frac{24}{24}$$

$$= \frac{24}{24} = 1 > 0$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ which converges.

Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and by Limit Comparison Test,

$\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$ converges.

26) $\displaystyle\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$ "L.C."

let $\displaystyle\sum_{n=3}^{\infty} a_n = \sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3} \to a_n = \frac{n+2}{(n+1)^3}$ $\quad \displaystyle\sum_{n=3}^{\infty} b_n = \sum_{n=3}^{\infty} \frac{1}{n^2} \to b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{\frac{n+2}{(n+1)^3}}{\frac{1}{n^2}} = \left(\frac{n+2}{(n+1)^3}\right)\left(\frac{n^2}{1}\right) = \frac{n^3+2n^2}{(n+1)(n^2+2n+1)} = \frac{n^3+2n^2}{n^3+3n^2+3n+1}$$

option 1:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+2}{(n+1)^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+3n^2+3n+1} = \lim_{n\to\infty} \frac{\frac{n^3}{n^3}+\frac{2n^2}{n^3}}{\frac{n^3}{n^3}+\frac{3n^2}{n^3}+\frac{3n}{n^3}+\frac{1}{n^3}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}} = \frac{1+0}{1+0+0+0} = 1 > 0$$

option 2:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+2}{(n+1)^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+3n^2+3n+1} \stackrel{L}{=} \lim_{n\to\infty} \frac{3n^2+4n}{3n^2+6n+3}$$

$$\stackrel{L}{=} \lim_{n\to\infty} \frac{6n+4}{6n+6} \stackrel{L}{=} \lim_{n\to\infty} \frac{6}{6} = \frac{6}{6} = 1 > 0$$

$\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$ is a (partial) p-series with $p = 2 > 1$ which converges.

Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and by Limit Comparison Test, $\displaystyle\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$ converges.

28) $\sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n}$ option 1: "D.C."

for $n \geq 1$ $\frac{n+3^n}{n+2^n} > \frac{3^n}{n+2^n} > \frac{3^n}{2^n+2^n} = \frac{3^n}{2(2^n)} = \frac{1}{2}\left(\frac{3}{2}\right)^n \Rightarrow \sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n} > \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{2}\right)^n$

$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{2}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^{n-1}$ is a geometric series

with $a = \frac{3}{2}$ and $r = \frac{3}{2}$, $|r| = \frac{3}{2} > 1$ which diverges.

Since $\sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n} > \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{2}\right)^n$ and by Direct Comparison

Test, $\sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n}$ diverges.

option 2: "L.C."

let $a_n = \frac{n+3^n}{n+2^n}$ $\qquad b_n = \left(\frac{3}{2}\right)^n = \frac{3^n}{2^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+3^n}{n+2^n}}{\frac{3^n}{2^n}} = \lim_{n\to\infty}\left(\frac{n+3^n}{n+2^n}\right)\left(\frac{2^n}{3^n}\right) = \lim_{n\to\infty} \frac{n\,2^n + (3^n)(2^n)}{n\,3^n + (2^n)(3^n)}$$

$$= \lim_{n\to\infty} \frac{\frac{n\,2^n}{6^n} + \frac{6^n}{6^n}}{\frac{n\,3^n}{6^n} + \frac{6^n}{6^n}} = \lim_{n\to\infty} \frac{\frac{n}{3^n} + 1}{\frac{n}{2^n} + 1} = \frac{\lim_{n\to\infty} \frac{n}{3^n} + \lim_{n\to\infty} 1}{\lim_{n\to\infty} \frac{n}{2^n} + \lim_{n\to\infty} 1} = \frac{0+1}{0+1} = 1 > 0$$

$$\lim_{n\to\infty} \frac{n}{3^n} \stackrel{L}{=} \lim_{n\to\infty} \frac{1}{(\ln 3)3^n} = 0 \qquad \lim_{n\to\infty} \frac{n}{2^n} \stackrel{L}{=} \lim_{n\to\infty} \frac{1}{(\ln 2)2^n} = 0$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ is a divergent geometric series, $|r| = \frac{3}{2} > 1$, by Limit Comparison Test $\sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n}$ diverges.

30) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ "L.C."

$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}} \to a_n = \frac{1}{n\sqrt{n^2-1}}$ $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^2} \to b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \left(\frac{1}{n\sqrt{n^2-1}}\right)\left(\frac{n^2}{1}\right) = \frac{n^2}{n\sqrt{n^2-1}} = \frac{n}{\sqrt{n^2-1}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{\sqrt{n^2-1}}{n}} = \lim_{n\to\infty} \frac{1}{\frac{\sqrt{n^2-1}}{\sqrt{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^2}{n^2} - \frac{1}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}} = \frac{1}{\sqrt{1-0}} = 1 > 0$$

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a (partial) p-series with $p=2>1$ which converges.

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and by Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ is converges.

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32) $\sum_{n=1}^{\infty} \frac{n^2+\cos^2 n}{n^3}$ "D.C."

for $n \geq 1$, $\frac{n^2+\cos^2 n}{n^3} \geq \frac{n^2}{n^3} = \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} \frac{n^2+\cos^2 n}{n^3} \geq \sum_{n=1}^{\infty} \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p=1\leq 1$ which diverges

Since $\sum_{n=1}^{\infty} \frac{n^2+\cos^2 n}{n^3} \geq \sum_{n=1}^{\infty} \frac{1}{n}$ and by Direct Comparison Test $\sum_{n=1}^{\infty} \frac{n^2+\cos^2 n}{n^3}$ diverges.

34) $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$

for $n \geq 1$, $\frac{e^{\frac{1}{n}}}{n} > \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n} > \sum_{n=1}^{\infty} \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1 \leq 1$ which diverges.

Since $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n} > \sum_{n=1}^{\infty} \frac{1}{n}$ and by Direct comparison Test,

$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$ diverges.

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36) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\frac{n!}{n^n} = \frac{\overbrace{(1)(2)(3)\cdots(n-1)(n)}^{n\text{-terms}}}{\underbrace{(n)(n)(n)\cdots(n)(n)(n)}_{n\text{-terms}}} \leq \overbrace{\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)(1)\cdots(1)(1)(1)}^{n\text{-terms}} = \frac{2}{n^2} \text{ for } n \geq 2$$

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ which converges.

Since $\sum_{n=1}^{\infty} \frac{n!}{n^n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ and by Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

38) $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$

let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right) \longrightarrow a_n = \sin^2\left(\frac{1}{n}\right)$ $\quad \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin^2\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\left(\sin\left(\frac{1}{n}\right)\right)^2}{\left(\frac{1}{n}\right)^2} = \lim_{n\to\infty} \left(\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right)^2 = (1)^2 = 1 > 0$$

because as $n\to\infty$ $\lim_{n\to\infty} \frac{1}{n} = 0$ so $\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x\to 0} \frac{\sin x}{x} = 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ which converges.

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and by Limit Comparison Test, $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$ converges.

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40) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \rightarrow a_n = \frac{1}{n^{1+\frac{1}{n}}}$ $\quad \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow b_n = \frac{1}{n}$

$$\frac{a_n}{b_n} = \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \left(\frac{1}{n^{1+\frac{1}{n}}}\right)\left(\frac{n}{1}\right) = \frac{n}{n^{1+\frac{1}{n}}} = \frac{n}{(n^1)(n^{\frac{1}{n}})} = \frac{1}{n^{\frac{1}{n}}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = \frac{1}{\lim_{n\to\infty} n^{\frac{1}{n}}} = \frac{1}{1} = 1 > 0$$

40) continued...

$$\lim_{n\to\infty} n^{\frac{1}{n}}$$

let $y = n^{\frac{1}{n}}$

$\ln y = \ln\left(n^{\frac{1}{n}}\right)$

$\ln y = \frac{1}{n}\ln n$

$\ln y = \frac{\ln n}{n}$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln n}{n} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\ln y = 0$

$\Downarrow$

$y = e^0 = 1$

$\therefore \lim_{n\to\infty} n^{\frac{1}{n}} = 1$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1 \leq 1$ which diverges.

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ diverges.