The Integral Test

Suppose f is a continuous, positive, decreasing function of $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

$\boxed{1}$ The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

$\boxed{2}$ Remainder Estimate for the Integral Test

Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \ge n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$$

$\boxed{3}$ $\boxed{s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_n^{\infty} f(x)\,dx}$ because $s_n + R_n = s$

On page 2, are descriptions from Thomas's Calculus textbook.

Corollary of Theorem 6

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ this is harmonic series and this series diverges.

Theorem 9 - The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \ge N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ converges if $p > 1$, diverges if $p \le 1$.

Bounds for the Remainder in the Integral Test

Suppose $\{a_n\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \ge n$, and that $\sum a_n$ converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$$

4) $\sum_{n=1}^{\infty} n^{-0.3}$ let $f(x) = x^{-0.3}$

for interval $[1, \infty)$
① $f(x)$ is continuous
② $f(x)$ is positive
③ $f(x)$ is decreasing as x increases

these 3 criterias satisfies that we can use the Integral Test.

$$\int_1^{\infty} x^{-0.3}\,dx = \lim_{U\to\infty} \int_1^{U} x^{-0.3}\,dx = \lim_{U\to\infty} \left[\frac{x^{0.7}}{0.7} + C\right]_1^U$$

$$= \lim_{U\to\infty} \left\{ \left[\frac{U^{0.7}}{0.7} + C\right] - \left[\frac{(1)^{0.7}}{0.7} + C\right] \right\} = +\infty$$

Since $\int_1^{\infty} x^{-0.3}\,dx = +\infty$ diverges, by the Integral Test $\sum_{n=1}^{\infty} n^{-0.3}$ diverges

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6) $\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$ let $f(x) = \frac{1}{(3x-1)^4}$

for interval $[1, \infty)$
① $f(x)$ is continuous
② $f(x)$ is positive
③ $f(x)$ is decreasing as x increases

these 3 criterias satisfies that we can use the Integral Test.

$$\int \frac{1}{(3x-1)^4}\,dx = \int \frac{1}{p^4}\left(\frac{1}{3}dp\right) = \frac{1}{3}\int p^{-4}\,dp = \frac{1}{3}\left[\frac{p^{-3}}{-3}\right] + C$$

$p = 3x-1$
$dp = 3\,dx$
$\frac{1}{3}dp = dx$

$$= \frac{-1}{9(3x-1)^3} + C$$

6) continued...

$$\int_1^\infty \frac{1}{(3x-1)^4}\,dx = \lim_{v\to\infty}\int_1^v \frac{1}{(3x-1)^4}\,dx = \lim_{v\to\infty}\left[\frac{-1}{9(3x-1)^3}+C\right]_1^v$$

$$= \lim_{v\to\infty}\left\{\left[\frac{-1}{9(3v-1)^3}+C\right]-\left[\frac{-1}{9(3(1)-1)^3}+C\right]\right\}$$

$$= \left\{[0]-\left[\frac{-1}{9(2)^3}\right]\right\} = \frac{1}{72}$$

Since $\int_1^\infty \frac{1}{(3x-1)^4}\,dx = \frac{1}{72}$ converges, by the Integral Test, $\sum_{n=1}^\infty \frac{1}{(3n-1)^4}$ converges

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8) $\sum_{n=1}^\infty n^2 e^{-n^3} = \sum_{n=1}^\infty \frac{n^2}{e^{n^3}}$ let $f(x) = \frac{x^2}{e^{x^3}} = x^2 e^{-x^3}$

for interval $[1,\infty)$

1. $f(x)$ is continuous
2. $f(x)$ is positive
3. $f(x)$ is decreasing as x increases

these 3 criterias satisfies that we can use the Integral Test.

$$\int x^2 e^{-x^3}\,dx = \int e^{-x^3}(x^2\,dx) = \int e^{p}\left(\frac{-1}{3}\,dp\right) = \frac{-1}{3}\left[e^{p}\right]+C$$

$$= \frac{-1}{3}e^{-x^3}+C = \frac{-1}{3e^{x^3}}+C$$

$p = -x^3$

$dp = -3x^2\,dx$

$\frac{-1}{3}dp = x^2\,dx$

8) continued...

$$\int_1^\infty x^2 e^{-x^3} dx = \lim_{U\to\infty} \int_1^U x^2 e^{-x^3} dx = \lim_{U\to\infty} \left[\frac{-1}{3e^{x^3}} + C\right]_1^U$$

$$= \lim_{U\to\infty} \left\{ \left[\frac{-1}{3e^{U^3}} + C\right] - \left[\frac{-1}{3e^{(1)^3}} + C\right] \right\} = [0] - \left[\frac{-1}{3e}\right] = \frac{1}{3e}$$

Since $\int_1^\infty x^2 e^{-x^3} dx = \frac{1}{3e}$ converges, by the Integral Test,

$\sum_{n=1}^\infty n^2 e^{-n^3}$ converges

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10) $\sum_{n=1}^\infty \frac{\tan^{-1} n}{1+n^2}$ let $f(x) = \frac{\tan^{-1} x}{1+x^2}$

for interval $[1,\infty)$
1. $f(x)$ is continuous
2. $f(x)$ is positive
3. $f(x)$ is decreasing as x increases

these 3 criterias satisfies that we can use the Integral Test.

$$\int \frac{\tan^{-1} x}{1+x^2} dx = \int \tan^{-1} x \left(\frac{1}{1+x^2} dx\right) = \int p \, dp = \left[\frac{p^2}{2}\right] + C$$

$p = \tan^{-1} x$

$dp = \frac{1}{1+x^2} dx$

$$= \frac{1}{2}(\tan^{-1} x)^2 + C$$

$$\int_1^\infty \frac{\tan^{-1} x}{1+x^2} dx = \lim_{U\to\infty} \int_1^U \frac{\tan^{-1} x}{1+x^2} dx = \lim_{U\to\infty} \left[\frac{1}{2}(\tan^{-1} x)^2 + C\right]_1^U$$

$$= \lim_{U\to\infty} \left\{ \left[\frac{1}{2}(\tan^{-1} U)^2 + C\right] - \left[\frac{1}{2}(\tan^{-1}(1))^2 + C\right] \right\}$$

10) continued...

$$= \left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2\right] - \left[\frac{1}{2}\left(\frac{\pi}{4}\right)^2\right] = \left[\frac{\pi^2}{8}\right] - \left[\frac{\pi^2}{32}\right]$$

$$= \frac{4\pi^2}{32} - \frac{\pi^2}{32} = \frac{3\pi^2}{32}$$

Since $\int_1^\infty \frac{\tan^{-1} x}{1+x^2}\,dx = \frac{3\pi^2}{32}$ converges, by the Integral Test

$\sum_{n=1}^\infty \frac{\tan^{-1} n}{1+n^2}$ converges.

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12) $\sum_{n=3}^\infty n^{-0.9999} = \sum_{n=3}^\infty \frac{1}{n^{0.9999}}$ is a p-series with $p = 0.9999 \le 1$.

It is divergent by $\boxed{1}$

Note: this series starting with $n=3$ is irrelevant when determining convergence or divergence.

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14) $\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \ldots = \sum_{n=1}^\infty \frac{1}{2n+3}$ let $f(x) = \frac{1}{2x+3}$

for interval $[1, \infty)$

1. $f(x)$ is continuous
2. $f(x)$ is positive
3. $f(x)$ is decreasing as x increases

these 3 criterias satisfies that we can use the Integral Test

$$\int_1^\infty \frac{1}{2x+3}\,dx = \lim_{v\to\infty} \int_1^v \frac{1}{\underset{\substack{\downarrow \\ p=2x+3}}{2x+3}}\,dx = \lim_{v\to\infty} \left[\frac{1}{2}\ln|2x+3| + C\right]_1^v$$

14) continued...

$$= \lim_{v \to \infty} \left\{ \left[\frac{1}{2} \ln |2v+3| + C \right] - \left[\frac{1}{2} \ln |2(1)+3| + C \right] \right\} = +\infty$$

Since $\int_1^\infty \frac{1}{2x+3} dx = +\infty$ diverges, by the Integral Test,

$\sum_{n=1}^{\infty} \frac{1}{2n+3}$ diverges

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16) $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

is a p-series with $p = \frac{3}{2} > 1$. It is convergent by $\boxed{1}$.

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18) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{3/2}}$ let $f(x) = \frac{\sqrt{x}}{1+x^{3/2}} = \frac{x^{\frac{1}{2}}}{1+x^{\frac{3}{2}}}$

for interval $[1, \infty)$ ① $f(x)$ is continuous

② $f(x)$ is positive

$$\frac{df}{dx} = \frac{(1+x^{\frac{3}{2}})\left[\frac{1}{2\sqrt{x}}\right] - (\sqrt{x})\left[\frac{3}{2}\sqrt{x}\right]}{(1+x^{\frac{3}{2}})^2} = \frac{\frac{1}{2\sqrt{x}} + \frac{x}{2} - \frac{3x}{2}}{(1+x^{\frac{3}{2}})^2}$$

$$= \frac{\frac{1}{2\sqrt{x}} - x}{(1+x^{\frac{3}{2}})^2} = \frac{\frac{1}{2\sqrt{x}} - x\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)}{(1+(\sqrt{x})^3)^2} = \frac{\frac{1-2(\sqrt{x})^3}{2\sqrt{x}}}{(1+(\sqrt{x})^3)^2} = \frac{1-2(\sqrt{x})^3}{2\sqrt{x}(1+(\sqrt{x})^3)^2}$$

for $x \geq 1$, denominator is positive and numerator negative so for $x \geq 1$ $\frac{df}{dx} < 0$ which indicates that $f(x)$ is decreasing for $x \geq 1$

③ $f(x)$ decreases as x increases

these 3 criterias satisfy that we can use the Integral Test.

18) continued...

$$\int \frac{\sqrt{x}}{1+x^{\frac{3}{2}}}dx = \int \frac{1}{1+x^{\frac{3}{2}}}\left(\sqrt{x}\,dx\right) = \int \frac{1}{p}\left(\frac{2}{3}dp\right) = \frac{2}{3}\left[\ln|p|\right] + C$$

$p = 1 + x^{\frac{3}{2}}$

$dp = \frac{3}{2}x^{\frac{1}{2}}dx$

$\frac{2}{3}dp = \sqrt{x}\,dx$

$$= \frac{2}{3}\ln\left|1+x^{\frac{3}{2}}\right| + C$$

$$= \frac{2}{3}\ln\left|1+(\sqrt{x})^3\right| + C$$

$$\int_1^{\infty} \frac{\sqrt{x}}{1+x^{\frac{3}{2}}}dx = \lim_{v\to\infty}\int_1^{v} \frac{\sqrt{x}}{1+x^{\frac{3}{2}}}dx = \lim_{v\to\infty}\left[\frac{2}{3}\ln\left|1+(\sqrt{x})^3\right| + C\right]_1^{v}$$

$$= \lim_{v\to\infty}\left\{\left[\frac{2}{3}\ln\left|1+(\sqrt{v})^3\right| + C\right] - \left[\frac{2}{3}\ln\left|1+(\sqrt{(1)})^3\right| + C\right]\right\}$$

$$= +\infty$$

Since $\int_1^{\infty} \frac{\sqrt{x}}{1+x^{3/2}}dx = +\infty$ diverges, by the Integral Test,

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{3/2}}$ diverges

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20) $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$ let $f(x) = \frac{1}{x^2+2x+2}$

for interval $[1,\infty)$

1. $f(x)$ is continuous
2. $f(x)$ is positive
3. $f(x)$ is decreasing as x increases

these 3 criterias satisfy that we can use the Integral Test.

20) continued...

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x^2+2x+1)+1}\,dx = \int \frac{1}{(x+1)^2+(1)^2}\,dx$$

use completing square

$$= \left[\frac{1}{(1)}\tan^{-1}\left(\frac{(x+1)}{(1)}\right)\right] + C = \tan^{-1}(x+1) + C$$

$$\int_1^\infty \frac{1}{x^2+2x+2}\,dx = \lim_{v\to\infty}\int_1^v \frac{1}{x^2+2x+2}\,dx = \lim_{v\to\infty}\left[\tan^{-1}(x+1)+C\right]_1^v$$

$$= \lim_{v\to\infty}\left\{\left[\tan^{-1}(v+1)+C\right] - \left[\tan^{-1}((1)+1)+C\right]\right.$$

$$= \left[\frac{\pi}{2}\right] - \left[\tan^{-1}(2)\right] = \frac{\pi}{2} - \tan^{-1}(2)$$

Since $\int_1^\infty \frac{1}{x^2+2x+2}\,dx = \frac{\pi}{2} - \tan^{-1}(2)$ converges, by the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$ converges

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22) $\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$ let $f(x) = \frac{3x-4}{x^2-2x}$ using partial fraction

$$\frac{3x-4}{x^2-2x} = \frac{3x-4}{(x)^1(x-2)^1} = \frac{A}{(x)^1} + \frac{B}{(x-2)^1}$$

$$3x-4 = A(x-2) + B(x)$$

const. term

$-4 = -2A$

$A = 2$

x - term

$3 = A + B$

$3 = (2) + B$

$1 = B$

$$f(x) = \frac{3x-4}{x^2-2x} = \frac{(2)}{(x)^1} + \frac{(1)}{(x-2)^1}$$

$$f(x) = \frac{2}{x} + \frac{1}{x-2}$$

22) continued...

for interval $[3, \infty)$ ① $f(x)$ is continuous

② $f(x)$ is positive

③ $f(x)$ is decreasing as x increases

these 3 criterias satisfy that we can use the Integral Test.

$$\int_3^{\infty} \frac{3x-4}{x^2-2x}\,dx = \lim_{u\to\infty} \int_3^{u} \frac{3x-4}{x^2-2x}\,dx = \lim_{u\to\infty} \int_3^{u} \left(\frac{2}{x} + \frac{1}{x-2}\right) dx = \lim_{u\to\infty} \Big[2\ln|x| + \ln|x-2| + C\Big]_3^u$$

$$= \lim_{u\to\infty} \Big\{ \big[2\ln|u| + \ln|u-2| + C\big] - \big[2\ln|(3)| - \ln|(3)-2| + C\big] \Big\}$$

$$= +\infty$$

Since $\int_3^{\infty} \frac{3x-4}{x^2-2x}\,dx = +\infty$ diverges, by the Integral Test,

$\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$ diverges

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24) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ let $f(x) = \frac{\ln x}{x^2}$

for interval $[2, \infty)$ ① $f(x)$ is continuous

② $f(x)$ is positive

$$\frac{df}{dx} = \frac{(x^2)\left[\frac{1}{x}(1)\right] - (\ln x)\left[2x\right]}{(x^2)^2} = \frac{x - 2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4}$$

$$\frac{df}{dx} = \frac{1-2\ln x}{x^3}$$

24) continued...

$$0 = \frac{df}{dx} = \frac{1-2\ln x}{x^3}$$

$$0 = \frac{1-2\ln x}{x^3}$$

$$0 = 1 - 2\ln x$$

$$2\ln x = 1$$

$$\ln x = \frac{1}{2}$$

$$\Downarrow$$

$$x = e^{\frac{1}{2}}$$

$e^{\frac{1}{2}} < 2$

and

for $x > e^{\frac{1}{2}}$ $\frac{df}{dx} < 0$

which means that $f(x)$ is decreasing

③ $f(x)$ is decreasing as x increases

these 3 criterias satisfy that we can use the Integral Test.

$$\int \frac{\ln x}{x^2}\,dx = \int (\ln x)\left(\frac{1}{x^2}\,dx\right) = (\ln x)\left(\frac{-1}{x}\right) - \int \left(\frac{-1}{x}\right)\left(\frac{1}{x}\,dx\right)$$

$u = \ln x \quad dv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}\,dx \quad v = \frac{-1}{x}$

$$= \frac{-\ln x}{x} + \int \frac{1}{x^2}\,dx = \frac{-\ln x}{x} + \left[\frac{-1}{x}\right] + C$$

$$= \frac{-\ln x}{x} - \frac{1}{x} + C$$

$$\int_2^\infty \frac{\ln x}{x^2}\,dx = \lim_{U\to\infty} \int_2^U \frac{\ln x}{x^2}\,dx = \lim_{U\to\infty} \left[\frac{-\ln x}{x} - \frac{1}{x} + C\right]_2^U$$

$$= \lim_{U\to\infty} \left\{\left[\frac{-\ln U}{U} - \frac{1}{U} + C\right] - \left[\frac{-\ln(2)}{(2)} - \frac{1}{(2)} + C\right]\right\}$$

$$\lim_{U\to\infty} \frac{-\ln U}{U} \overset{L}{=} \lim_{U\to\infty} \frac{-\left[\frac{1}{U}\right]}{1} = \lim_{U\to\infty} \frac{-1}{U} = 0$$

$$= [0 - 0] - \left[\frac{-\ln 2}{2} - \frac{1}{2}\right] = \frac{\ln 2}{2} + \frac{1}{2} = \frac{1+\ln 2}{2}$$

Since $\int_2^\infty \frac{\ln x}{x^2}\,dx = \frac{1+\ln 2}{2}$ converges, by the Integral Test, $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges

26) $\sum_{k=1}^{\infty} k e^{-k^2}$ let $f(x) = x e^{-x^2} = \frac{x}{e^{x^2}}$

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for interval $[1, \infty)$

1 $f(x)$ is continuous

2 $f(x)$ is positive

$$\frac{df}{dx} = \frac{(e^{x^2})[1] - (x)[e^{x^2}(2x)]}{(e^{x^2})^2} = \frac{e^{x^2} - 2x^2 e^{x^2}}{(e^{x^2})^2} = \frac{e^{x^2}(1-2x^2)}{(e^{x^2})^2}$$

$$\frac{df}{dx} = \frac{1-2x^2}{e^{x^2}}$$

$$0 = \frac{df}{dx} = \frac{1-2x^2}{e^{x^2}}$$

$$0 = \frac{1-2x^2}{e^{x^2}}$$

$$0 = 1 - 2x^2$$

$$0 = (1+\sqrt{2}x)(1-\sqrt{2}x)$$

$1 + \sqrt{2}x = 0$ not needed | $1 - \sqrt{2}x = 0$, $1 = \sqrt{2}x$, $\frac{1}{\sqrt{2}} = x$

$\frac{1}{\sqrt{2}} < 1$

and

for $x > \frac{1}{\sqrt{2}}$ $\frac{df}{dx} < 0$

which means that $f(x)$ is decreasing

3 $f(x)$ is decreasing as x increases

these 3 criterias satisfy that we can use the Integral Test.

$$\int x e^{-x^2} dx = \int e^{-x^2}(x\,dx) = \int e^{p}\left(\frac{-1}{2}dp\right) = \frac{-1}{2}\left[e^{p}\right] + C$$

$$= \frac{-1}{2} e^{-x^2} + C = \frac{-1}{2e^{x^2}} + C$$

$p = -x^2$

$dp = -2x\,dx$

$\frac{-1}{2}dp = x\,dx$

$$\int_1^{\infty} x e^{-x^2} dx = \lim_{v\to\infty} \int_1^{v} x e^{-x^2} dx = \lim_{v\to\infty} \left[\frac{-1}{2e^{x^2}} + C\right]_1^{v}$$

$$= \lim_{v\to\infty} \left\{\left[\frac{-1}{2e^{v^2}} + C\right] - \left[\frac{-1}{2e^{(1)^2}} + C\right]\right\} = [0] - \left[\frac{-1}{2e}\right] = \frac{1}{2e}$$

26) continued...

Since $\int_1^\infty x e^{-x^2} dx = \frac{1}{2e}$ converges, by the Integral Test, $\sum_{k=1}^{\infty} ke^{-k^2}$ converges

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28) $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ let $f(x) = \frac{x}{x^4+1}$

for interval $[1, \infty)$ ① $f(x)$ is continuous

② $f(x)$ is positive

$$\frac{df}{dx} = \frac{(x^4+1)[1] - (x)[4x^3]}{(x^4+1)^2} = \frac{x^4+1-4x^4}{(x^4+1)^2} = \frac{1-3x^4}{(x^4+1)^2}$$

$$0 = \frac{df}{dx} = \frac{1-3x^4}{(x^4+1)^2}$$

$$0 = \frac{1-3x^4}{(x^4+1)^2}$$

$$0 = 1-3x^4$$

$$0 = (1+\sqrt{3}x^2)(1-\sqrt{3}x^2)$$

$$0 = (1+\sqrt{3}x^2)(1+\sqrt{\sqrt{3}}x)(1-\sqrt{\sqrt{3}}x)$$

$$1-\sqrt{\sqrt{3}}x = 0$$

$$1 = \sqrt{\sqrt{3}}x$$

$$\frac{1}{\sqrt{\sqrt{3}}} = x$$

$1 < \sqrt{\sqrt{3}}$ so $\frac{1}{\sqrt{\sqrt{3}}} < 1$

and

for $x > \frac{1}{\sqrt{\sqrt{3}}}$ $\frac{df}{dx} < 0$

which means that $f(x)$ is decreasing

③ $f(x)$ is decreasing as x increases

these 3 criterias satisfy that we can use the Integral Test.

28) continued...

$$\int \frac{x}{x^4+1}\,dx = \int \frac{1}{(x^2)^2+1}(x\,dx) = \int \frac{1}{p^2+1}\left(\frac{1}{2}dp\right) = \frac{1}{2}\int \frac{1}{p^2+(1)^2}dp$$

$p = x^2$

$dp = 2x\,dx$

$\frac{1}{2}dp = x\,dx$

$$= \frac{1}{2}\left[\frac{1}{(1)}\tan^{-1}\left(\frac{p}{(1)}\right)\right] + C = \frac{1}{2}\tan^{-1}(x^2) + C$$

$$\int_1^\infty \frac{x}{x^4+1}dx = \lim_{v\to\infty}\int_1^v \frac{x}{x^4+1}dx = \lim_{v\to\infty}\left[\frac{1}{2}\tan^{-1}(x^2)+C\right]_1^v$$

$$= \lim_{v\to\infty}\left\{\left[\frac{1}{2}\tan^{-1}(v^2)+C\right] - \left[\frac{1}{2}\tan^{-1}((1)^2)+C\right]\right\}$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{2}\right)\right] - \left[\frac{1}{2}\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{4} - \frac{\pi}{8} = \frac{2\pi}{8} - \frac{\pi}{8} = \frac{\pi}{8}$$

Since $\int_1^\infty \frac{x}{x^4+1}dx = \frac{\pi}{8}$ converges, by the Integral Test,

$\sum_{n=1}^\infty \frac{n}{n^4+1}$ converges

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30) $\sum_{n=1}^\infty \frac{\cos^2 n}{1+n^2}$ Let $f(x) = \frac{\cos^2 x}{1+x^2}$

for interval $[1,\infty)$

1. $f(x)$ is continuous
2. $f(x)$ is positive

for $x > 1$ $\cos x$ will oscillates from -1 to 1 and

$\cos^2 x$ will oscillate from 0 to 1

therefore $f(x)$ is not decreasing (overall) as x increases

Since 3 criterias are not completely satisfied we can't use the Integral Test

11.3/15

32) $\sum_{n=3}^{\infty} \frac{1}{n \ln n \left[\ln(\ln n)\right]^{p}}$ let $f(x) = \frac{1}{x \ln x \left[\ln(\ln x)\right]^{p}}$

for interval $[3, \infty)$

① $f(x)$ is continuous

② $f(x)$ is positive

for $p \geq 0$, $f(x)$ is decreasing as x increases because only the denominator is getting larger.

for $p < 0$, $f(x) = \frac{\left[\ln(\ln x)\right]^{(-p)}}{x \ln x}$ for a large N and $x > N$

$x > \ln x$ and eventually

$x \ln x > \left[\ln(\ln x)\right]^{(-p)}$

so for $x > N$ $f(x)$ is decreasing as x increases and $x > N$

③ $f(x)$ is decreasing as $x > N$ and N sufficiently large

these 3 criterias satisfy that we can use the Integral Test,

$$\int \frac{1}{x \ln x \left[\ln(\ln x)\right]^{p}}\, dx = \int \frac{1}{\left[\ln(\ln x)\right]^{p}} \left(\frac{1}{x \ln x}\, dx\right) = \int \frac{1}{q^{p}}\, dq$$

$q = \ln(\ln x)$

$dq = \frac{1}{x \ln x}\, dx$

if $p = 1$

$= \int \frac{1}{q^{1}}\, dq$

$= \ln|q| + C$

$= \ln\left|\ln(\ln x)\right| + C$

if $p \neq 1$

$= \int q^{-p}\, dp = \left[\frac{q^{(-p+1)}}{(-p+1)}\right] + C$

$= \frac{\left[\ln(\ln x)\right]^{(-p+1)}}{1-p} + C$

$= \frac{1}{(1-p)\left[\ln(\ln x)\right]^{(p-1)}} + C$

32) continued...

for $p=1$;

$$\int_3^\infty \frac{1}{x\ln x[\ln(\ln x)]^p}\,dx = \lim_{v\to\infty}\int_3^v \frac{1}{x\ln x[\ln(\ln x)]^1}\,dx = \lim_{v\to\infty}\Big[\ln|\ln(\ln x)|+C\Big]_3^v$$

$$= \lim_{v\to\infty}\Big\{\Big[\ln|\ln(\ln v)|+C\Big]-\Big[\ln|\ln(\ln(3))|+C\Big]\Big\} = +\infty$$

diverges

for $p \neq 1$

$$\int_3^\infty \frac{1}{x\ln x[\ln(\ln x)]^p}\,dx = \lim_{v\to\infty}\int_3^v \frac{1}{x\ln x[\ln(\ln x)]^p}\,dx = \lim_{v\to\infty}\left[\frac{1}{(1-p)[\ln(\ln x)]^{(p-1)}}+C\right]_3^v$$

$$= \lim_{v\to\infty}\left\{\left[\frac{1}{(1-p)[\ln(\ln v)]^{(p-1)}}+C\right]-\left[\frac{1}{(1-p)[\ln(\ln(3))]^{(p-1)}}+C\right]\right\}$$

$$= \lim_{v\to\infty}\left\{\frac{1}{(1-p)[\ln(\ln v)]^{(p-1)}}-\frac{1}{(1-p)[\ln(\ln 3)]^{(p-1)}}\right\}$$

$$= 0 - \frac{1}{(1-p)[\ln(\ln 3)]^{(p-1)}} \quad \text{if } p-1>0, \; p>1$$

$\int_3^\infty \frac{1}{x\ln x[\ln(\ln x)]^p}\,dx$ converges for $p>1$,

by the Integral Test,

$\sum_{n=3}^{\infty}\frac{1}{n\ln n[\ln(\ln n)]^p}$ converges for $p>1$, 11.3/17

34) $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

for $p \leq 0$, $a_n = \frac{\ln n}{n^p}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{\ln n}{n^p}\right) \stackrel{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{p n^{(p-1)}} = \lim_{n\to\infty} \frac{1}{p n^p}$$

if $p = 0$ then $\lim_{n\to\infty} \frac{1}{p n^p} = +\infty$

if $p < 0$ then $\lim_{n\to\infty} \frac{1}{p n^p} = \lim_{n\to\infty} \frac{n^{(-p)}}{p} = +\infty$ ← makes exponent positive

so for $p \leq 0$, $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ diverges

for $p > 0$ let $f(x) = \frac{\ln x}{x^p}$

for interval $[1, \infty)$ ① $f(x)$ is continuous

② $f(x)$ is positive

$$\frac{df}{dx} = \frac{(x^p)\left[\frac{1}{x}(1)\right] - (\ln x)\left[p x^{(p-1)}\right]}{(x^p)^2} = \frac{x^{(p-1)} - p x^{(p-1)}(\ln x)}{(x^p)(x^p)}$$

$$= \frac{x^{(p-1)}(1 - p \ln x)}{(x^p)(x^p)} = \frac{(1 - p \ln x)}{x(x^p)}$$

$$0 = \frac{df}{dx} = \frac{(1-p\ln x)}{x(x^p)}$$

$$0 = \frac{1 - p\ln x}{x(x^p)}$$

$$0 = 1 - p \ln x$$

$$p \ln x = 1$$

$\ln x = \frac{1}{p}$

$\Downarrow$

$x = e^{\frac{1}{p}}$

so for $x > e^{\frac{1}{p}}$ $\frac{df}{dx} < 0$

③ $f(x)$ is decreasing for $x > e^{\frac{1}{p}}$

34) continued...

11.3/18

these 3 criterias satisfy that we can use the Integral Test.

$$\int \frac{\ln x}{x^p}\,dx = \int (\ln x)\left(\frac{1}{x^p}\,dx\right) = (\ln x)\left(\frac{1}{(1-p)\,x^{(p-1)}}\right) - \int\left(\frac{1}{(1-p)\,x^{(p-1)}}\right)\left(\frac{1}{x}\,dx\right)$$

$u = \ln x$ $\quad dv = \frac{1}{x^p}\,dx = x^{-p}\,dp$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{x^{-p+1}}{-p+1} = \frac{1}{(1-p)\,x^{(p-1)}}$

$$= \frac{\ln x}{(1-p)\,x^{(p-1)}} - \frac{1}{(1-p)}\int \frac{1}{x^p}\,dx$$

$$= \frac{\ln x}{(1-p)\,x^{(p-1)}} - \frac{1}{(1-p)}\left[\frac{1}{(1-p)\,x^{(p-1)}}\right] + C$$

$$= \frac{\ln x}{(1-p)\,x^{(p-1)}} - \frac{1}{(1-p)^2\,x^{(p-1)}} + C$$

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \lim_{U\to\infty}\int_1^U \frac{\ln x}{x^p}\,dx = \lim_{U\to\infty}\left[\frac{\ln x}{(1-p)\,x^{(p-1)}} - \frac{1}{(1-p)^2\,x^{(p-1)}} + C\right]_1^\infty$$

$$= \lim_{U\to\infty}\left\{\left[\frac{\ln U}{(1-p)\,U^{(p-1)}} - \frac{1}{(1-p)^2\,U^{(p-1)}} + C\right] - \left[\frac{\ln(1)}{(1-p)(1)^{(p-1)}} - \frac{1}{(1-p)^2(1)^{(p-1)}} + C\right]\right\}$$

for $p = 1$:

$\int_1^\infty \frac{\ln x}{x^p}\,dx =$ is divergent because $\frac{1}{(1-p)^2} = +\infty$

for $0 < p < 1$:

$\int_1^\infty \frac{\ln x}{x^p}\,dx$ is divergent because $U^{(p-1)}$ will have negative exponent which $\lim_{U\to\infty}\frac{1}{U^{(p-1)}} = +\infty$

34) continued (part 2) ....

for $p > 1 \iff p-1 > 0$

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \lim_{U\to\infty} \left\{ \frac{\ln U}{(1-p)\,U^{(p-1)}} - \frac{1}{(1-p)^2 U^{(p-1)}} - 0 + \frac{1}{(1-p)^2} \right\}$$

$$\lim_{U\to\infty} \frac{\ln U}{(1-p)\,U^{(p-1)}} \overset{L}{=} \lim_{U\to\infty} \frac{\frac{1}{U}}{(1-p)(p-1)\,U^{(p-2)}} = \lim_{U\to\infty} \frac{1}{-(1-p)^2\,U^{(p-1)}} = 0$$

$$= \left\{ 0 - 0 - 0 + \frac{1}{(1-p)^2} \right\} = \frac{1}{(1-p)^2}$$

$\int_1^\infty \frac{\ln x}{x^p}\,dx = \frac{1}{(1-p)^2}$ converges for $p > 1$,

by the Integral Test,

$\sum_{n=1}^\infty \frac{\ln n}{n^p}$ converges for $p > 1$.