$\boxed{2}$ **Definition**

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its *n*th partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series.
If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r. (The series that arises from Zeno's paradox is the special case where $a = \frac{1}{2}$ and $r = \frac{1}{2}$.)

If $r = 1$, then $s_n = a + a + a + \cdots + a = na \to \pm\infty$. Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case.

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Subtracting these equations (and solving for s_n), we get

$$s_n - rs_n = a - ar^n$$
$$s_n(1-r) = a(1-r^n)$$
$$s_n = \frac{a(1-r^n)}{1-r}$$

$\boxed{3}$ $\boxed{s_n = \frac{a(1-r^n)}{1-r}}$

If $-1 < r < 1$, we know from (11.1.9) that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \lim_{n\to\infty}\left(\frac{a - ar^n}{1-r}\right) = \lim_{n\to\infty}\left(\frac{a}{1-r}\right) - \lim_{n\to\infty}\left(\frac{ar^n}{1-r}\right)$$
$$= \frac{a}{1-r} - \left(\frac{a}{1-r}\right)\lim_{n\to\infty}\left(r^n\right) = \frac{a}{1-r} - \left(\frac{a}{1-r}\right)(0) = \frac{a}{1-r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $\dfrac{a}{1-r}$.

If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent by (11.1.9) and so, by Equation $\boxed{3}$, $\lim_{n\to\infty} s_n$ does not exist.

$\boxed{4}$ The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

$\boxed{5}$ $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$\boxed{6}$ **Theorem**

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

$\boxed{7}$ **Test for Divergence**

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

$\boxed{8}$ **Theorem**

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

On pages 3 and 4 are definitions and descriptions from Thomas's Calculus textbook

Definitions

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$s_1 = a_1$$

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} .$$

In which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$.

If $|r| \neq 1$, we can determine the convergence or divergence of the Geometric series in the following way:

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + ar^3 + \cdots + ar^n \\ s_n - rs_n &= a - ar^n \\ s_n(1-r) &= a(1-r^n) \\ s_n &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$, so $s_n \to \dfrac{a}{1-r}$ in this case.

On the other hand, if $|r| > 1$, then $|r^n| \to \infty$ and the series diverge.

If $|r|<1$, the geometric series $a+ar+ar^2+\cdots+ar^{n-1}+\cdots$ converges to $\dfrac{a}{1-r}$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r|<1$$

If $|r|\geq 1$, the series diverges.

Theorem 7

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

The *n*th-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Theorem 8

If $\sum a_n = A$: and $\sum b_n = B$ are convergent series, then

1. Sum Rule: $\sum (a_n + b_n) = \sum a_n + \sum b_n = A+B$
2. Difference Rule: $\sum (a_n - b_n) = \sum a_n - \sum b_n = A-B$
3. Constant Multiple Rule: $\sum ka_n = k\sum a_n = kA$ (any number k)

1. Every nonzero constant multiple of a divergent series diverges.

2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverge.

16-a) $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$

$\sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n$

both represent the sum of the first nth terms of the sequence $\{a_n\}$ (nth partial sum).

16-b) $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$ (nth partial sum)

$$\sum_{i=1}^{n} a_j = \underbrace{a_j + a_j + a_j + \cdots + a_j}_{n \text{ terms}} = n\, a_j$$

summation notation of a basic multiplication

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18) $\sum_{n=4}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$

$$s_n = \sum_{i=4}^{n} \left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i+1}} \right)$$

$$= \left(\frac{1}{\sqrt{(4)}} - \frac{1}{\sqrt{(4)+1}} \right) + \left(\frac{1}{\sqrt{(5)}} - \frac{1}{\sqrt{(5)+1}} \right) + \cdots + \left(\frac{1}{\sqrt{(n-1)}} - \frac{1}{\sqrt{(n-1)+1}} \right) + \left(\frac{1}{\sqrt{(n)}} - \frac{1}{\sqrt{(n)+1}} \right)$$

$$= \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} \right) + \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} \right) + \cdots + \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} \right) + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$$

$$= \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{n+1}} \quad \text{[telescoping series]}$$

$$\sum_{n=4}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{n+1}} \right) = \frac{1}{\sqrt{4}} - 0 = \frac{1}{2}$$

Converges

20) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

$s_n = \sum_{i=1}^{n} \ln\left(\frac{i}{i+1}\right) = \sum_{i=1}^{\infty} \left(\ln(i) - \ln(i+1)\right)$

$= \left(\ln(1) - \ln((1)+1)\right) + \left(\ln(2) - \ln((2)+1)\right) + \left(\ln(3) - \ln((3)+1)\right) + \dots$

$+ \left(\ln(n-1) - \ln((n-1)+1)\right) + \left(\ln(n) - \ln((n)+1)\right)$

$= (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \dots$

$+ \left(\ln(n-1) - \ln(n)\right) + \left(\ln(n) - \ln(n+1)\right)$

$= \ln 1 - \ln(n+1) = -\ln(n+1)$ {note $\ln 1 = 0$} [telescoping series]

$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(-\ln(n+1)\right) = -\infty$ diverges

---

22) $\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$

$s_n = \sum_{i=2}^{n} \frac{1}{i^3 - i}$ use partial fraction

$\frac{1}{i^3 - i} = \frac{1}{i(i^2-1)} = \frac{1}{(i)'(i+1)'(i-1)'} = \frac{A}{(i)'} + \frac{B}{(i+1)'} + \frac{C}{(i-1)'}$

$1 = A(i+1)(i-1) + B(i)(i-1) + C(i)(i+1)$

$1 = A(i^2-1) + B(i^2 - i) + C(i^2 + i)$

const. term.	i term	i^2 term
$1 = -A$	$0 = -B + C$	$0 = A + B + C$
$\underline{A = -1}$	$B = C$	$0 = (-1) + B + (B)$
	$\underline{C = \frac{1}{2}}$	$1 = 2B$
		$\underline{\frac{1}{2} = B}$

22) continued...

switch terms and factor $\frac{1}{2}$ to get

11.2/7

$$\frac{1}{i^3-i} = \frac{(-1)}{(i)^1} + \frac{\left(\frac{1}{2}\right)}{(i+1)^1} + \frac{\left(\frac{1}{2}\right)}{(i-1)^1} = \frac{1}{2}\left(\frac{1}{i-1} - \frac{2}{i} + \frac{1}{i+1}\right)$$

$$s_n = \sum_{i=2}^{\infty} \frac{1}{i^3-i} = \sum_{i=2}^{\infty} \frac{1}{2}\left(\frac{1}{i-1} - \frac{2}{i} + \frac{1}{i+1}\right) = \frac{1}{2}\sum_{i=2}^{\infty}\left(\frac{1}{i-1} - \frac{2}{i} + \frac{1}{i+1}\right)$$

$$= \frac{1}{2}\left\{\left(\frac{1}{(2)-1} - \frac{2}{(2)} + \frac{1}{(2)+1}\right) + \left(\frac{1}{(3)-1} - \frac{2}{(3)} + \frac{1}{(3)+1}\right) + \left(\frac{1}{(4)-1} - \frac{2}{(4)} + \frac{1}{(4+1)}\right)\right.$$

$$+ \left(\frac{1}{(5)-1} - \frac{2}{(5)} + \frac{1}{(5)+1}\right) + \left(\frac{1}{(6)-1} - \frac{2}{(6)} + \frac{1}{(6)+1}\right) + \cdots$$

$$\left. + \left(\frac{1}{(n-2)-1} - \frac{2}{(n-2)} + \frac{1}{(n-2)+1}\right) + \left(\frac{1}{(n-1)-1} - \frac{2}{(n-1)} + \frac{1}{(n-1)+1}\right) + \left(\frac{1}{(n)-1} - \frac{2}{(n)} + \frac{1}{(n)+1}\right)\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{1}{1} - \frac{2}{2} + \underset{A}{\frac{1}{3}}\right) + \left(\frac{1}{2} - \underset{A}{\frac{2}{3}} + \underset{B}{\frac{1}{4}}\right) + \left(\underset{A}{\frac{1}{3}} - \underset{B}{\frac{2}{4}} + \underset{C}{\frac{1}{5}}\right)\right.$$

$$+ \left(\underset{B}{\frac{1}{4}} - \underset{C}{\frac{2}{5}} + \underset{D}{\frac{1}{6}}\right) + \left(\underset{C}{\frac{1}{5}} - \underset{D}{\frac{2}{6}} + \underset{E}{\frac{1}{7}}\right) + \underset{D\ E\ \ \gamma\ \beta}{\cdots}$$

$$\left. + \left(\underset{\gamma}{\frac{1}{n-3}} - \underset{\beta}{\frac{2}{n-2}} + \underset{\alpha}{\frac{1}{n-1}}\right) + \left(\underset{\beta}{\frac{1}{n-2}} - \underset{\alpha}{\frac{2}{n-1}} + \frac{1}{n}\right) + \left(\underset{\alpha}{\frac{1}{n-1}} - \frac{2}{n} + \frac{1}{n+1}\right)\right\}$$

Note: for 3 consecutive expressions in parenthesis, the 3rd term of left plus 2nd term of middle and 1st term of right sum to 0.

$$= \frac{1}{2}\left\{\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1}\right\} = \frac{1}{2}\left\{\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right\}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^3-n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{2}\left\{\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right\}\right) = \frac{1}{2}\left\{\frac{1}{2} - 0 + 0\right\} = \frac{1}{4}$$

Converges

for ex. 23-32, it is stated that it is geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots + ar^{(n-1)} + \dots$$
$$= ar^0 + ar^1 + ar^2 + \dots + ar^{(n-1)} + \dots$$

---

24) $\underset{a}{\overset{\shortparallel}{4}} + 3 + \frac{9}{4} + \frac{27}{16} + \dots$

$$= 4 + 4\left(\tfrac{1}{4}\right)(3) + 4\left(\tfrac{1}{4}\right)\left(\tfrac{9}{4}\right) + 4\left(\tfrac{1}{4}\right)\left(\tfrac{27}{16}\right) + \dots$$

$$= 4\left(\tfrac{3}{4}\right)^0 + 4\left(\tfrac{3}{4}\right)^1 + 4\left(\tfrac{3}{4}\right)^2 + 4\left(\tfrac{3}{4}\right)^3 + \dots = \sum_{n=1}^{\infty} 4\left(\tfrac{3}{4}\right)^{(n-1)}$$

this is a geometric series with $a = 4$ and $r = \frac{3}{4}$.

Since $|r| = \frac{3}{4} < 1$, the series converges to $\frac{a}{1-r} = \frac{(4)}{1-\left(\frac{3}{4}\right)} = \frac{4}{\frac{1}{4}} = 16$

---

26) $\underset{a}{\overset{\shortparallel}{2}} + 0.5 + 0.125 + 0.03125 + \dots$

$$= 2 + 2\left(\tfrac{1}{2}\right)(0.5) + 2\left(\tfrac{1}{2}\right)(0.125) + 2\left(\tfrac{1}{2}\right)(0.03125) + \dots$$

$$= 2 + 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{8}\right) + 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{32}\right) + \dots$$

$$= 2 + 2\left(\tfrac{1}{4}\right) + 2\left(\tfrac{1}{16}\right) + 2\left(\tfrac{1}{64}\right) + \dots$$

$$= 2\left(\tfrac{1}{4}\right)^0 + 2\left(\tfrac{1}{4}\right)^1 + 2\left(\tfrac{1}{4}\right)^2 + 2\left(\tfrac{1}{4}\right)^3 + \dots = \sum_{n=1}^{\infty} 2\left(\tfrac{1}{4}\right)^{(n-1)}$$

this is a geometric series with $a = 2$ and $r = \frac{1}{4}$.

Since $|r| = \frac{1}{4} < 1$, the series converges to $\frac{a}{1-r} = \frac{(2)}{1-\left(\frac{1}{4}\right)} = \frac{2}{\frac{3}{4}} = \frac{8}{3}$

28) $\sum_{n=1}^{\infty} \frac{5}{\pi^n} = \frac{5}{\pi^1} + \frac{5}{\pi^2} + \frac{5}{\pi^3} + \cdots + \frac{5}{\pi^n} + \cdots$

$= \frac{5}{\pi} + \frac{5}{\pi}\left(\frac{1}{\pi}\right)^1 + \frac{5}{\pi}\left(\frac{1}{\pi}\right)^2 + \cdots + \frac{5}{\pi}\left(\frac{1}{\pi}\right)^{(n-1)} + \cdots = \sum_{n=1}^{\infty} \left(\frac{5}{\pi}\right)\left(\frac{1}{\pi}\right)^{(n-1)}$

this is a geometric series with $a = \frac{5}{\pi}$ and $r = \frac{1}{\pi}$.

Since $|r| = \frac{1}{\pi} < 1$, the series converges to

$$\frac{a}{1-r} = \frac{\left(\frac{5}{\pi}\right)}{1-\left(\frac{1}{\pi}\right)} = \left(\frac{\frac{5}{\pi}}{1-\frac{1}{\pi}}\right)\left(\frac{\frac{\pi}{1}}{\frac{\pi}{1}}\right) = \frac{5}{\pi-1}$$

---

30) $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n} = \frac{3^{(0)+1}}{(-2)^{(0)}} + \frac{3^{(1)+1}}{(-2)^{(1)}} + \frac{3^{(2)+1}}{(-2)^{(2)}} + \frac{3^{(3)+1}}{(-2)^{(3)}} + \cdots + \frac{3^{n+1}}{(-2)^n} + \cdots$

$= 3 + \frac{3^2}{(-2)} + \frac{3^3}{(-2)^2} + \frac{3^4}{(-2)^3} + \cdots + \frac{3^{n+1}}{(-2)^n} + \cdots$

$= 3 + 3\left(\frac{-3}{2}\right)^1 + 3\left(\frac{-3}{2}\right)^2 + 3\left(\frac{-3}{2}\right)^3 + \cdots + 3\left(\frac{-3}{2}\right)^n + \cdots = \sum_{n=1}^{\infty} 3\left(\frac{-3}{2}\right)^{(n-1)}$

this is a geometric series with $a = 3$ and $r = \frac{-3}{2}$.

Since $|r| = \frac{3}{2} > 1$, the series diverges.

---

32) $\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$ use the properties of exponents so that we can rewrite the sum

$= \sum_{n=1}^{\infty} \frac{(6)(2^{2n})(2^{-1})}{3^n} = \sum_{n=1}^{\infty} \frac{(6)(2^{-1})\ (2^2)^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{6}{2}\right)\frac{(4)^n}{3^n} = \sum_{n=1}^{\infty} 3\left(\frac{4}{3}\right)^n$

this is a geometric series with $a = 3$ and $r = \frac{4}{3}$.

Since $|r| = \frac{4}{3} > 1$, the series diverges.

34) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}+\dots$

$$=\frac{1}{1+1}+\frac{2}{2+1}+\frac{3}{3+1}+\frac{4}{4+1}+\frac{5}{5+1}+\frac{6}{6+1}+\dots=\sum_{n=1}^{\infty}\frac{n}{n+1}=\sum_{n=1}^{\infty}a_n$$

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n}{n+1}\overset{L}{=}\lim_{n\to\infty}\frac{1}{1}=1\neq 0$$

Since $\lim\limits_{n\to\infty} a_n \neq 0$, by the Test for Divergence, this series diverges.

---

36) $\frac{1}{3}+\frac{2}{9}+\frac{1}{27}+\frac{2}{81}+\frac{1}{243}+\frac{2}{729}+\dots$

$$=\left(\frac{1}{3}+\frac{1}{27}+\frac{1}{243}+\dots\right)+\left(\frac{2}{9}+\frac{2}{81}+\frac{2}{729}+\dots\right)$$

$$=\left(\frac{1}{3}+\frac{1}{3}\left(\frac{1}{9}\right)+\frac{1}{3}\left(\frac{1}{81}\right)+\dots\right)+\left(\frac{2}{9}+\frac{2}{9}\left(\frac{1}{9}\right)+\frac{2}{9}\left(\frac{1}{81}\right)+\dots\right)$$

$$=\left(\frac{1}{3}+\frac{1}{3}\left(\frac{1}{9}\right)^1+\frac{1}{3}\left(\frac{1}{9}\right)^2+\dots\right)+\left(\frac{2}{9}+\frac{2}{9}\left(\frac{1}{9}\right)^1+\frac{2}{9}\left(\frac{1}{9}\right)^2+\dots\right)$$

$$=\sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{9}\right)^{(n-1)}+\sum_{n=1}^{\infty}\frac{2}{9}\left(\frac{1}{9}\right)^{(n-1)}$$ both geometric series

$a=\frac{1}{3}$ $r=\frac{1}{9}$ $\quad$ $a=\frac{2}{9}$ $r=\frac{1}{9}$ $\quad$ $|r|=\frac{1}{9}<1$, converges to $\frac{a}{1-r}$

$$=\left(\frac{\left(\frac{1}{3}\right)}{1-\left(\frac{1}{9}\right)}\right)+\left(\frac{\left(\frac{2}{9}\right)}{1-\left(\frac{1}{9}\right)}\right)=\left(\frac{\frac{1}{3}}{\frac{8}{9}}\right)+\left(\frac{\frac{2}{9}}{\frac{8}{9}}\right)=\left(\frac{1}{3}\right)\left(\frac{9}{8}\right)+\left(\frac{2}{9}\right)\left(\frac{9}{8}\right)$$

$$=\frac{3}{8}+\frac{2}{8}=\frac{5}{8}$$

this series converges.

38) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^2-2k+5} = \sum_{k=1}^{\infty} a_k$

$\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{k^2-2k+5} = \lim_{k\to\infty}\left(\frac{\frac{k^2}{k^2}}{\frac{k^2}{k^2}-\frac{2k}{k^2}+\frac{5}{k^2}}\right) = \lim_{k\to\infty}\left(\frac{1}{1-\frac{2}{k}+\frac{5}{k^2}}\right)$

$\displaystyle = \frac{1}{1-0+0} = 1 \neq 0$

by the Test for Divergence, diverges

---

40) $\displaystyle\sum_{n=1}^{\infty}\left[(-0.2)^n + (0.6)^{n-1}\right] = \sum_{n=1}^{\infty}(-0.2)^n + \sum_{n=1}^{\infty}(0.6)^{(n-1)}$

$\displaystyle = \sum_{n=1}^{\infty}(-0.2)(-0.2)^{(n-1)} + \sum_{n=1}^{\infty} 1(0.6)^{(n-1)}$

geometric with $a=-0.2$, $r=-0.2$
$|r|=0.2<1$, converges

geometric with $a=1$, $r=0.6$
$|r|=0.6<1$, converges

$\displaystyle = \left(\frac{(-0.2)}{1-(-0.2)}\right) + \left(\frac{(1)}{1-(0.6)}\right) = \left(\frac{-0.2}{1.2}\right) + \left(\frac{1}{0.4}\right) = \frac{-2}{12} + \frac{10}{4}$

$\displaystyle = \frac{-1}{6} + \frac{5}{2} = \frac{-1}{6} + \frac{15}{6} = \frac{14}{6} = \frac{7}{3}$ converges

---

42) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n} = \sum_{n=1}^{\infty} a_n$

$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{2^n+4^n}{e^n}\right) \geq \lim_{n\to\infty}\left(\frac{4^n}{e^n}\right) = \lim_{n\to\infty}\left(\frac{4}{e}\right)^n = +\infty \neq 0$ because $\left(\frac{4}{e}\right)>1$
see (11.1, 9)

by the Test for Divergence, diverges

44) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n} = \sum_{n=1}^{\infty} a_n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{1+\left(\frac{2}{3}\right)^n}\right) = \frac{1}{1+\lim_{n\to\infty}\left(\frac{2}{3}\right)^n} = \frac{1}{1+0} = 1 \neq 0$$

see (11.1.9), $\frac{2}{3}<1$

by the Test for Divergence, diverges

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46) $\displaystyle\sum_{k=0}^{\infty} (\sqrt{2})^{-k} = \sum_{k=0}^{\infty} \frac{1}{(\sqrt{2})^k} = \frac{1}{(\sqrt{2})^0} + \frac{1}{(\sqrt{2})^1} + \frac{1}{(\sqrt{2})^2} + \cdots$

$$= 1 + 1\cdot\left(\frac{1}{\sqrt{2}}\right)^1 + 1\left(\frac{1}{\sqrt{2}}\right)^2 + \cdots = \sum_{n=1}^{\infty} 1\left(\frac{1}{\sqrt{2}}\right)^{(n-1)}$$

this is a geometric series with $a=1$ and $r=\frac{1}{\sqrt{2}}$.

Since $|r| = \frac{1}{\sqrt{2}} < 1$, the series converges to

$$\frac{(1)}{1-\left(\frac{1}{\sqrt{2}}\right)} = \left(\frac{1}{1-\frac{1}{\sqrt{2}}}\right)\left(\frac{\frac{\sqrt{2}}{1}}{\frac{\sqrt{2}}{1}}\right) = \frac{\sqrt{2}}{\sqrt{2}-1}$$

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48) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right) = \sum_{n=1}^{\infty}\left(\frac{3}{5^n}\right) + \sum_{n=1}^{\infty}\frac{2}{n} = 3\sum_{n=1}^{\infty}\frac{1}{5^n} + 2\sum_{n=1}^{\infty}\frac{1}{n}$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is a harmonic series which diverges (see sec 11.2 example 8)

Since $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n} = 2\sum_{n=1}^{\infty}\frac{1}{n}$ diverges,

$\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right)$ also diverges.

50) $\sum_{n=1}^{\infty} \frac{e^n}{n^2} = a_n$ | 11.2/13

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{e^n}{n^2}\right) \overset{L}{=} \lim_{n\to\infty} \left(\frac{e^n}{2n}\right) \overset{L}{=} \lim_{n\to\infty} \left(\frac{e^n}{2}\right) = +\infty \neq 0$

by the Test for Divergence, diverges

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60) $\sum_{n=1}^{\infty} (x+2)^n = \sum_{n=1}^{\infty} (x+2)(x+2)^{(n-1)}$

is a geometric series with $a = (x+2)$ and $r = (x+2)$

and it converges if

$|r| < 1$

$|(x+2)| < 1$

$-1 < x+2 < 1$

$-3 < x < -1$

$\frac{a}{1-r} = \frac{(x+2)}{1-(x+2)} = \frac{x+2}{1-x-2} = \frac{x+2}{-x-1}$

the series will converge for $-3 < x < -1$

And its sum is $\frac{x+2}{-x-1}$.

62) $\sum_{n=0}^{\infty} (-4)^n (x-5)^n = \sum_{n=0}^{\infty} \left(-4(x-5)\right)^n = \sum_{n=1}^{\infty} 1 \left(-4(x-5)\right)^{(n-1)}$ 11.2/14

is a geometric series with $a=1$ and $r=-4(x-5)$

and it converges if

$$|r|<1 \qquad \frac{a}{1-r} = \frac{(1)}{1-(-4(x-5))}$$

$$|-4(x-5)|<1$$

$$|-4||x-5|<1 \qquad = \frac{1}{1+4(x-5)} = \frac{1}{1+4x-20}$$

$$|x-5|<\frac{1}{4}$$

$$-\frac{1}{4}<x-5<\frac{1}{4} \qquad = \frac{1}{4x-19}$$

$$5-\frac{1}{4}<x<5+\frac{1}{4}$$

$$\frac{19}{4}<x<\frac{21}{4}$$

the series will converge if $\frac{19}{4}<x<\frac{21}{4}$

and its sum is $\frac{1}{4x-19}$.

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64) $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=1}^{\infty} 1 \left(\frac{x}{2}\right)^{(n-1)}$

is a geometric series with $a=1$ and $r=\frac{x}{2}$

$$|r|<1 \qquad \frac{a}{1-r} = \frac{(1)}{1-\left(\frac{x}{2}\right)} = \left(\frac{1}{1-\frac{x}{2}}\right)\left(\frac{\frac{2}{1}}{\frac{2}{1}}\right) = \frac{2}{2-x}$$

$$\left|\frac{x}{2}\right|<1$$

$-1<\frac{x}{2}<1$ the series will converge if $-2<x<2$

$-2<x<2$ and its sum is $\frac{2}{2-x}$.

66) $\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n} = \sum_{n=1}^{\infty} 1 \left(\frac{\sin x}{3}\right)^{(n-1)}$

11.2/15

is a geometric series with $a = 1$ and $r = \frac{\sin x}{3}$

and it converges if

$|r| < 1$

$\left|\frac{\sin x}{3}\right| < 1$

$\frac{|\sin x|}{|3|} < 1$

$\frac{|\sin x|}{3} < 1$

$|\sin x| < 3$

$$\frac{a}{1-r} = \frac{(1)}{1 - \left(\frac{\sin x}{3}\right)}$$

$$= \left(\frac{1}{1 - \frac{\sin x}{3}}\right)\left(\frac{\frac{3}{1}}{\frac{3}{1}}\right)$$

$$= \frac{3}{3 - \sin x}$$

since the range of $\sin x$ is $[-1, 1]$, $|\sin x| < 3$ is true for all values of x.

the series will converge for all values of x $(-\infty, \infty)$ and its sum is $\frac{3}{3 - \sin x}$.