1 Intuitive Definition of a Limit of a Sequence

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty$$

if we can make the terms a_n as close to L as we line by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

2 Precise Definition of a Limit of a Sequence

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

3 Precise Definition of an Infinite Limit

The notation $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad a_n > M$$

4 Theorem

If $\lim_{n\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Limit Laws of Sequences

Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then

1. Sum Law: $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. Difference Law: $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. Constant Multiple Law: $\lim_{n\to\infty}(c \cdot a_n) = c \lim_{n\to\infty} a_n$
4. Product Law: $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
5. Quotient Law: $\lim_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$

Power Laws

$$\lim_{n\to\infty} a_n{}^p = \left[\lim_{n\to\infty} a_n\right]^p \qquad \text{if} \qquad p > 0 \quad \text{and} \quad a_n > 0$$

Squeeze Theorem for Sequences

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.

$\boxed{6}$ **Theorem**

If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

$\boxed{7}$ **Theorem**

If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} a_n = f(L).$$

$\boxed{9}$ The sequence $\{r_n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1 \\ 1 & \text{if} \quad r = 1 \end{cases}.$$

$\boxed{10}$ **Definition**

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \ge 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \ge 1$. A sequence is called **monotonic** if it is either increasing or decreasing.

$\boxed{11}$ **Definition**

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n < M \qquad \text{for all } n \ge 1$$

A sequence $\{a_n\}$ is **bounded below** if there is a number m such that

$$m < a_n \qquad \text{for all } n \ge 1$$

If a sequence is bounded above and below, then it is called a **bounded sequence**.

$\boxed{12}$ **Monotonic Sequence Theorem**

Every bounded, monotonic sequence is convergent.

In particular, a sequence that is increasing and bounded above converges, and a sequence that is decreasing and bounded below converges.

Definitions, theorems and notes on page 3 and 4 are from Thomas's Calculus textbook.

Useful list that our text did not have are Theorem 5 on page 4.

Definitions

The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that

$$|a_n - L < \varepsilon| \qquad \text{whenever} \qquad n > N .$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{x\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence.

Definition

The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty .$$

Similarly, if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \quad \text{or} \quad a_n \to -\infty .$$

Theorem 1

Let $\{a_n\}$ and $\{a_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. Sum Rule: $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. Difference Rule: $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. Constant Multiple Rule: $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. Product Rule: $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. Quotient Rule: $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ if $B \neq 0$

Theorem 2 - The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Theorem 3 - The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Theorem 4

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $x \geq n_0$. Then

$$\lim_{n\to\infty} a_n = L \quad \text{whenever} \quad \lim_{x\to\infty} f(x) = L.$$

Theorem 5

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Definition

A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n. The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

Definitions

A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all n. That is, $a_1 \leq a_2 \leq a_3 \leq \cdots$. The sequence is **nonincreasing** if $a_n \geq a_{n+1}$ for all n. The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

Theorem 6 - The Monotonic Sequence Theorem

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

4) $a_n = \dfrac{1}{3^n+1}$

$$\left\{ \frac{1}{3^{(1)}+1}, \frac{1}{3^{(2)}+1}, \frac{1}{3^{(3)}+1}, \frac{1}{3^{(4)}+1}, \frac{1}{3^{(5)}+1}, \dots \right\}$$

$$= \left\{ \frac{1}{3+1}, \frac{1}{9+1}, \frac{1}{27+1}, \frac{1}{81+1}, \frac{1}{243+1}, \dots \right\}$$

$$= \left\{ \frac{1}{4}, \frac{1}{10}, \frac{1}{28}, \frac{1}{82}, \frac{1}{244}, \dots \right\}$$

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6) $\left\{ \dfrac{n^2-1}{n^2+1} \right\}_{n=3}^{\infty}$

$$\left\{ \frac{(3)^2-1}{(3)^2+1}, \frac{(4)^2-1}{(4)^2+1}, \frac{(5)^2-1}{(5)^2+1}, \frac{(6)^2-1}{(6)^2+1}, \frac{(7)^2-1}{(7)^2+1}, \dots \right\}$$

$$= \left\{ \frac{9-1}{9+1}, \frac{16-1}{16+1}, \frac{25-1}{25+1}, \frac{36-1}{36+1}, \frac{49-1}{49+1}, \dots \right\}$$

$$= \left\{ \frac{8}{10}, \frac{15}{17}, \frac{24}{26}, \frac{35}{37}, \frac{48}{50}, \dots \right\}$$

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8) $a_n = \dfrac{(-1)^n}{4^n}$

$$\left\{ \frac{(-1)^{(1)}}{4^{(1)}}, \frac{(-1)^{(2)}}{4^{(2)}}, \frac{(-1)^{(3)}}{4^{(3)}}, \frac{(-1)^{(4)}}{4^{(4)}}, \frac{(-1)^{(5)}}{4^{(5)}}, \dots \right\}$$

$$= \left\{ \frac{-1}{4}, \frac{1}{16}, \frac{-1}{64}, \frac{1}{256}, \frac{-1}{1024}, \dots \right\}$$

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10) $a_n = 1 + (-1)^n$

$$\left\{ 1+(-1)^{(1)}, 1+(-1)^{(2)}, 1+(-1)^{3}, 1+(-1)^{(4)}, 1+(-1)^{(5)}, \dots \right\}$$

$$= \{ 1-1, 1+1, 1-1, 1+1, 1-1, \dots \} = \{0, 2, 0, 2, 0, \dots\}$$

11.1/6

12) $a_n = \frac{2n+1}{n!+1}$

$\left\{ \frac{2(1)+1}{(1)!+1}, \frac{2(2)+1}{(2)!+1}, \frac{2(3)+1}{(3)!+1}, \frac{2(4)+1}{(4)!+1}, \frac{2(5)+1}{(5)!+1}, \ldots \right\}$

$= \left\{ \frac{2+1}{1+1}, \frac{4+1}{(2)(1)+1}, \frac{6+1}{(3)(2)(1)+1}, \frac{8+1}{(4)(3)(2)(1)+1}, \frac{10+1}{(5)(4)(3)(2)(1)+1}, \ldots \right\}$

$= \left\{ \frac{3}{2}, \frac{5}{3}, \frac{7}{7}, \frac{9}{25}, \frac{11}{121}, \ldots \right\}$

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14) $a_1 = 6$, $a_{n+1} = \frac{a_n}{n}$

$a_2 = \frac{a_1}{(1)} = \frac{6}{1} = 6$

$a_3 = \frac{a_2}{(2)} = \frac{6}{2} = 3$

$a_4 = \frac{a_3}{(3)} = \frac{3}{3} = 1$

$a_5 = \frac{a_4}{(4)} = \frac{1}{4}$

$\{a_1, a_2, a_3, a_4, a_5, \ldots\} = \{6, 6, 3, 1, \frac{1}{4}, \ldots\}$

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16) $a_1 = 2$, $a_2 = 1$, $a_{n+1} = a_n - a_{n-1}$

$a_3 = a_2 - a_1 = (1) - (2) = -1$

$a_4 = a_3 - a_2 = (-1) - (1) = -2$

$a_5 = a_4 - a_3 = (-2) - (-1) = -2 + 1 = -1$

$\{a_1, a_2, a_3, a_4, a_5, \ldots\} = \{2, 1, -1, -2, -1, \ldots\}$

18) $\{4, -1, \frac{1}{4}, \frac{-1}{16}, \frac{1}{64}, \dots\}$

① alternating sign ② from 3rd term on, denominator increases by multiple of 4

option 1: $a_1 = 4$, $a_2 = -1$, $a_n = \frac{(-1)^{(n-1)}}{4^{(n-2)}}$ for $n \ge 3$

option 2;

$$\{4, -1, \frac{1}{4}, \frac{-1}{16}, \frac{1}{64}, \dots\}$$

$$= \{4(1), 4\left(\frac{-1}{4}\right), 4\left(\frac{1}{16}\right), 4\left(\frac{-1}{64}\right), 4\left(\frac{1}{256}\right), \dots\}$$

$$= \{\underset{a_1}{4\left(\frac{-1}{4}\right)^0}, \underset{a_2}{4\left(\frac{-1}{4}\right)^1}, \underset{a_3}{4\left(\frac{-1}{4}\right)^2}, \underset{a_4}{4\left(\frac{-1}{4}\right)^3}, \underset{a_5}{4\left(\frac{-1}{4}\right)^4}, \dots\}$$

$a_n = 4\left(\frac{-1}{4}\right)^{(n-1)}$ for $n \ge 1$

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20) $\{5, 8, 11, 14, 17, \dots\}$

$$= \{\underset{a_1}{5}, \underset{a_2}{5+3}, \underset{a_3}{8+3}, \underset{a_4}{11+3}, \underset{a_5}{14+3}, \dots\}$$

$a_5 = 14 + 3$
$= (11+3) + 3 = 11 + 2(3)$
$= (8+3) + 2(3) = 8 + 3(3)$
$= (5+3) + 3(3) = 5 + 4(3)$

since $5 = 2+3$

$a_5 = 5 + 4(3)$
$= (2+3) + 4(3)$
$a_5 = 2 + 5(3)$
$\Downarrow$
$a_n = 2 + n(3)$

so $a_n = 2 + 3n = 3n + 2$ for $n \ge 1$

22) $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

① signs alternating and values passing the midpoint 0. It is tracing lowest, middle, and highest values of trigonometric functions ($\sin\theta$ and $\cos\theta$).

option 1: $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

$$a_1 = \sin\left(\tfrac{\pi}{2}\right) = 1, \quad a_2 = \sin(\pi) = \sin\left(\tfrac{2\pi}{2}\right) = 0, \quad a_3 = \sin\left(\tfrac{3\pi}{2}\right) = -1, \quad a_4 = \sin(2\pi) = \sin\left(\tfrac{4\pi}{2}\right) = 0,$$

$$a_5 = \sin\left(\tfrac{5\pi}{2}\right) = 1, \quad a_6 = \sin(3\pi) = \sin\left(\tfrac{6\pi}{2}\right) = 0, \quad a_7 = \sin\left(\tfrac{7\pi}{2}\right) = -1, \quad a_8 = \sin(4\pi) = \sin\left(\tfrac{8\pi}{2}\right) = 0$$

so $a_n = \sin\left(\frac{n\pi}{2}\right)$ for $n \geq 1$

option 2: $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

$$a_1 = \cos(0) = \cos\left(\tfrac{0}{2}\right) = \cos\left(\tfrac{0\pi}{2}\right) = 1, \quad a_2 = \cos\left(\tfrac{\pi}{2}\right) = 0, \quad a_3 = \cos(\pi) = \cos\left(\tfrac{2\pi}{2}\right) = -1, \quad a_4 = \cos\left(\tfrac{3\pi}{2}\right) = 0,$$

$$a_5 = \cos(2\pi) = \cos\left(\tfrac{4\pi}{2}\right) = 1, \quad a_6 = \cos\left(\tfrac{5\pi}{2}\right) = 0, \quad a_7 = \cos(3\pi) = \cos\left(\tfrac{6\pi}{2}\right) = -1, \quad a_8 = \cos\left(\tfrac{7\pi}{2}\right) = 0$$

so $a_n = \cos\left(\frac{(n-1)\pi}{2}\right)$ for $n \geq 1$

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28) $a_n = 5\sqrt{n+2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 5\sqrt{n+2} = +\infty$ "Pg1 $\boxed{1}$" Diverges

30) $a_n = \dfrac{4n^2-3n}{2n+1}$

option 1:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{4n^2-3n}{2n+1}\right) = \lim_{n\to\infty}\left(\frac{\frac{4n^2}{n^2}-\frac{3n}{n^2}}{\frac{2n}{n^2}+\frac{1}{n^2}}\right) = \lim_{n\to\infty}\left(\frac{4-\frac{3}{n}}{\frac{2}{n}+\frac{1}{n^2}}\right)$$

$$= \frac{4-0^+}{0^+ + 0^+} = +\infty$$

option 2:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{4n^2-3n}{2n+1}\right) \overset{L}{=} \lim_{n\to\infty}\left(\frac{8n-3}{2}\right) = +\infty$$

Diverges

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32) $a_n = 2+(0.86)^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(2+(0.86)^n\right) = \lim_{n\to\infty} 2 + \lim_{n\to\infty}(0.86)^n$$

$$= 2 + \lim_{n\to\infty}(0.86)^n = 2+0 = 2 \quad \text{Converges}$$

because $-1 < r = 0.86 < 1$ and by $\boxed{9}$ "Page 2", $\lim_{n\to\infty}(0.86)^n = 0$

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34) $a_n = \dfrac{3\sqrt{n}}{\sqrt{n}+2}$

option 1:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3\sqrt{n}}{\sqrt{n}+2}\right) = \lim_{n\to\infty}\left(\frac{\frac{3\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n}}{\sqrt{n}}+\frac{2}{\sqrt{n}}}\right) = \lim_{n\to\infty}\left(\frac{3}{1+\frac{2}{\sqrt{n}}}\right)$$

$$= \frac{3}{1+0} = 3$$

34) continued...

option 2:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3\sqrt{n}}{\sqrt{n}+2}\right) \overset{L}{=} \lim_{n\to\infty}\left(\frac{\frac{3}{2\sqrt{n}}}{\frac{1}{2\sqrt{n}}}\right) = \lim_{n\to\infty}\left(\frac{3}{2\sqrt{n}}\right)\left(\frac{2\sqrt{n}}{1}\right)$$

$$= \lim_{n\to\infty} 3 = 3$$

Converges

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36) $a_n = \frac{4^n}{1+9^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{4^n}{1+9^n}\right) = \lim_{n\to\infty}\left(\frac{\frac{4^n}{9^n}}{\frac{1}{9^n}+\frac{9^n}{9^n}}\right) = \lim_{n\to\infty}\left(\frac{\left(\frac{4}{9}\right)^n}{\frac{1^n}{9^n}+1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\left(\frac{4}{9}\right)^n}{\left(\frac{1}{9}\right)^n+1}\right) = \frac{0}{0+1} = 0 \quad \text{Converges}$$

because $-1 < \left(\frac{1}{9}\right) < \left(\frac{4}{9}\right) < 1$ and by $\boxed{9}$ $\lim_{n\to\infty}\left(\frac{1}{9}\right)^n = 0$ and $\lim_{n\to\infty}\left(\frac{4}{9}\right)^n = 0$

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38) $a_n = \cos\left(\frac{n\pi}{n+1}\right)$

option 1:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\cos\left(\frac{n\pi}{n+1}\right)\right) = \cos\left(\lim_{n\to\infty}\left(\frac{n\pi}{n+1}\right)\right) = \cos\left(\lim_{n\to\infty}\left(\frac{\frac{n\pi}{n}}{\frac{n}{n}+\frac{1}{n}}\right)\right)$$

$$= \cos\left(\lim_{n\to\infty}\left(\frac{\pi}{1+\frac{1}{n}}\right)\right) = \cos\left(\frac{\pi}{1+0}\right) = \cos\pi = -1$$

38) continued...

option 2:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\cos\left(\frac{n\pi}{n+1}\right)\right) = \cos\left(\lim_{n\to\infty}\left(\frac{n\pi}{n+1}\right)\right) \stackrel{L}{=} \cos\left(\lim_{n\to\infty}\left(\frac{\pi}{1}\right)\right)$$

$$= \cos(\pi) = -1$$

Converges

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40) $a_n = e^{\frac{2n}{n+2}}$ let $b_n = \frac{2n}{n+2}$

option 1:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{2n}{n+1}\right) = \lim_{n\to\infty}\left(\frac{\frac{2n}{n}}{\frac{n}{n}+\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{2}{1+\frac{1}{n}}\right) = \frac{2}{1+0} = 2$$

option 2:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{2n}{n+1}\right) \stackrel{L}{=} \lim_{n\to\infty}\left(\frac{2}{1}\right) = 2$$

Since the natural exponential function (e) is continuous at 2, by $\boxed{7}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(e^{\frac{2n}{n+1}}\right) = e^{\left(\lim_{n\to\infty}\left(\frac{2n}{n+1}\right)\right)} = e^{\left(\lim_{n\to\infty} b_n\right)} = e^2$$

Converges

42) $a_n = \frac{(-1)^{(n+1)} n}{n+\sqrt{n}}$ since the sign alternates, we must check $|a_n| = \frac{n}{n+\sqrt{n}}$

option 1

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left(\frac{n}{n+\sqrt{n}}\right) = \lim_{n\to\infty} \left(\frac{\frac{n}{n}}{\frac{n}{n}+\frac{\sqrt{n}}{n}}\right) = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{\sqrt{n}}}\right) = \frac{1}{1+0} = 1$$

option 2:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left(\frac{n}{n+\sqrt{n}}\right) \overset{L}{=} \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{2\sqrt{n}}}\right) = \frac{1}{1+0} = 1$$

for $n > 1$, $a_n = \frac{(-1)^{(n+1)} n}{n+\sqrt{n}}$

has odd-numbered terms that approaches 1
and even-numbered terms that approaches -1
Therefore this sequence $\{a_n\}$ is divergent

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44) $\left\{\frac{\ln n}{\ln(2n)}\right\} \Rightarrow a_n = \frac{\ln n}{\ln(2n)}$

option 1:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{\ln n}{\ln(2n)}\right) = \lim_{n\to\infty} \left(\frac{\ln n}{\ln 2 + \ln n}\right) = \lim_{n\to\infty} \left(\frac{\frac{\ln n}{\ln n}}{\frac{\ln 2}{\ln n}+\frac{\ln n}{\ln n}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{\frac{\ln 2}{\ln n}+1}\right) = \frac{1}{0+1} = 1$$

option 2:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{\ln n}{\ln(2n)}\right) \overset{L}{=} \lim_{n\to\infty} \left(\frac{\frac{1}{n}}{\frac{1}{(2n)}(2)}\right) = \lim_{n\to\infty} (1) = 1$$

Converges

46) $a_n = \dfrac{\tan^{-1} n}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\tan^{-1} n}{n}\right) = \frac{\left(\frac{\pi}{2}\right)}{+\infty} = 0$$

Converges

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48) $a_n = \ln(n+1) - \ln(n) = \ln\left(\dfrac{n+1}{n}\right)$

option 1:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\ln\left(\frac{n+1}{n}\right)\right) = \ln\left(\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\right) = \ln\left(\lim_{n\to\infty}\left(\frac{\frac{n}{n}+\frac{1}{n}}{\frac{n}{n}}\right)\right)$$

$$= \ln\left(\lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1}\right)\right) = \ln\left(\frac{1+0}{1}\right) = \ln(1) = 0$$

option 2:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\ln\left(\frac{n+1}{n}\right)\right) = \ln\left(\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\right) \stackrel{L}{=} \ln\left(\lim_{n\to\infty}\left(\frac{1}{1}\right)\right) = \ln(1) = 0$$

Converges

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50) $a_n = \sqrt[n]{2^{(1+3n)}} = \left(2^{(1+3n)}\right)^{\frac{1}{n}} = \left((2)^{(1)}(2)^{(3n)}\right)^{\frac{1}{n}} = \left((2)^{\frac{1}{n}}(2)^{\frac{3n}{n}}\right)$

$$= (2)^{\frac{1}{n}}(2)^3 = (2)^{\frac{1}{n}}(8) = 8(2)^{\frac{1}{n}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(8(2)^{\frac{1}{n}}\right) = 8\lim_{n\to\infty}(2)^{\frac{1}{n}} = 8(2)^{\lim_{n\to\infty}\left(\frac{1}{n}\right)} = 8(2)^{(0)} = 8$$

$\lim_{n\to\infty}\frac{1}{n} = 0$ and $f(x) = 2^x$ is continuous at 0, so by $\boxed{7}$

Converges

52) $a_n = 2^{-n} \cos n\pi = \dfrac{\cos(n\pi)}{2^n}$

for $n \geq 1$, $\quad 0 \leq \left|\dfrac{\cos(n\pi)}{2^n}\right| \leq \dfrac{1}{2^n} = \left(\dfrac{1^n}{2^n}\right) = \left(\dfrac{1}{2}\right)^n$

since $-1 < \frac{1}{2} < 1$ and $\boxed{9}$ $\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

now use the Squeeze Theorem for Sequences

$$0 \leq \left|\frac{\cos(n\pi)}{2^n}\right| < \left(\frac{1}{2}\right)^n$$

$$\lim_{n\to\infty}(0) \leq \lim_{n\to\infty}\left|\frac{\cos(n\pi)}{2^n}\right| < \lim_{n\to\infty}\left(\frac{1}{2}\right)^n$$

$$0 \leq \lim_{n\to\infty}\left|\frac{\cos(n\pi)}{2^n}\right| < 0$$

so $\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty}\left|\dfrac{\cos(n\pi)}{2^n}\right| = 0$

Therefore by Theorem $\boxed{6}$, we get $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(\dfrac{\cos(n\pi)}{2^n}\right) = 0$

Converges

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54) $a_n = n^{\frac{1}{n}}$ $\quad$ let $y = x^{\frac{1}{x}} \Rightarrow \ln y = \ln\left(x^{\frac{1}{x}}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}$

$\lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty}\left(\dfrac{\ln x}{x}\right) \overset{L}{=} \lim\limits_{x\to\infty}\left(\dfrac{\frac{1}{x}}{1}\right) = \lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right) = 0$

$\ln y = 0 \Rightarrow y = e^0 = 1$ and by Theorem $\boxed{4}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(n^{\frac{1}{n}}\right) = e^0 = 1$ $\quad$ Converges

11.1/15

56) $a_n = \frac{(\ln n)^2}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{(\ln n)^2}{n}\right) \overset{L}{=} \lim_{n\to\infty}\left(\frac{\frac{2(\ln n)}{n}}{1}\right) = \lim_{n\to\infty}\left(\frac{2(\ln n)}{n}\right)$$

$$\overset{L}{=} \lim_{n\to\infty}\left(\frac{2\left(\frac{1}{n}\right)}{1}\right) = \lim_{n\to\infty}\left(\frac{2}{n}\right) = 0 \quad \text{Converges}$$

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58) $a_n = n - \sqrt{n+1}\sqrt{n+3} = n - \sqrt{(n+1)(n+3)} = n - \sqrt{n^2+4n+3}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(n - \sqrt{n^2+4n+3}\right) = \lim_{n\to\infty}\left(\frac{n-\sqrt{n^2+4n+3}}{1}\right)\left(\frac{n+\sqrt{n^2+4n+3}}{n+\sqrt{n^2+4n+3}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(n)^2-\left(\sqrt{n^2+4n+3}\right)^2}{n+\sqrt{n^2+4n+3}}\right) = \lim_{n\to\infty}\left(\frac{n^2-(n^2+4n+3)}{n+\sqrt{n^2+4n+3}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{-4n-3}{n+\sqrt{n^2+4n+3}}\right) = \lim_{n\to\infty}\left(\frac{\frac{-4n}{n}-\frac{3}{n}}{\frac{n}{n}+\frac{\sqrt{n^2+4n+3}}{n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{-4-\frac{3}{n}}{1+\frac{\sqrt{n^2+4n+3}}{\sqrt{n^2}}}\right) = \lim_{n\to\infty}\left(\frac{-4-\frac{3}{n}}{1+\sqrt{\frac{n^2+4n+3}{n^2}}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{-4-\frac{3}{n}}{1+\sqrt{\frac{n^2}{n^2}+\frac{4n}{n^2}+\frac{3}{n^2}}}\right) = \lim_{n\to\infty}\left(\frac{-4-\frac{3}{n}}{1+\sqrt{1+\frac{4}{n}+\frac{3}{n^2}}}\right)$$

$$= \frac{-4-0}{1+\sqrt{1+0+0}} = \frac{-4}{1+\sqrt{1}} = \frac{-4}{2} = -2$$

Converges

60) $\left\{ \frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \dots \right\}$
$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8$

we have 2 patterns in this sequence.

odd a_k $(a_1, a_3, a_5, a_7, \dots)$: $a_{(2n-1)} = b_n = \frac{1}{n}$ for $n \geq 1$

even a_k $(a_2, a_4, a_6, a_8, \dots)$: $a_{(2n)} = c_n = \frac{1}{n+2}$ for $n \geq 1$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \left(\frac{1}{n+2}\right) = 0$$

Therefore for sufficiently large n,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n + \lim_{n\to\infty} c_n = 0 + 0 = 0 \quad \text{Converges}$$

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62) $a_n = \frac{(-3)^n}{n!}$ since the sign alternates, we must check $|a_n| = \frac{3^n}{n!}$

for $n > 2$, $0 < |a_n| = \left(\frac{3}{1}\right)\left(\frac{3}{2}\right)\left(\frac{3}{3}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{3}{n-1}\right)\left(\frac{3}{n}\right) \leq \left(\frac{3}{1}\right)\left(\frac{3}{2}\right)\left(\frac{3}{n}\right) = \frac{27}{2n}$

$\lim_{n\to\infty}\left(\frac{27}{2n}\right) = 0$ now using Squeeze Theorem for Sequences

$$0 < |a_n| \leq \frac{27}{2n}$$

$$\lim_{n\to\infty} 0 < \lim_{n\to\infty} |a_n| \leq \lim_{n\to\infty}\left(\frac{27}{2n}\right)$$

$$0 < \lim_{n\to\infty} |a_n| \leq 0$$

and $\lim_{n\to\infty} |a_n| = 0$

Since $\lim_{n\to\infty} |a_n| = 0$, by Theorem $\boxed{6}$,

$\lim_{n\to\infty} a_n = 0$

Converges