Definition

Integrals with infinite limits of integration are **improper Integrals of Type I**.

1. If $f(x)$ is continuous on $[a,\infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{U\to\infty}\int_a^U f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty,b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{L\to-\infty}\int_L^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty,\infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$ where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Definition

Integrals of functions that become infinite at a point within the interval of integration are **improper Integrals of Type II**.

1. If $f(x)$ is continuous on $(a,b]$ and discontinuous at a, then
$$\int_a^b f(x)\,dx = \lim_{L\to a^+}\int_L^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a,b)$ and discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{U\to b^+}\int_a^U f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a<c<b$, and continuous on $[a,c)\cup(c,b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$ where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Theorem 2

Let f and g be continuous on $[a,\infty)$ with $f(x)\ge g(x)\ge 0$ for all $x\ge a$. Then

1. If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ also converges.

2. If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ also diverges.

Theorem 2-a

Let f and g be continuous on $(0,a]$ with $f(x) \geq g(x) \geq 0$ for all $0 < x \leq a$. Then

1. If $\int_0^a f(x)\,dx$ converges, then $\int_0^a g(x)\,dx$ also converges.
2. If $\int_0^a g(x)\,dx$ diverges, then $\int_0^a f(x)\,dx$ also diverges.

Reference for Comparison Theorem:

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{convergent if} & p > 1 \\ \text{divergent if} & p \leq 1 \end{cases} \qquad \int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{divergent if} & p \geq 1 \\ \text{convergent if} & p < 1 \end{cases}$$

Theorem 3

If the positive functions f and g are continuous on $[a,\infty)$ with $f(x) \geq g(x) \geq 0$, and if

$$\lim_{x\to\infty} \frac{\text{smaller}}{\text{larger}} = \lim_{x\to\infty} \frac{g(x)}{f(x)} = L, \quad 0 < L < \infty$$

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either *both converge* or *both diverge*.

Theorem 3-a

If the positive functions f and g are continuous on $(0,a]$ with $f(x) \geq g(x) \geq 0$, and if

$$\lim_{x\to 0} \frac{\text{smaller}}{\text{larger}} = \lim_{x\to 0} \frac{g(x)}{f(x)} = L, \quad 0 < L < \infty$$

then $\int_0^a f(x)\,dx$ and $\int_0^a g(x)\,dx$ either *both converge* or *both diverge*.

2-a) $\int_0^{\pi} \sec x \, dx$ sec x is discontinuous at $\frac{\pi}{2}$

Type II

2-b) $\int_0^4 \frac{dx}{x-5}$ $x-5 \neq 0$ on $[0,4]$ ~~so~~ it is continuous on $[0,4]$

it is regular definite integral

2-c) $\int_{-1}^{3} \frac{dx}{x+x^3}$ $x+x^3=0$ on $[-1,3]$ so it is discontinuous

at $x=0$ Type II

2-d) $\int_1^{\infty} \frac{dx}{x+x^3}$ since $U=\infty$ it is improper integral Type I

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6) $$\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}} dx = \lim_{L\to-\infty} \int_L^{-1} x^{-\frac{1}{3}} dx = \lim_{L\to-\infty} \left[\frac{x^{\frac{2}{3}}}{\frac{2}{3}} + C\right]_L^{-1}$$

$$= \lim_{L\to-\infty} \left[\frac{3}{2}\left(\sqrt[3]{x}\right)^2 + C\right]_L^{-1} = \lim_{L\to-\infty} \left\{\left[\frac{3}{2}\left(\sqrt[3]{(-1)}\right)^2 + C\right] - \left[\frac{3}{2}\left(\sqrt[3]{L}\right)^2 + C\right]\right\}$$

$$= \lim_{L\to-\infty} \left\{\left[\frac{3}{2}(1)\right] - \left[\frac{3}{2}\left(\sqrt[3]{L}\right)^2\right]\right\} = \left[\frac{3}{2}\right] - \left[\frac{3}{2}(-\infty)^2\right] = \underline{\underline{-\infty}}$$

Divergent

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8) $$\int_1^{\infty} \left(\frac{1}{3}\right)^x dx = \lim_{U\to\infty} \int_1^U \left(\frac{1}{3}\right)^x dx = \lim_{U\to\infty} \left[\frac{\left(\frac{1}{3}\right)^x}{\ln\left(\frac{1}{3}\right)} + C\right]_1^U$$

$$= \lim_{U\to\infty} \left\{\left[\frac{\left(\frac{1}{3}\right)^U}{\ln\left(\frac{1}{3}\right)} + C\right] - \left[\frac{\left(\frac{1}{3}\right)^{(1)}}{\ln\left(\frac{1}{3}\right)} + C\right]\right\} = \left[\frac{0}{\ln\left(\frac{1}{3}\right)}\right] - \left[\frac{\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{3}\right)}\right] = [0] - \left[\frac{\frac{1}{3}}{-\ln 3}\right]$$

$$= \underline{\underline{\frac{1}{3\ln 3}}}$$ Convergent

10) $\displaystyle\int_1^{\infty}\frac{1}{x^2+4}\,dx=\lim_{U\to\infty}\int_1^{U}\frac{1}{x^2+(2)^2}\,dx=\lim_{U\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C\right]_1^{U}$ 7.8/4

$\displaystyle=\lim_{U\to\infty}\left\{\left[\frac{1}{2}\tan^{-1}\left(\frac{U}{2}\right)+C\right]-\left[\frac{1}{2}\tan^{-1}\left(\frac{(1)}{2}\right)+C\right]\right\}=\left[\frac{1}{2}\left(\frac{\pi}{2}\right)\right]-\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)\right]$

$\displaystyle=\underline{\underline{\frac{\pi}{4}-\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)}}$ Convergent

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12) $\displaystyle\int_0^{\infty}\frac{1}{\sqrt[4]{1+x}}\,dx=\lim_{U\to\infty}\int_0^{U}\frac{1}{\sqrt[4]{1+x}}\,dx=\lim_{U\to\infty}\left[\frac{4}{3}\left(\sqrt[4]{1+x}\right)^3+C\right]_0^{U}$

$\displaystyle\int\frac{1}{\sqrt[4]{1+x}}\,dx=\int\frac{1}{\sqrt[4]{p}}\,dp=\int p^{-\frac{1}{4}}\,dp$

$p=1+x$

$dp=dx$

$\displaystyle=\left[\frac{p^{\frac{3}{4}}}{\frac{3}{4}}\right]+C=\frac{4}{3}\left(\sqrt[4]{1+x}\right)^3+C$

$\displaystyle=\lim_{U\to\infty}\left\{\left[\frac{4}{3}\left(\underbrace{\sqrt[4]{1+U}}_{+\infty}\right)^3+C\right]-\left[\frac{4}{3}\left(\sqrt[4]{1+(0)}\right)+C\right]\right\}$

$=+\infty$ Divergent

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14) $\displaystyle\int_{-\infty}^{-3}\frac{x}{4-x^2}\,dx=\lim_{L\to-\infty}\int_L^{-3}\frac{x}{4-x^2}\,dx=\lim_{L\to-\infty}\left[\frac{-1}{2}\ln|4-x^2|+C\right]_L^{-3}$

$\displaystyle\int\frac{x}{4-x^2}\,dx=\int\frac{1}{4-x^2}(x\,dx)=\int\frac{1}{p}\left(\frac{-1}{2}dp\right)$

$p=4-x^2$

$dp=-2x\,dx$

$\frac{-1}{2}dp=dx$

$\displaystyle=\frac{-1}{2}\left[\ln|p|\right]+C$

$\displaystyle=\frac{-1}{2}\ln|4-x^2|+C$

$\displaystyle=\lim_{L\to-\infty}\left\{\left[\frac{-1}{2}\ln|4-(-3)^2|+C\right]-\left[\frac{-1}{2}\ln\underbrace{|4-L^2|}_{+\infty}+C\right]\right\}$

$=+\infty$ Divergent

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16) $\displaystyle\int_2^{\infty}\frac{x}{\sqrt{x^2-1}}\,dx=\lim_{U\to\infty}\int_2^{U}\frac{x}{\sqrt{x^2-1}}\,dx=\lim_{U\to\infty}\left[\sqrt{x^2-1}+C\right]_2^{U}$

$\displaystyle\int\frac{x}{\sqrt{x^2-1}}\,dx=\int\frac{1}{\sqrt{x^2-1}}(x\,dx)=\int\frac{1}{\sqrt{p}}\left(\frac{1}{2}dp\right)$

$p=x^2-1$

$dp=2x\,dx$

$\frac{1}{2}dp=x\,dx$

$\displaystyle=\frac{1}{2}\int p^{-\frac{1}{2}}\,dp=\frac{1}{2}\left[\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right]+C$

$=\sqrt{x^2-1}+C$

$\displaystyle=\lim_{U\to\infty}\left\{\left[\underbrace{\sqrt{U^2-1}}_{+\infty}+C\right]-\left[\sqrt{(2)^2-1}+C\right]\right\}$

$=+\infty$ Divergent

18) $\int_{-\infty}^{-1} \frac{x^2+x}{x^3} dx = \lim_{L\to-\infty} \int_{L}^{-1} \frac{x^2+x}{x^3} dx = \lim_{L\to-\infty} \left[\ln|x| - \frac{1}{x} + C\right]_{L}^{-1}$ 7.8/5

$\int \frac{x^2+x}{x^3} dx = \int \left(\frac{x^2}{x^3} + \frac{x}{x^3}\right) dx$

$= \int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx = \int \left(\frac{1}{x} + x^{-2}\right) dx$

$= [\ln|x|] + \left[\frac{x^{-1}}{-1}\right] + C = \ln|x| - \frac{1}{x} + C$

$= \lim_{L\to-\infty} \left\{ \left[\ln|(-1)| - \frac{1}{(-1)} + C\right] - \left[\underbrace{\ln|L|}_{+\infty} - \underbrace{\frac{1}{L}}_{0} + C\right] \right\}$

$= -\infty$

Divergent

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20) $\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx = \int_{-\infty}^{0} \frac{x}{x^2+1} dx + \int_{0}^{\infty} \frac{x}{x^2+1} dx$

$\int \frac{x}{x^2+1} dx = \int \frac{1}{x^2+1} (x\,dx) = \int \frac{1}{p}\left(\frac{1}{2}dp\right) = \frac{1}{2}[\ln|p|] + C$

$= \frac{1}{2} \ln|x^2+1| + C$

$p = x^2+1 \to dp = 2x\,dx$

$\frac{1}{2}dp = x\,dx$

$\int_{0}^{\infty} \frac{x}{x^2+1} dx = \lim_{U\to\infty} \int_{0}^{U} \frac{x}{x^2+1} dx = \lim_{U\to\infty} \left[\frac{1}{2}\ln|x^2+1| + C\right]_{0}^{U}$

$= \lim_{U\to\infty} \left\{ \left[\frac{1}{2}\underbrace{\ln|U^2+1|}_{+\infty} + C\right] - \left[\frac{1}{2}\ln|(0)^2+1| + C\right] \right\} = +\infty$

Divergent

Since $\int_{0}^{\infty} \frac{x}{x^2+1} dx$ is Divergent, $\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx$ is Divergent

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22) $\int_{1}^{\infty} \frac{e^{\frac{1}{x}}}{x^2} dx = \lim_{U\to\infty} \int_{1}^{U} \frac{e^{\frac{1}{x}}}{x^2} dx = \lim_{U\to\infty} \left[-e^{\frac{1}{x}} + C\right]_{1}^{U}$

$\int \frac{e^{\frac{1}{x}}}{x^2} dx = \int e^{\frac{1}{x}}\left(\frac{1}{x^2}dx\right) = \int e^{p}(-1\,dp)$

$p = \frac{1}{x}$

$dp = \frac{-1}{x^2}dx$

$-1\,dp = \frac{1}{x^2}dx$

$= -1[e^{p}] + C$

$= -e^{\frac{1}{x}} + C$

$= \lim_{U\to\infty} \left\{ \left[-e^{\frac{1}{U}} + C\right] - \left[-e^{\frac{1}{(1)}} + C\right] \right\}$

$= [-e^{(0)}] - [-e^{1}] = -1 + e = \underline{e - 1}$

Convergent

24) $$\int_0^\infty \sin\theta\, e^{\cos\theta}\,d\theta = \lim_{v\to\infty}\int_0^v \sin\theta\, e^{\cos\theta}\,d\theta$$

$$\int \sin\theta\, e^{\cos\theta}\,d\theta = \int e^{\cos\theta}(\sin\theta\, d\theta)$$

$p=\cos\theta$

$dp=-\sin\theta\, d\theta$

$-1\,dp=\sin\theta\, d\theta$

$$= \int e^{p}(-1\,dp)$$
$$= -1[e^{p}]+C$$
$$= -e^{\cos\theta}+C$$

$$= \lim_{v\to\infty}\left[-e^{\cos\theta}+C\right]_0^v$$
$$= \lim_{v\to\infty}\left\{\left[-e^{\cos v}+C\right]-\left[-e^{\cos(0)}+C\right]\right\}$$
$$= \lim_{v\to\infty}\left\{-e^{\cos v}+e^{1}\right\}$$

as $v\to\infty$, $\cos v$ will oscillate between $[-1,1]$ and will make $e^{\cos v}$ to be also oscillating between $[e^{-1}, e^{1}]$. Since $e^{\cos v}$ does not converge to a single value $\int_0^\infty \sin\theta\, e^{\cos\theta}\,d\theta$ is Divergent.

26) $$\int_2^\infty \frac{dv}{v^2+2v-3} = \lim_{v\to\infty}\int_2^v \frac{dv}{v^2+2v-3} = \lim_{v\to\infty}\left[\frac{1}{4}\ln\left(1+\frac{(-4)}{v+3}\right)+C\right]_2^v$$

$$\int \frac{dv}{v^2+2v-3} = \int \frac{1}{(v+3)(v-1)}\,dv = \int\left(\frac{(\ )}{(v+3)^1}+\frac{(\ )}{(v-1)^1}\right)dv$$

$$\frac{1}{(v+3)^1(v-1)^1} = \frac{A}{(v+3)^1}+\frac{B}{(v-1)^1}$$

$1=A(v-1)+B(v+3)$

const term	v-term
$1=-A+3B$	$0=A+B$
$1=B+3B$	$-A=B$
$1=4B$	$A=-B$
$\frac{1}{4}=B$	$A=\frac{-1}{4}$

$$v+3\,\overline{)\,v-1}\quad 1,\quad -(v+3),\quad -4$$

$$= \frac{-1}{4}\left[\ln|v+3|\right]+\frac{1}{4}\left[\ln|v-1|\right]+C$$
$$= \frac{1}{4}\ln|v-1|-\frac{1}{4}\ln|v+3|+C$$
$$= \frac{1}{4}\left(\ln(|v-1|)-\ln(|v+3|)\right)+C$$
$$\frac{1}{4}\ln\frac{(|v-1|)}{(|v+3|)}+C$$
$$= \frac{1}{4}\ln\left(1+\frac{(-4)}{v+3}\right)+C$$

$$= \lim_{v\to\infty}\left\{\left[\frac{1}{4}\ln\left(1-\frac{4}{v+3}\right)+C\right]-\left[\frac{1}{4}\ln\left(1-\frac{4}{(2)+3}\right)+C\right]\right.$$
$$= \left[\frac{1}{4}\ln(1)\right]-\left[\frac{1}{4}\ln\left(1-\frac{4}{5}\right)\right]$$
$$= \left[\frac{1}{4}(0)\right]-\left[\frac{1}{4}\ln\left(\frac{1}{5}\right)\right]$$
$$= -\frac{1}{4}\ln\left(\frac{1}{5}\right) = \frac{1}{4}\ln(5)$$

Convergent

7.8/7

28) $\int_2^{\infty} y e^{-3y}\, dy = \lim_{U\to\infty} \int_2^{U} y e^{-3y}\, dy$

$\int y e^{-3y}\, dy = (y)\left(\frac{-1}{3}e^{-3y}\right) - \int \left(\frac{-1}{3}e^{-3y}\right)(dy)$

$u = y \quad dv = e^{-3y}dy$

$du = dy \quad v = \frac{-1}{3}e^{-3y}$

$= \frac{-1}{3} y e^{-3y} + \frac{1}{3}\int e^{-3y}\, dy$

$= \frac{-1}{3} y e^{-3y} + \frac{1}{3}\left[\frac{-1}{3}e^{-3y}\right] + C$

$= \frac{-1}{3} y e^{-3y} - \frac{1}{9}e^{-3y} + C = \frac{-y}{3e^{3y}} - \frac{1}{9e^{3y}} + C$

$= \lim_{U\to\infty} \left[\frac{-y}{3e^{3y}} - \frac{1}{9e^{3y}} + C\right]_2^U$

$= \lim_{U\to\infty} \left\{\left[\frac{-U}{3e^{3U}} - \frac{1}{9e^{3U}} + C\right] - \left[\frac{-(2)}{3e^{3(2)}} - \frac{1}{9e^{3(2)}} + C\right]\right\}$

$= \left[0 - 0\right] - \left[\frac{-2}{3e^6} - \frac{1}{9e^6}\right] = \left[0\right] - \left[\frac{-6}{9e^6} - \frac{1}{9e^6}\right]$

$= -\left[\frac{-7}{9e^6}\right] = \underline{\underline{\frac{7}{9e^6}}}$ Convergent

$\lim_{U\to\infty} \frac{\overset{-\infty}{-U}}{\underset{+\infty}{3e^{3U}}} \overset{L}{=} \lim_{U\to\infty} \frac{-1}{9e^{3U}} = 0$

7.8/8

30) $\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{U\to\infty}\int_1^U \frac{\ln x}{x^2}\,dx = \lim_{U\to\infty}\left[\frac{-\ln x}{x} - \frac{1}{x} + C\right]_1^U$

$\int \frac{\ln x}{x^2}\,dx = (\ln x)\left(\frac{-1}{x}\right) - \int\left(\frac{-1}{x}\right)\left(\frac{1}{x}\,dx\right) = \frac{-1}{x}\ln x + \int \frac{1}{x^2}\,dx$

$u = \ln x \quad dv = \frac{1}{x^2}dx$

$du = \frac{1}{x}dx \quad v = \frac{-1}{x}$

$= \frac{-\ln x}{x} + \left[\frac{-1}{x}\right] + C = \frac{-\ln x}{x} - \frac{1}{x} + C$

$= \lim_{U\to\infty}\left\{\left[\frac{-\ln U}{U} - \frac{1}{U} + C\right] - \left[\frac{-\ln(1)}{(1)} - \frac{1}{(1)} + C\right]\right\} = [0 - 0] - \left[\frac{-(0)}{1} - 1\right] = \underline{\underline{1}}$

Convergent

$\lim_{U\to\infty}\frac{\overset{-\infty}{-\ln U}}{\underset{+\infty}{U}} \overset{L}{=} \lim_{U\to\infty}\frac{\frac{-1}{U}}{1} = \lim_{U\to\infty}\frac{-1}{U} = 0$

32) $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{U\to\infty}\int_e^U \frac{1}{x(\ln x)^2}\,dx = \lim_{U\to\infty}\left[\frac{-1}{\ln x} + C\right]_e^U$

$\int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{(\ln x)^2}\left(\frac{1}{x}\,dx\right)$

$p = \ln x$

$dp = \frac{1}{x}dx$

$= \int \frac{1}{p^2}\,dp = \int p^{-2}\,dp$

$= \frac{-1}{p} + C = \frac{-1}{\ln x} + C$

$= \lim_{U\to\infty}\left\{\left[\frac{-1}{\ln U} + C\right] - \left[\frac{-1}{\ln(e)} + C\right]\right\}$

$= [0] - \left[\frac{-1}{(1)}\right] = \underline{\underline{1}}$ Convergent

34) $\int_1^\infty \frac{dx}{\sqrt{x} + x\sqrt{x}} = \lim_{U\to\infty}\int_1^U \frac{dx}{\sqrt{x} + x\sqrt{x}} = \lim_{U\to\infty}\left[2\tan^{-1}(\sqrt{x}) + C\right]_1^U$

$\int \frac{dx}{\sqrt{x} + x\sqrt{x}} = \int \frac{1}{\sqrt{x}(1+x)}\,dx = \int \frac{1}{p(1+p^2)}(2p\,dp)$

$p = \sqrt{x} \to p^2 = x$

$2p\,dp = dx$

$= \int \frac{2}{1+p^2}\,dp = \int \frac{2}{(1)^2 + p^2}\,dp$

$= 2\left[\frac{1}{1}\tan^{-1}\left(\frac{p}{1}\right)\right] + C$

$= 2\tan^{-1}(\sqrt{x}) + C$

$= \lim_{U\to\infty}\left\{\left[2\tan^{-1}\sqrt{U} + C\right] - \left[2\tan^{-1}(1) + C\right]\right\}$

$= \left[2\left(\frac{\pi}{2}\right)\right] - \left[2\left(\frac{\pi}{4}\right)\right] = [\pi] - \left[\frac{\pi}{2}\right]$

$= \frac{\pi}{2}$ Convergent

36) $\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,dx = \lim_{U\to 5^-} \int_0^U \frac{1}{\sqrt[3]{5-x}}\,dx = \lim_{U\to 5^-} \left[\frac{-3}{2}\left(\sqrt[3]{5-x}\right)^2 + C\right]_0^U$ | 7.8/9

$\int \frac{1}{\sqrt[3]{5-x}}\,dx = \int \frac{1}{\sqrt[3]{p}}(-1\,dp)$

$p = 5-x$ | $= -1\int p^{-\frac{1}{3}}\,dp$

$dp = -1\,dx$ | $= -1\left[\frac{p^{\frac{2}{3}}}{\frac{2}{3}}\right] + C$

$-1\,dp = dx$ | $= \frac{-3}{2}\left(\sqrt[3]{5-x}\right)^2 + C$

$= \lim_{U\to 5^-} \left\{\left[\frac{-3}{2}\left(\sqrt[3]{5-U}\right)^2 + C\right] - \left[\frac{-3}{2}\left(\sqrt[3]{5-(0)}\right)^2 + C\right]\right\}$

$= \left[\frac{-3}{2}\left(\sqrt[3]{0^+}\right)^2\right] - \left[\frac{-3}{2}\left(\sqrt[3]{5}\right)^2\right] = \underline{\underline{\frac{3}{2}\left(\sqrt[3]{5}\right)^2}}$

Convergent

38) $\int_{-1}^2 \frac{x}{(x+1)^2}\,dx = \lim_{L\to -1^+} \int_L^2 \frac{x}{(x+1)^2}\,dx = \lim_{L\to -1^+} \left[\ln|x+1| + \frac{1}{x+1} + C\right]_L^2$

$\int \frac{x}{(x+1)^2}\,dx = \int \frac{(p-1)}{p^2}\,dp = \int \left(\frac{p}{p^2} - \frac{1}{p^2}\right)dp$

$p = x+1 \to p-1 = x$

$dp = dx$

$= \int \left(\frac{1}{p} - p^{-2}\right)dp$

$= \left[\ln|p|\right] - \left[\frac{-1}{p}\right] + C$

$= \ln|x+1| + \frac{1}{x+1} + C$

$= \lim_{L\to -1^+} \left\{\left[\ln|(2)+1| + \frac{1}{(2)+1} + C\right] - \left[\underbrace{\ln|L+1| + \frac{1}{L+1}}_{+\infty} + C\right]\right\}$

$= -\infty$ Divergent

$\lim_{L\to -1^+} \left(\ln|L+1| + \frac{1}{L+1}\right) = \lim_{L\to -1^+} \left(\ln(L+1)\left(\frac{L+1}{L+1}\right) + \frac{1}{L+1}\right)$

$= \lim_{L\to -1^+} \frac{(L+1)\ln(L+1) + 1}{(L+1)} = \frac{0+1}{0^+} = +\infty$

$\lim_{L\to -1^+} (L+1)\ln(L+1) = \lim_{L\to -1^+} \frac{\overset{-\infty}{\ln(L+1)}}{\underset{+\infty}{\frac{1}{(L+1)}}} \overset{L}{=} \lim_{L\to -1^+} \frac{\frac{1}{L+1}}{\frac{-1}{(L+1)^2}} = \lim_{L\to -1^+} -(L+1) = 0$

40) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{U\to 1^-}\int_0^U \frac{dx}{\sqrt{(1)^2-x^2}} = \lim_{U\to 1^-}\left[\sin^{-1}\left(\frac{x}{1}\right)+C\right]_0^U$

$\displaystyle= \lim_{U\to 1^-}\left\{\left[\sin^{-1}U + C\right] - \left[\sin^{-1}(0)\right]\right\} = \left[\sin^{-1}(1)\right] - \left[0\right] = \underline{\underline{\frac{\pi}{2}}}$

Convergent

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42) $\displaystyle\int_0^5 \frac{w}{w-2}\,dw = \int_0^2 \frac{w}{w-2}\,dw + \int_2^5 \frac{w}{w-2}\,dw$

$\displaystyle\int \frac{w}{w-2}\,dw = \int\left(1+\frac{(+2)}{w-2}\right)dw = [w] + 2\left[\ln|w-2|\right]+C$

$= w + 2\ln|w-2| + C$

$$\begin{array}{r} 1 \\ w-2\,\overline{)\,w-0} \\ -(w-2) \\ \hline +2 \end{array}$$

$\displaystyle\int_0^2 \frac{w}{w-2}\,dw = \lim_{U\to 2^-}\int_0^U \frac{w}{w-2}\,dw = \lim_{U\to 2^-}\left[w+2\ln|w-2|+C\right]_0^U$

$\displaystyle= \lim_{U\to 2^-}\left\{\left[U + \underbrace{2\ln|\underbrace{U-2}_{0^+}|}_{-\infty} + C\right] - \left[(0)+2\ln|(0)-2|+C\right]\right\}$

$= -\infty$ Divergent

Since $\displaystyle\int_0^2 \frac{w}{w-2}\,dw$ is Divergent,

$\displaystyle\int_0^5 \frac{w}{w-2}\,dw$ is Divergent

44) $\displaystyle\int_0^4 \frac{dx}{x^2-x-2} = \int_0^2 \frac{dx}{x^2-x-2} + \int_2^4 \frac{dx}{x^2-x-2}$

$$\int \frac{dx}{x^2-x-2} = \int \frac{1}{(x+1)(x-2)}\,dx = \int \left(\frac{\left(\frac{-1}{3}\right)}{(x+1)^1} + \frac{\left(\frac{1}{3}\right)}{(x-2)^1} \right) dx = \frac{-1}{3}\left[\ln|x+1|\right] + \frac{1}{3}\left[\ln|x-2|\right] + C$$

$$\frac{1}{(x+1)^1(x-2)^1} = \frac{A}{(x+1)^1} + \frac{B}{(x-2)^1}$$

$$1 = A(x-2) + B(x+1)$$

const term	x-term
$1 = -2A + B$	$0 = A + B$
$1 = 2B + B$	$-A = B$
$\frac{1}{3} = B$	$A = \frac{-1}{3}$

$$= \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C$$

$$\int_0^2 \frac{dx}{x^2-x-2} = \lim_{U\to 2^-} \int_0^U \frac{dx}{x^2-x-2} = \lim_{U\to 2^-} \left[\frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C \right]_0^U$$

$$= \lim_{U\to 2^-} \left\{ \left[\frac{1}{3}\underbrace{\ln|U-2|}_{-\infty} - \frac{1}{3}\ln|U+1| + C \right] - \left[\frac{1}{3}\ln|(0)-2| - \frac{1}{3}\ln|(0)+1| + C \right] \right\}$$

$= -\infty$ Divergent

Since $\displaystyle\int_0^2 \frac{dx}{x^2-x-2}$ is Divergent, $\displaystyle\int_0^4 \frac{dx}{x^2-x-2}$ is Divergent

46) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta = \lim_{L\to 0^+} \int_L^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta = \lim_{L\to 0^+} \left[2\sqrt{\sin\theta} + C \right]_L^{\frac{\pi}{2}}$

$$\int \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta = \int \frac{1}{\sqrt{p}}\,dp = \int p^{\frac{-1}{2}}\,dp$$

$$= \left[\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right] + C = 2\sqrt{\sin\theta} + C$$

$p = \sin\theta$

$dp = \cos\theta\, d\theta$

$$= \lim_{L\to 0^+} \left\{ \left[2\sqrt{\sin\left(\tfrac{\pi}{2}\right)} + C \right] - \left[2\sqrt{\sin L} + C \right] \right\}$$

$$= \left[2\sqrt{(1)}\right] - \left[2\sqrt{(0)}\right] = 2 - 0$$

$= 2$

Convergent

48) $\displaystyle\int_0^1 \frac{e^{\frac{1}{x}}}{x^3}\,dx = \lim_{L\to 0^+}\int_L^1 \frac{e^{\frac{1}{x}}}{x^3}\,dx = \lim_{L\to 0^+}\left[\frac{-e^{\frac{1}{x}}}{x}+e^{\frac{1}{x}}+C\right]_L^1$ 7.8/12

$$\int \frac{e^{\frac{1}{x}}}{x^3}\,dx = \int e^{\frac{1}{x}}\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\,dx\right) = \int e^{p}(p)(-1\,dp) = \int -p\,e^{p}\,dp$$

$p = \frac{1}{x}$

$dp = \frac{-1}{x^2}\,dx$

$-1\,dp = \frac{1}{x^2}\,dx$

$u = -p \quad dv = e^{p}\,dp$

$du = -1\,dp \quad v = e^{p}$

$$= (-p)(e^{p}) - \int (e^{p})(-1\,dp)$$

$$= -pe^{p} + \int e^{p}\,dp$$

$$= -pe^{p} + \left[e^{p}\right] + C$$

$$= -\left(\frac{1}{x}\right)\left(e^{\frac{1}{x}}\right) + e^{\frac{1}{x}} + C = \frac{-e^{\frac{1}{x}}}{x} + e^{\frac{1}{x}} + C$$

$$= \lim_{L\to 0^+}\left\{\left[\underbrace{\frac{-e^{\frac{1}{L}}}{L} + e^{\frac{1}{L}}}_{-\infty} + C\right] - \left[\frac{-e^{\frac{1}{(1)}}}{(1)} + e^{\frac{1}{(1)}} + C\right]\right\} = -\infty$$

$$\lim_{L\to 0^+}\left(\frac{-e^{\frac{1}{L}}}{L} + e^{\frac{1}{L}}\right) = \lim_{L\to 0^+}\underbrace{\left(\frac{-1}{L}+1\right)}_{-\infty}\underbrace{e^{\frac{1}{L}}}_{\infty} = -\infty$$

Divergent

58) $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$

for $x \geq 1$, $\displaystyle\frac{1+\sin^2 x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} \Rightarrow \int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx \geq \int_1^\infty \frac{1}{\sqrt{x}}\,dx$

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{U\to\infty}\int_1^U x^{-\frac{1}{2}}\,dx = \lim_{U\to\infty}\left[2\sqrt{x}+C\right]_1^U = \lim_{U\to\infty}\left\{\left[2\sqrt{U}+C\right]-\left[2\sqrt{(1)}+C\right]\right\} = +\infty$$

or by Theorem 2 with $p = \frac{1}{2} \leq 1$ $\quad \displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is Divergent

Since $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx \geq \int_1^\infty \frac{1}{\sqrt{x}}\,dx$ and $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is Divergent,

by the Comparison Theorem, $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$ is Divergent.

7.8/13

60) $\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$

for $x \ge 0$, $\arctan x = \tan^{-1} x < \frac{\pi}{2} < 2$

which makes $\frac{\arctan x}{2+e^x} < \frac{(\frac{\pi}{2})}{2+e^x} < \frac{2}{2+e^x} < \frac{2}{e^x} = 2e^{-x}$

$$\frac{\arctan x}{2+e^x} < 2e^{-x} \Rightarrow \int_0^\infty \frac{\arctan x}{2+e^x}\,dx < \int_0^\infty 2e^{-x}\,dx$$

$$\int_0^\infty 2e^{-x}\,dx = \lim_{U\to\infty}\int_0^U 2e^{-x}\,dx = \lim_{U\to\infty}\left[-2e^{-x}+C\right]_0^U = \lim_{U\to\infty}\left\{\left[-2e^{-U}+C\right]-\left[-2e^{-(0)}+C\right]\right\}$$

$$= \lim_{U\to\infty}\left\{\left[\frac{-2}{e^U}\right]-\left[-2(1)\right]\right\} = [0]-[-2] = 2 \quad \text{Convergent}$$

Since $\int_0^\infty \frac{\arctan x}{2+e^x}\,dx < \int_0^\infty 2e^{-x}\,dx$ and $\int_0^\infty 2e^{-x}\,dx$ is Convergent, by the Comparison Theorem, $\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$ is Convergent.

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62) $\int_1^\infty \frac{2+\cos x}{\sqrt{x^4+x^2}}\,dx$

for $x \ge 1$, $\frac{2+\cos x}{\sqrt{x^4+x^2}} \le \frac{2+1}{\sqrt{x^4+x^2}} < \frac{3}{\sqrt{x^4+x^2}} < \frac{3}{\sqrt{x^4}} = \frac{3}{x^2}$

$$\frac{2+\cos x}{\sqrt{x^4+x^2}} < \frac{3}{x^2} \Rightarrow \int_1^\infty \frac{2+\cos x}{\sqrt{x^4+x^2}}\,dx < \int_1^\infty \frac{3}{x^2}\,dx$$

$$\int_1^\infty \frac{3}{x^2}\,dx = \lim_{U\to\infty}\int_1^U \frac{3}{x^2}\,dx = \lim_{U\to\infty}\left[\frac{-3}{x}+C\right]_1^U = \lim_{U\to\infty}\left\{\left[\frac{-3}{U}+C\right]-\left[\frac{-3}{(1)}+C\right]\right\}$$

$$= [0]-[-3] = 3 \quad \text{Convergent}$$

or by Theorem 2 with $p = 2 > 1$ "$\int_1^\infty \frac{3}{x^2}\,dx = 3\int_1^\infty \frac{1}{x^2}\,dx$"

62) continued...

Since $\int_1^\infty \frac{2+\cos x}{\sqrt{x^4+x^2}}\,dx < \int_1^\infty \frac{3}{x^2}\,dx$ and $\int_1^\infty \frac{3}{x^2}\,dx$ is Convergent, by the Comparison Theorem, $\int_1^\infty \frac{2+\cos x}{\sqrt{x^4+x^2}}\,dx$ is Convergent.

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64) $\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$

for $0 < x \leq \pi$, $\frac{\sin^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \Rightarrow \int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx \leq \int_0^\pi \frac{1}{\sqrt{x}}\,dx$

$$\int_0^\pi \frac{1}{\sqrt{x}}\,dx = \lim_{L\to 0^+} \int_L^\pi \frac{1}{\sqrt{x}}\,dx = \lim_{L\to 0^+} \left[2\sqrt{x} + C\right]_L^\pi$$

$$= \lim_{L\to 0^+} \left\{\left[2\sqrt{(\pi)} + C\right] - \left[2\sqrt{L} + C\right]\right\} = \left[2\sqrt{\pi}\right] - \left[2(0)\right] = 2\sqrt{\pi}$$

Convergent

Since $\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx < \int_0^\pi \frac{1}{\sqrt{x}}\,dx$ and $\int_0^\pi \frac{1}{\sqrt{x}}\,dx$ is Convergent, by the Comparison Theorem, $\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$ is Convergent.