21200 {11} section 7.4

2-a) $$\frac{x-6}{x^2+x-6} = \frac{x-6}{(x+3)^1(x-2)^1} = \frac{A}{(x+3)^1} + \frac{B}{(x-2)^1}$$

2-b) $$\frac{1}{x^2+x^4} = \frac{1}{x^2(1+x^2)} = \frac{1}{(x)^2(x^2+1)^1} = \frac{A}{(x)^1} + \frac{B}{(x)^2} + \frac{(Cx+D)}{(x^2+1)^1}$$

4-a) $$\frac{5}{x^4-1} = \frac{5}{(x^2+1)(x^2-1)} = \frac{5}{(x^2+1)^1(x+1)^1(x-1)^1} = \frac{A}{(x+1)^1} + \frac{B}{(x-1)^1} + \frac{(Cx+D)}{(x^2+1)^1}$$

4-b) $$\frac{x^4+x+1}{(x^3-1)(x^2-1)} = \frac{x^4+x+1}{(x-1)(x^2+x+1)(x+1)(x-1)} = \frac{x^4+x+1}{(x+1)^1(x-1)^2(x^2+x+1)^1}$$

$$= \frac{A}{(x+1)^1} + \frac{B}{(x-1)^1} + \frac{C}{(x-1)^2} + \frac{(Dx+E)}{(x^2+x+1)^1}$$

6-a) improper fraction (perform long division 1st)

$$\begin{array}{r l}
 & x^4 + 4x^2 + 16 \\
x^2+0x-4 \,\big) & x^6+0x^5+0x^4+0x^3+0x^2+0x+0 \\
 & -(x^6+0x^5-4x^4) \\
\hline
 & +4x^4+0x^3+0x^2 \\
 & -(+4x^4+0x^3-16x^2) \\
\hline
 & +16x^2+0x+0 \\
 & -(+16x^2+0x-64) \\
\hline
 & +64
\end{array}$$

6-a) continued...

$$\frac{x^6}{x^2-4} = x^4+4x^2+16+\frac{(+64)}{x^2-4}$$

$$= x^4+4x^2+16+\frac{64}{(x+2)^1(x-2)^1}$$

$$= x^4+4x^2+16+\frac{A}{(x+2)^1}+\frac{B}{(x-2)^1}$$

$$6\text{-b)}\ \frac{x^4}{(x^2-x+1)(x^2+2)^2} = \frac{x^4}{(x^2+2)^2(x^2-x+1)^1} = \frac{(Ax+B)}{(x^2+2)^1}+\frac{(Cx+D)}{(x^2+2)^2}+\frac{(Ex+F)}{(x^2-x+1)^1}$$

$$8)\ \frac{x-12}{x^2-4x} = \frac{x-12}{(x)^1(x-4)^1} = \frac{A}{(x)^1}+\frac{B}{(x-4)^1}$$

$$x-12 = A(x-4)+B(x)$$

const term	x-term
$-12=-4A$	$1=A+B$
$3=A$	$1=(3)+B$
	$-2=B$

$$\int\frac{x-12}{x^2-4x}\,dx = \int\left(\frac{(3)}{(x)^1}+\frac{(-2)}{(x-4)}\right)dx = \int\left(\frac{3}{x}-\frac{2}{x-4}\right)dx$$

$$= 3\left[\ln|x|\right]-2\left[\ln|x-4|\right]+C$$

$$\underline{\underline{= 3\ln|x|-2\ln|x-4|+C}}$$

10) $$\frac{y}{(y+4)'(2y-1)'} = \frac{A}{(y+4)'} + \frac{B}{(2y-1)'}$$

$$y = A(2y-1) + B(y+4)$$

constant term	y-term
$0 = -A + 4B$	$1 = 2A + B$
$A = 4B$	$1 = 2(4B) + B$
	$1 = 8B + B$
$A = 4\left(\frac{1}{9}\right) = \frac{4}{9}$	$1 = 9B$
	$\frac{1}{9} = B$

$$\int \frac{y}{(y+4)(2y-1)}\,dy = \int \left(\frac{\left(\frac{4}{9}\right)}{(y+4)'} + \frac{\left(\frac{1}{9}\right)}{(2y-1)'} \right) dy$$

$\downarrow$

$u = 2y-1$

$$= \frac{4}{9}\left[\ln|y+4|\right] + \frac{1}{9}\left[\frac{1}{2}\ln|2y-1|\right] + C$$

$$= \frac{4}{9}\ln|y+4| + \frac{1}{18}\ln|2y-1| + C$$

12) $\int_0^1 \frac{x-4}{x^2-5x+6}\,dx$

$$\frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-2)^1(x-3)^1} = \frac{A}{(x-2)^1} + \frac{B}{(x-3)^1}$$

$$x-4 = A(x-3) + B(x-2)$$

const term

$-4 = -3A - 2B$

x-term

$1 = A + B$

$-4 = -3A - 2B \rightarrow -4 = -3A - 2B$

$1 = A + B \xrightarrow{2} 2 = 2A + 2B$

$-2 = -A$

$2 = A$

$1 = (2) + B$

$-1 = B$

$$\int \frac{x-4}{x^2-5x+6}\,dx = \int \left(\frac{(2)}{(x-2)^1} + \frac{(-1)}{(x-3)^1} \right) dx$$

$$= 2\left[\ln|x-2|\right] - 1\left[\ln|x-3|\right] + C$$

$$= 2\ln|x-2| - \ln|x-3| + C$$

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,dx = \left[2\ln|x-2| - \ln|x-3| + C\right]_0^1$$

$$= \left[2\ln|(1)-2| - \ln|(1)-3| + C\right] - \left[2\ln|(0)-2| - \ln|(0)-3| + C\right]$$

$$= \left[2\ln|-1| - \ln|-2|\right] - \left[2\ln|-2| - \ln|-3|\right]$$

$$= \left[2\ln(1) - \ln(2)\right] - \left[2\ln(2) - \ln(3)\right]$$

$$= \left[2(0) - \ln(2)\right] - \left[2\ln(2) - \ln(3)\right]$$

$$= -\ln(2) - 2\ln(2) + \ln 3 = \underline{\underline{\ln 3 - 3\ln 2}}$$

14) $$\frac{1}{(x+a)'(x+b)'} = \frac{A}{(x+a)'} + \frac{B}{(x+b)'}$$

$$1 = A(x+b) + B(x+a)$$

const term

$$1 = bA + ab$$

x - term

$$0 = A + B$$

$$B = -A$$

$$1 = bA + a(-A)$$

$$1 = A(b-a)$$

$$\frac{1}{b-a} = A$$

$$B = -A = -\left(\frac{1}{b-a}\right) = \frac{-1}{b-a} = \frac{1}{a-b}$$

$$\int \frac{1}{(x+a)(x+b)}\,dx = \int\left(\frac{\left(\frac{1}{b-a}\right)}{(x+a)'} + \frac{\left(\frac{1}{a-b}\right)}{(x+b)'}\right)dx$$

$$= \frac{1}{(b-a)}\Big[\ln|x+a|\Big] + \frac{1}{(a-b)}\Big[\ln|x+b|\Big] + C$$

$$= \frac{1}{b-a}\ln|x+a| + \frac{1}{a-b}\ln|x+b| + C$$

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16) improper, must divide 1st

$$\begin{array}{r} 3 \\ t+1\,\overline{)\,3t-2} \\ -(3t+3) \\ \hline -5 \end{array}$$

$$\int \frac{3t-2}{t+1}\,dt = \int\left(3 + \frac{(-5)}{t+1}\right)dt$$

$$= 3\big[t\big] - 5\big[\ln|t+1|\big] + C$$

$$= 3t - 5\ln|t+1| + C$$

7.4/6

18) $\int_1^2 \frac{3x^2+6x+2}{x^3+3x+2}\,dx$

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improper, must divide 1st

$$\begin{array}{r} 3 \\ x^2+3x+2 \,\overline{)\, 3x^2+6x+2} \\ -(3x^2+9x+6) \\ \hline -3x-4 \end{array}$$

$$\frac{3x^2+6x+2}{x^2+3x+2} = 3 + \frac{(-3x-4)}{x^2+3x+2}$$

$$= 3 + \frac{(-3x-4)}{(x+2)'(x+1)'}$$

$$\frac{(-3x+4)}{(x+2)'(x+1)'} = \frac{A}{(x+2)'} + \frac{B}{(x+1)'}$$

$$-3x+4 = A(x+1) + B(x+2)$$

const term

$4 = A + 2B$

x - term

$-3 = A + B$

$4 = A + 2B \rightarrow 4 = A + 2B$

$-3 = A + B \xrightarrow{-1} 3 = -A - B$

$7 = B$

$-3 = A + (7)$

$-10 = A$

$$\int \frac{3x^2+6x+2}{x^2+3x+2}\,dx = \int \left(3 + \frac{(-10)}{(x+2)'} + \frac{(7)}{(x+1)'}\right) dx$$

$$= 3[x] - 10[\ln|x+2|] + 7[\ln|x+1|] + C$$

$$= 3x - 10\ln|x+2| + 7\ln|x+1| + C$$

$$\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}\,dx = \Big[3x - 10\ln|x+2| + 7\ln|x+1| + C\Big]_1^2$$

$$= \Big[3(2) - 10\ln|(3)+2| + 7\ln|(2)+1| + C\Big] - \Big[3(1) - 10\ln|(1)+2| + 7\ln|(1)+1| + C\Big]$$

$$= \Big[6 - 10\ln|5| + 7\ln|3|\Big] - \Big[3 - 10\ln|3| + 7\ln|2|\Big]$$

$$= \underline{\underline{3 - 10\ln|5| + 17\ln|3| - 7\ln|2|}}$$

20) $\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}dx$

$$\frac{x(3-5x)}{(3x-1)(x-1)^2} = \frac{3x-5x^2}{(3x-1)^1(x-1)^2} = \frac{A}{(3x-1)^1} + \frac{B}{(x-1)^1} + \frac{C}{(x-1)^2}$$

$$3x-5x^2 = A(x-1)^2 + B(3x-1)(x-1) + C(3x-1)$$

$$3x-5x^2 = A(x^2-2x+1) + B(3x^2-4x+1) + C(3x-1)$$

const term	x-term	x^2-term
$0 = A + B - C$	$3 = -2A - 4B + 3C$	$0 = A + 3B$
$0 = (-3B) + B - C$	$3 = -2(-3B) - 4B + 3(-2B)$	$A = -3B$
$C = -2B$	$3 = 6B - 4B - 6B$	$A = -3\left(\frac{-3}{4}\right) = \frac{9}{4}$
$C = -2\left(\frac{-3}{4}\right) = \frac{3}{2}$	$3 = -4B$	
	$\frac{-3}{4} = B$	

$$\int \frac{x(3-5x)}{(3x-1)(x-1)^2}dx = \int\left(\frac{\left(\frac{9}{4}\right)}{\underset{\substack{\downarrow \\ u=3x-1}}{(3x-1)^1}} + \frac{\left(\frac{-3}{4}\right)}{(x-1)^1} + \frac{\left(\frac{3}{2}\right)}{\underset{\substack{\downarrow \\ u=x-1}}{(x-1)^2}}\right)dx$$

$$= \frac{9}{4}\left[\frac{1}{3}\ln|3x-1|\right] - \frac{3}{4}\left[\ln|x-1|\right] + \frac{3}{2}\left[\frac{(x-1)^{-1}}{-1}\right] + C$$

$$= \frac{3}{4}\ln|3x-1| - \frac{3}{4}\ln|x-1| - \frac{3}{2(x-1)} + C$$

$$\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}dx = \left[\frac{3}{4}\ln|3x-1| - \frac{3}{4}\ln|x-1| - \frac{3}{2(x-1)} + C\right]_2^3$$

$$= \left[\frac{3}{4}\ln|3(3)-1| - \frac{3}{4}\ln|(3)-1| - \frac{3}{2((3)-1)} + C\right] - \left[\frac{3}{4}\ln|3(2)-1| - \frac{3}{4}\ln|(2)-1| - \frac{3}{2((2)-1)} + C\right]$$

20) continued...

$$=\left[\frac{3}{4}\ln|8|-\frac{3}{4}\ln|2|-\frac{3}{4}\right]-\left[\frac{3}{4}\ln|5|-\frac{3}{4}\ln|1|-\frac{3}{2}\right]$$

$$=\left[\frac{3}{4}\ln(8)-\frac{3}{4}\ln(2)-\frac{3}{4}\right]-\left[\frac{3}{4}\ln(5)-\frac{3}{4}(0)-\frac{3}{2}\right]=\underline{\underline{\frac{3}{4}\ln(8)-\frac{3}{4}\ln(2)-\frac{3}{4}\ln(5)+\frac{3}{4}}}$$

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22) $$\frac{3x^2+12x-20}{x^4-8x^2+16}=\frac{3x^2+12x-20}{(x^2-4)(x^2-4)}=\frac{3x^2+12x-20}{(x+2)(x-2)(x+2)(x-2)}$$

$$=\frac{3x^2+12x-20}{(x+2)^2(x-2)^2}=\frac{A}{(x+2)^1}+\frac{B}{(x+2)^2}+\frac{C}{(x-2)^1}+\frac{D}{(x-2)^2}$$

$$3x^2+12x-20=A(x+2)(x-2)^2+B(x-2)^2+C(x+2)^2(x-2)+D(x+2)^2$$

$$3x^2+12x-20=A(x^3-2x^2-4x+8)+B(x^2-4x+4)+C(x^3+2x^2-4x-8)+D(x^2+4x+4)$$

const term

$-20=8A+4B-8C+4D$

$-20=8A+4B-8(-A)+4D$

$-20=16A+4B+4D$

$-5=4A+B+D$

$-5=4A+B+(B+3)$

$-8=4A+2B$ ☆

$-8=4(-1)+2B$

$-4=2B$

$\underline{-2=B}$

x-term

$12=-4A-4B-4C+4D$

$12=-4A-4B-4(-A)+4D$

$12=-4B+4D$

$3=-B+D$

$D=B+3$

$D=(-2)+3=1$

$\underline{D=1}$

x^2-term

$3=-2A+B+2C+D$

$3=-2A+B+2(-A)+D$

$3=-4A+B+D$

$3=-4A+B+(B+3)$

$0=-4A+2B$ ☆

mult. by -1

$$\begin{array}{r} -8=4A+2B \\ 0=4A-2B \\ \hline -8=8A \\ \underline{-1=A} \end{array}$$

x^3-term

$0=A+C$

$A=-C$

$C=-A$

$C=-(-1)$

$\underline{C=1}$

22) continued...

$$\int \frac{3x^2+12x-20}{x^4-8x^2+16}\,dx = \int \left(\frac{(-1)}{(x+2)^1} + \frac{(-2)}{(x+2)^2} + \frac{(1)}{(x-2)^1} + \frac{(1)}{(x-2)^2}\right)dx$$

$$= -1\left[\ln|x+2|\right] - 2\left[\frac{(x+2)^{-1}}{-1}\right] + \left[\ln|x-2|\right] + \left[\frac{(x-2)^{-1}}{-1}\right] + C$$

$$= -\ln|x+2| + \frac{1}{2(x+2)} + \ln|x-2| - \frac{1}{(x-2)} + C$$

$$= \frac{1}{2(x+2)} - \frac{1}{x-2} + \ln|x-2| - \ln|x+2| + C$$

24) $$\frac{3x^2-x+8}{x^3+4x} = \frac{3x^2-x+8}{(x)^1(x^2+4)^1} = \frac{A}{(x)^1} + \frac{(Bx+C)}{(x^2+4)^1}$$

$$3x^2-x+8 = A(x^2+4) + (Bx+C)(x)$$

$$3x^2-x+8 = A(x^2+4) + B(x^2) + C(x)$$

const term

$8 = 4A$

$2 = A$

x-term

$-1 = C$

x^2-term

$3 = A+B$

$3 = (2)+B$

$1 = B$

$$\int \frac{3x^2-x+8}{x^3+4x}\,dx = \int\left(\frac{(2)}{(x)^1} + \frac{(1)x+(-1)}{(x^2+4)^1}\right)dx = \int\left(\frac{2}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\right)dx$$

$$= 2\left[\ln|x|\right] + \left[\frac{1}{2}\ln|x^2+4|\right] - \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right] + C$$

$$= 2\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

26) $$\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx$$

improper, must divide first

$$\begin{array}{r|l} & 1 \\ x^3+x^2+0x+0 & x^3+4x^2+x-1 \\ & -(x^3+x^2+0x+0) \\ \hline & +3x^2+x-1 \end{array}$$

$$\frac{x^3+4x^2+x-1}{x^3+x^2} = 1+\frac{(+3x^2+x-1)}{x^3+x^2}$$

$$\frac{3x^2+x-1}{x^3+x^2} = \frac{3x^2+x-1}{(x)^2(x+1)^1} = \frac{A}{(x)^1}+\frac{B}{(x)^2}+\frac{C}{(x+1)^1}$$

$$3x^2+x-1 = A(x)(x+1)+B(x+1)+C(x)^2$$

$$3x^2+x-1 = A(x^2+x)+B(x+1)+C(x^2)$$

constant term	x-term	x^2-term
$\underline{-1=B}$	$1=A+B$	$3=A+C$
	$1=A+(-1)$	$3=(2)+C$
	$\underline{2=A}$	$\underline{1=C}$

$$\int \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx = \int\left(1+\frac{(2)}{(x)^1}+\frac{(-1)}{(x)^2}+\frac{(1)}{(x+1)^1}\right)dx$$

$$=[x]+2[\ln|x|]-1\left[\frac{x^{-1}}{-1}\right]+[\ln|x+1|]+C$$

$$=x+2\ln|x|+\frac{1}{x}+\ln|x+1|+C$$

$$\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx = \left[x+2\ln|x|+\frac{1}{x}+\ln|x+1|+C\right]_1^2$$

$$=\left[(2)+2\ln|(2)|+\frac{1}{(2)}+\ln|(2)+1|+C\right]-\left[(1)+2\ln|(1)|+\frac{1}{(1)}+\ln|(1)+1|+C\right]$$

$$=\left[2+2\ln(2)+\frac{1}{2}+\ln(3)\right]-\left[1+2(0)+1+\ln(2)\right]=\underline{\underline{\frac{1}{2}+\ln(2)+\ln(3)}}$$

28) $\frac{x^2+x+1}{(x^2+1)^2} = \frac{(Ax+B)}{(x^2+1)^1} + \frac{(Cx+D)}{(x^2+1)^2}$

$x^2+x+1 = (Ax+B)(x^2+1) + (Cx+D)$

$x^2+x+1 = A(x^3+x) + B(x^2+1) + C(x) + D$

const. term	x-term	x^2-term	x^3-term
$1 = B+D$	$1 = A+C$	$\underline{1 = B}$	$\underline{0 = A}$
$1 = (1)+D$	$1 = (0)+C$		
$\underline{0 = D}$	$\underline{1 = C}$		

$$\int \frac{x^2+x+1}{(x^2+1)^2}\,dx = \int \left(\frac{(0)+(1)}{(x^2+1)^1} + \frac{(1)x+(0)}{(x^2+1)^2}\right)dx$$

$$= \int \left(\frac{1}{x^2+(1)^2}\right)dx + \int \left(\frac{x}{(x^2+1)^2}\right)dx$$

$$= \left[\frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right)\right] + \left[\frac{-1}{2(x^2+1)}\right] + C$$

$$\underline{\underline{= \tan^{-1}(x) - \frac{1}{2(x^2+1)} + C}}$$

$$\int \frac{x}{(x^2+1)^2}\,dx = \int \frac{1}{(x^2+1)^2}(x\,dx) = \int \frac{1}{p^2}\left(\frac{1}{2}\,dp\right) = \frac{1}{2}\int p^{-2}\,dp =$$

$p = x^2+1$

$dp = 2x\,dx$

$\frac{1}{2}\,dp = x\,dx$

$$= \frac{1}{2}\left[\frac{p^{-1}}{-1}\right] + C_* = \underbrace{\frac{-1}{2(x^2+1)}} + C_*$$

30) $\frac{x^3+6x-2}{x^4+6x^2} = \frac{x^3+6x-2}{(x)^2(x^2+6)^1} = \frac{A}{(x)^1}+\frac{B}{(x)^2}+\frac{(Cx+D)}{(x^2+6)^1}$

$x^3+6x-2 = A(x)(x^2+6)+B(x^2+6)+(Cx+D)(x^2)$

$x^3+6x-2 = A(x^3+6x)+B(x^2+6)+C(x^3)+D(x^2)$

const term	x-term	x^2-term	x^3-term
$-2=6B$	$6=6A$	$0=B+D$	$1=A+C$
$\underline{\frac{-1}{3}=B}$	$\underline{1=A}$	$0=(\frac{-1}{3})+D$	$1=(1)+C$
		$\underline{\frac{1}{3}=D}$	$\underline{0=C}$

$$\int \frac{x^3+6x-2}{x^4+6x^2}\,dx = \int\left(\frac{(1)}{(x)^1}+\frac{(\frac{-1}{3})}{(x)^2}+\frac{(0)x+(\frac{1}{3})}{(x^2+6)^1}\right)dx$$

$$=\int\left(\frac{1}{x}-\frac{1}{3}\left(\frac{1}{x^2}\right)+\frac{(\frac{1}{3})}{x^2+(\sqrt{6})^2}\right)dx$$

$$=\left[\ln|x|\right]-\frac{1}{3}\left[\frac{x^{-1}}{-1}\right]+\frac{1}{3}\left[\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right)\right]+C$$

$$=\underline{\underline{\ln|x|+\frac{1}{3x}+\frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right)+C}}$$

32) $\int_0^1 \frac{x}{x^2+4x+13}\,dx$

7.4/13

$$\int \frac{x}{x^2+4x+13}\,dx = \int \frac{x+2-2}{x^2+4x+13}\,dx$$

we need to insert constant for substitution and complete square.

$p = x^2+4x+13$

$dp = 2x+4\,dx$

$dp = 2(x+2)dx$

$\frac{1}{2}dp = (x+2)dx$

$$= \int\left(\frac{x+2}{x^2+4x+13} - \frac{2}{x^2+4x+13}\right)dx$$

$$= \int \frac{1}{x^2+4x+13}(x+2)dx - 2\int \frac{1}{x^2+4x+13}\,dx$$

complete square

$$= \int \frac{1}{p}\left(\frac{1}{2}dp\right) - 2\int \frac{1}{x^2+4x+4+9}\,dx$$

$$= \frac{1}{2}\left[\ln|p|\right] - 2\int \frac{1}{(x+2)^2+(3)^2}\,dx$$

$$= \frac{1}{2}\ln|x^2+4x+13| - 2\left[\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)\right] + C$$

$$= \frac{1}{2}\ln|x^2+4x+13| - \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C$$

$$\int_0^1 \frac{x}{x^2+4x+13}\,dx = \left[\frac{1}{2}\ln|x^2+4x+13| - \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C\right]_0^1$$

$$= \left[\frac{1}{2}\ln|(1)^2+4(1)+13| - \frac{2}{3}\tan^{-1}\left(\frac{(1)+2}{3}\right) + C\right] - \left[\frac{1}{2}\ln|(0)^2+4(0)+13| - \frac{2}{3}\tan^{-1}\left(\frac{(0)+2}{3}\right) + C\right]$$

$$= \left[\frac{1}{2}\ln|18| - \frac{2}{3}\tan^{-1}(1)\right] - \left[\frac{1}{2}\ln|13| - \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)\right]$$

$$= \left[\frac{1}{2}\ln(18) - \frac{2}{3}\left(\frac{\pi}{4}\right)\right] - \left[\frac{1}{2}\ln(13) - \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)\right]$$

$$\underline{\underline{= \frac{1}{2}\ln(18) - \frac{1}{2}\ln(13) - \frac{\pi}{6} + \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)}}$$

34) $$\frac{x^3-2x^2+2x-5}{x^4+4x^2+3} = \frac{x^3-2x^2+2x-5}{(x^2+3)^1(x^2+1)^1} = \frac{(Ax+B)}{(x^2+3)^1} + \frac{(Cx+D)}{(x^2+1)^1}$$

$$x^3-2x^2+2x-5 = (Ax+B)(x^2+1) + (Cx+D)(x^2+3)$$

$$x^3-2x^2+2x-5 = A(x^3+x) + B(x^2+1) + C(x^3+3x) + D(x^2+3)$$

const. term	x-term	x^2-term	x^3-term
$-5 = B+3D$	$2 = A+3C$	$-2 = B+D$	$1 = A+C$
$-5 = (-D-2)+3D$	$2 = (1-C)+3C$	$B = (-D-2)$	$A = (1-c)$
$-3 = 2D$	$1 = 2C$	$B = -(\frac{-3}{2})-2$	$A = 1-(\frac{1}{2})$
$-\frac{3}{2} = D$	$\frac{1}{2} = C$	$B = \frac{-1}{2}$	$A = \frac{1}{2}$

$$\int \frac{x^3-2x^2+2x-5}{x^4+4x+3}\,dx = \int\left(\frac{(\frac{1}{2})x+(\frac{-1}{2})}{(x^2+3)^1} + \frac{(\frac{1}{2})x+(\frac{-3}{2})}{(x^2+1)^1}\right)dx$$

$$= \frac{1}{2}\int\frac{x}{x^2+3}\,dx - \frac{1}{2}\int\frac{1}{x^2+(\sqrt{3})^2}\,dx + \frac{1}{2}\int\frac{x}{x^2+1}\,dx - \frac{3}{2}\int\frac{1}{x^2+(1)^2}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\ln|x^2+3|\right] - \frac{1}{2}\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right] + \frac{1}{2}\left[\frac{1}{2}\ln|x^2+1|\right] - \frac{3}{2}\left[\frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right)\right] + C$$

$$= \frac{1}{4}\ln|x^2+3| - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{1}{4}\ln|x^2+1| - \frac{3}{2}\tan^{-1}(x) + C$$

36) improper, must divide first

$$\begin{array}{r} x^2 \\ x^3+0x^2+0x+1 \overline{)\, x^5+0x^4+0x^3+0x^2+x-1} \\ -(x^5+0x^4+0x+x^2) \\ \hline -x^2+x-1 \end{array}$$

$$\frac{x^5+x-1}{x^3+1} = x^2+\frac{(-x^2+x-1)}{x^3+1}$$

$$\frac{-x^2+x-1}{x^3+1} = \frac{-x^2+x-1}{(x+1)^1(x^2-x+1)^1} = \frac{A}{(x+1)^1} + \frac{(Bx+C)}{(x^2-x+1)^1}$$

$$-x^2+x-1 = A(x^2-x+1)+(Bx+C)(x+1)$$

$$-x^2+x-1 = A(x^2-x+1)+B(x^2+x)+C(x+1)$$

const. term	x-term	x^2-term
$-1=A+C$	$1=-A+B+C$	$-1=A+B$
$C=(-A-1)$	$1=-A+(-A-1)+C$	$B=(-A-1)$
$C=-(-1)-1$	$2=-2A+C$	$B=(-(-1)-1)$
$\underline{C=0}$	$2=-2A+(-A-1)$	$\underline{B=0}$
	$3=-3A$	
	$\underline{-1=A}$	

$$\int \frac{x^5+x-1}{x^3+1}\,dx = \int\left(x^2+\frac{(-1)}{(x)^1}+\frac{(0)x+(0)}{(x^2-x+1)^1}\right)dx$$

$$=\int\left(x^2-\frac{1}{x}\right)dx = \left[\frac{x^3}{3}\right]-\left[\ln|x|\right]+C$$

$$=\frac{1}{3}x^3-\ln|x|+C$$

38) $\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}\,dx = \int \frac{1}{x^5+5x^3+5x}(x^4+3x^2+1)\,dx$

$p = x^5+5x^3+5x$

$dp = 5x^4+15x^2+5\,dx$

$dp = 5(x^4+3x^2+1)\,dx$

$\frac{1}{5}dp = (x^4+3x^2+1)\,dx$

$= \int \frac{1}{p}\left(\frac{1}{5}dp\right) = \frac{1}{5}\left[\ln|p|\right]+C$

$= \frac{1}{5}\ln|x^5+5x^3+5x|+C$

---

40) $\frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2} = \frac{(Ax+B)}{(x^2+2x+2)^1} + \frac{(Cx+D)}{(x^2+2x+2)^2}$

$x^3+2x^2+3x-2 = (Ax+B)(x^2+2x+2) + (Cx+D)$

$x^3+2x^2+3x-2 = A(x^3+2x^2+2x)+B(x^2+2x+2)+C(x)+D$

const. term	x-term	x^2-term	x^3-term
$-2 = 2B+D$	$3=2A+2B+C$	$2=2A+B$	$1=A$
$-2 = 2(0)+D$	$3=2(1)+2(0)+C$	$2=2(1)+B$	
$-2=D$	$1=C$	$0=B$	

$\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx = \int\left(\frac{(1)x+(0)}{(x^2+2x+2)^1} + \frac{(1)x+(-2)}{(x^2+2x+2)^2}\right)dx$

$= \int\left(\frac{x}{x^2+2x+2} + \frac{x-2}{(x^2+2x+2)^2}\right)dx$

we need to fix constants for both fractions

$p = x^2+2x+2$

$dp = 2x+2\,dx$

$dp = 2(x+1)\,dx$

$\frac{1}{2}dp = (x+1)\,dx$

$= \int\left(\frac{x+1-1}{x^2+2x+2} + \frac{x+1-1-2}{(x^2+2x+2)^2}\right)dx$

40) continued...

$$= \int \left(\frac{x+1}{x^2+2x+2} - \frac{1}{x^2+2x+2} + \frac{x+1}{(x^2+2x+2)^2} - \frac{3}{(x^2+2x+2)^2} \right) dx$$

complete square

$$= \int \frac{(x+1)}{x^2+2x+2}\,dx - \int \frac{1}{(x^2+2x+1)+1}\,dx + \int \frac{(x+1)}{(x^2+2x+2)^2}\,dx - \int \frac{3}{((x^2+2x+1)+1)^2}\,dx$$

$$= \begin{array}{cccc} \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ I_1 & I_2 & I_3 & I_4 \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow \end{array}$$

$$= \int \frac{1}{x^2+2x+2}(x+1)\,dx - \int \frac{1}{(x+1)^2+(1)^2}\,dx + \int \frac{1}{(x^2+2x+2)^2}(x+1)\,dx - \int \frac{3}{((x+1)^2+(1)^2)^2}\,dx$$

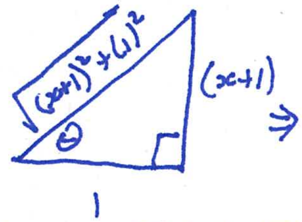
$$= \left[\frac{1}{2}\ln|x^2+2x+2|\right] - \left[\frac{1}{1}\tan^{-1}\left(\frac{x+1}{1}\right)\right] + \left[\frac{1}{2}\left(\frac{(x^2+2x+2)^{-1}}{-1}\right)\right] - \int \frac{3}{\left(\sqrt{(x+1)^2+(1)^2}\right)^4}\,dx$$

$$= \frac{1}{2}\ln|x^2+2x+2| - \tan^{-1}(x+1) - \frac{1}{2(x^2+2x+2)} - \left[\frac{3}{2}\tan^{-1}(x+1) + \frac{3(x+1)}{2(x^2+2x+2)}\right] + C$$

$$= \frac{1}{2}\ln|x^2+2x+2| - \tan^{-1}(x+1) - \frac{1}{2(x^2+2x+2)} - \frac{3}{2}\tan^{-1}(x+1) - \frac{3x}{2(x^2+2x+2)} - \frac{3}{2(x^2+2x+2)} + C$$

$$= \frac{1}{2}\ln|x^2+2x+2| - \frac{5}{2}\tan^{-1}(x+1) - \frac{3x}{2(x^2+2x+2)} - \frac{4}{2(x^2+2x+2)} + C$$

$$I_4 = \int \frac{3}{\left(\sqrt{(x+1)^2+(1)^2}\right)^4}\,dx = \int \frac{3}{(\sec\theta)^4}(\sec^2\theta\,d\theta) = \int \frac{3}{\sec^2\theta}\,d\theta = 3\int \cos^2\theta\,d\theta$$



$\Rightarrow$

$\frac{x+1}{1} = \tan\theta \rightarrow x = -1 + \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\frac{\sqrt{(x+1)^2+(1)^2}}{1} = \sec\theta \rightarrow \sqrt{(x+1)^2+(1)^2} = \sec\theta$

$$= 3\int \left(\frac{1+\cos(2\theta)}{2}\right) d\theta$$

$$= \frac{3}{2}\int (1+\cos(2\theta))\,d\theta$$

$$= \frac{3}{2}\left\{[\theta] + \left[\frac{1}{2}\sin(2\theta)\right]\right\} + C_*$$

$$= \frac{3}{2}\{\theta + \sin\theta\cos\theta\} + C_* = \frac{3}{2}\left\{\left(\tan^{-1}\left(\frac{x+1}{1}\right)\right) + \left(\frac{x+1}{\sqrt{(x+1)^2+(1)^2}}\right)\left(\frac{1}{\sqrt{(x+1)^2+(1)^2}}\right)\right\} + C_*$$

$$= \frac{3}{2}\left\{\tan^{-1}(x+1) + \frac{x+1}{(x+1)^2+(1)^2}\right\} + C_* = \underbrace{\frac{3}{2}\tan^{-1}(x+1) + \frac{3(x+1)}{2(x^2+2x+2)}}_{} + C_*$$

42) $\displaystyle\int \frac{dx}{2\sqrt{x+3}+x} = \int \frac{(2p\,dp)}{2p+(p^2-3)} = \int \frac{2p}{p^2+2p-3}\,dp$

$p = \sqrt{x+3}$

$p^2 = x+3$

$p^2-3 = x$

$2p\,dp = dx$

$\displaystyle\frac{2p}{p^2+2p-3} = \frac{2p}{(p+3)'(p-1)'} = \frac{A}{(p+3)'} + \frac{B}{(p-1)'}$

$2p = A(p-1) + B(p+3)$

const term	p-term
$0 = -A+3B$	$2 = A+B$
$A = 3B$	$2 = (3B)+B$
$A = 3(\frac{1}{2})$	$2 = 4B$
$\underline{A = \frac{3}{2}}$	$\underline{\frac{1}{2} = B}$

$\displaystyle = \int \frac{2p}{p^2+2p-3}\,dp = \int \left(\frac{(\frac{3}{2})}{(p+3)'} + \frac{(\frac{1}{2})}{(p-1)'} \right) dp$

$\displaystyle = \frac{3}{2}\left[\ln|p+3|\right] + \frac{1}{2}\left[\ln|p-1|\right] + C$

$\displaystyle = \frac{3}{2}\ln\left|\sqrt{x+3}+3\right| + \frac{1}{2}\ln\left|\sqrt{x+3}-1\right| + C$

44) $\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx$

$$\int \frac{1}{1+\sqrt[3]{x}}\,dx = \int \frac{1}{1+p}\left(3p^2dp\right) = \int \frac{3p^2}{p+1}\,dp$$

$p = \sqrt[3]{x}$

$p^3 = x$

$3p^2dp = dx$

improper, divide first

$$\begin{array}{r} 3p-3 \\ p+1\,\overline{)\,3p^2+0p+0} \\ -(3p^2+3p) \\ \hline -3p+0 \\ -(-3p-3) \\ \hline +3 \end{array}$$

$$\frac{3p^2}{p+1} = 3p-3+\frac{(+3)}{p+1}$$

$$= \int \frac{3p^2}{p+1}\,dp = \int \left(3p-3+\frac{(+3)}{p+1}\right)dp$$

$$= 3\left[\frac{p^2}{2}\right] - 3\left[p\right] + 3\left[\ln|p+1|\right] + C$$

$$= \frac{3}{2}\left(\sqrt[3]{x}\right)^2 - 3\left(\sqrt[3]{x}\right) + 3\ln\left|\sqrt[3]{x}+1\right| + C$$

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx = \left[\frac{3}{2}\left(\sqrt[3]{x}\right)^2 - 3\left(\sqrt[3]{x}\right) + 3\ln\left|\sqrt[3]{x}+1\right| + C\right]_0^1$$

$$= \left[\frac{3}{2}\left(\sqrt[3]{(1)}\right)^2 - 3\left(\sqrt[3]{(1)}\right) + 3\ln\left|\sqrt[3]{(1)}+1\right| + C\right] - \left[\frac{3}{2}\left(\sqrt[3]{(0)}\right)^2 - 3\left(\sqrt[3]{(0)}\right) + 3\ln\left|\sqrt[3]{(0)}+1\right| + C\right]$$

$$= \left[\frac{3}{2} - 3 + 3\ln|2|\right] - \left[0 - 0 + 3\ln|1|\right] = \left[\frac{-3}{2} + 3\ln(2)\right] - \left[0\right]$$

$$= \underline{\underline{\frac{-3}{2} + 3\ln(2) \quad +3\ln(2) - \frac{3}{2}}}$$

46) $\int \frac{dx}{(1+\sqrt{x})^2} = \int \frac{2(p-1)\,dp}{p^2} = 2\int \frac{p-1}{p^2}\,dp$

$p = 1+\sqrt{x}$

$p-1 = \sqrt{x}$

$(p-1)^2 = x$

$2(p-1)\,dp = dx$

$= 2\int \left(\frac{p}{p^2} - \frac{1}{p^2}\right) dp = 2\int \left(\frac{1}{p} - \frac{1}{p^2}\right) dp$

$= 2\int \frac{1}{p}\,dp - 2\int p^{-2}\,dp$

$= 2\left[\ln|p|\right] - 2\left[\frac{p^{-1}}{-1}\right] + C$

$$= 2\ln|1-\sqrt{x}| + \frac{2}{1+\sqrt{x}} + C$$

---

48) $\int \frac{1}{x - x^{\frac{1}{5}}}\,dx = \int \frac{1}{x - \sqrt[5]{x}}\,dx = \int \frac{1}{p^5 - p}\,(5p^4\,dp)$

$p = \sqrt[5]{x}$

$p^5 = x$

$5p^4\,dp = dx$

$= \int \frac{5p^4}{p(p^4-1)}\,dp = \int \frac{5p^3}{p^4-1}\,dp$

$= 5\int \frac{1}{p^4-1}(p^3\,dp) = 5\int \frac{1}{q}\left(\frac{1}{4}\,dq\right)$

$q = p^4 - 1$

$dq = 4p^3\,dp$

$\frac{1}{4}\,dq = p^3\,dp$

$= \frac{5}{4}\left[\ln|q|\right] + C = \frac{5}{4}\ln|p^4-1| + C$

$$= \frac{5}{4}\ln\left|(\sqrt[5]{x})^4 - 1\right| + C$$

50) $\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx = \int \frac{\sqrt{p}}{(p-1)^2}\left(2(p-1)\,dp\right) = \int \frac{2\sqrt{p}}{p-1}\,dp$

$p = 1+\sqrt{x}$

$p-1 = \sqrt{x}$

$(p-1)^2 = x$

$2(p-1)\,dp = dx$

$q = \sqrt{p}$

$q^2 = p$

$2q\,dq = dp$

$= \int \frac{2(q)}{(q^2)-1}\left(2q\,dq\right) = 4\int \frac{q^2}{q^2-1}\,dq$

improper, divide first

$$\begin{array}{r|l} & 1 \\ \hline q^2+0q-1 & q^2+0q+0 \\ & -(q^2+0q-1) \\ \hline & +1 \end{array}$$

$\frac{q^2}{q^2-1} = 1 + \frac{(+1)}{q^2-1}$

$\frac{1}{q^2-1} = \frac{1}{(q+1)^1(q-1)^1} = \frac{A}{(q+1)^1} + \frac{B}{(q-1)^1}$

$1 = A(q-1) + B(q+1)$

constant term	q-term
$1 = -A + B$	$0 = A + B$
$1 = (B) + B$	$-A = B$
$1 = 2B$	$A = -B$
$\frac{1}{2} = B$	$A = \frac{-1}{2}$

$= 4\int \frac{q^2}{q^2-1}\,dq = 4\int \left(1 + \frac{(1)}{q^2-1}\right)dq$

$= 4\int \left(1 + \frac{\left(\frac{-1}{2}\right)}{(q+1)^1} + \frac{\left(\frac{1}{2}\right)}{(q-1)^1}\right)dq$

$= 4\left\{[q] - \frac{1}{2}\left[\ln|q+1|\right] + \frac{1}{2}\left[\ln|q-1|\right]\right\} + C$

$= 4q - 2\ln|q+1| + 2\ln|q-1| + C$

$= 4(\sqrt{p}) - 2\ln|(\sqrt{p})+1| + 2\ln|(\sqrt{p})-1| + C$

$= 4\sqrt{1+\sqrt{x}} - 2\ln\left|\sqrt{1+\sqrt{x}}+1\right| + 2\ln\left|\sqrt{1+\sqrt{x}}-1\right| + C$

52) $\int \frac{\sin x}{\cos^2 x - 3\cos x}\,dx = \int \frac{1}{\cos^2 x - 3\cos x}(\sin x\,dx)$

$p = \cos x$

$dp = -\sin x\,dx$

$-1\,dp = \sin x\,dx$

$= \int \frac{1}{p^2-3p}(-1\,dp) = \int \frac{-1}{p^2-3p}\,dp$

$\frac{-1}{p^2-3p} = \frac{-1}{(p)^1(p-3)^1} = \frac{A}{(p)^1} + \frac{B}{(p-3)^1}$

$-1 = A(p-3) + B(p)$

const term

$-1 = -3A$

$\underline{\frac{1}{3} = A}$

p-term

$0 = A + B$

$B = -A \qquad \underline{B = -\frac{1}{3}}$

$= \int \frac{-1}{p^2-3p}\,dp = \int \left(\frac{\left(\frac{1}{3}\right)}{(p)^1} + \frac{\left(\frac{-1}{3}\right)}{(p-3)^1}\right)dp$

$= \frac{1}{3}\left[\ln|p|\right] - \frac{1}{3}\left[\ln|p-3|\right] + C$

$= \frac{1}{3}\ln|\cos x| - \frac{1}{3}\ln|\cos x - 3| + C$

54) $\displaystyle\int \frac{e^{x}}{(e^{x}-2)(e^{2x}+1)}\,dx = \int \frac{1}{(e^{x}-2)((e^{x})^{2}+1)}\,(e^{x}dx)$

$p = e^{x}$

$dp = e^{x}\,dx$

$$= \int \frac{1}{(p-2)(p^{2}+1)}\,dp = \int \left(\frac{\left(\frac{1}{5}\right)}{(p-2)^{1}} + \frac{\left(\frac{-1}{5}\right)p + \left(\frac{2}{5}\right)}{(p^{2}+1)^{1}} \right) dp$$

$$\frac{1}{(p-2)^{1}(p^{2}+1)^{2}} = \frac{A}{(p-2)^{1}} + \frac{(Bp+C)}{(p^{2}+1)^{2}}$$

$$1 = A(p^{2}+1) + (Bp+C)(p-2)$$

$$1 = A(p^{2}+1) + B(p^{2}-2p) + C(p-2)$$

const term

$1 = A - 2C$

$1 = A - 2(2B)$

$1 = A - 4B$

$1 = (-B) - 4B$

$1 = -5B$

$\underline{\frac{-1}{5} = B}$

p-term

$0 = -2B + C$

$2B = C$

$\underline{C = 2\left(\frac{-1}{5}\right) = \frac{-2}{5}}$

$\underline{C = \frac{-2}{5}}$

p^{2}-term

$0 = A + B$

$-B = A$

$\underline{A = \frac{1}{5}}$

$$= \int \frac{\left(\frac{1}{5}\right)}{p-2}\,dp + \int \frac{\left(\frac{-1}{5}\right)p}{p^{2}+1}\,dp + \int \frac{\left(\frac{2}{5}\right)}{p^{2}+(1)^{2}}\,dp$$

$$= \frac{1}{5}\Big[\ln|p-2|\Big] - \frac{1}{5}\Big[\frac{1}{2}\ln|p^{2}+1|\Big] + \frac{2}{5}\Big[\frac{1}{1}\tan^{-1}\Big(\frac{p}{1}\Big)\Big] + C$$

$$= \frac{1}{5}\ln|e^{x}-2| - \frac{1}{10}\ln|(e^{x})^{2}+1| + \frac{2}{5}\tan^{-1}(e^{x}) + C$$

$$\underline{= \frac{1}{5}\ln|e^{x}-2| - \frac{1}{10}\ln|e^{2x}+1| + \frac{2}{5}\tan^{-1}(e^{x}) + C}$$

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56) $\displaystyle\int \frac{\cosh t}{\sinh^2 t+\sinh^4 t}\,dt = \int \frac{1}{\sinh^2 t+\sinh^4 t}\left(\cosh t\,dt\right)$

$p = \sinh t$
$dp = \cosh t\,dt$

$$= \int \frac{1}{p^2+p^4}(dp) = \int \frac{1}{(p)^2(1+p^2)}\,dp = \int \frac{1}{(p)^2(p^2+1)^1}\,dp$$

$$\frac{1}{(p)^2(p^2+1)^1} = \frac{A}{(p)^1} + \frac{B}{(p)^2} + \frac{(Cp+D)}{(p^2+1)^1}$$

$$1 = A(p)(p^2+1) + B(p^2+1) + (Cp+D)(p)^2$$

$$1 = A(p^3+p) + B(p^2+1) + C(p^3) + D(p^2)$$

const. term	p-term	p^2-term	p^3-term
$\underline{1 = B}$	$\underline{0 = A}$	$0 = B + D$	$0 = A + C$
		$0 = (1) + D$	$0 = (0) + C$
		$\underline{-1 = D}$	$\underline{0 = C}$

$$= \int \frac{1}{(p)^2(p^2+1)^1}\,dp = \int \left(\frac{(0)}{(p)^1} + \frac{(1)}{(p)^2} + \frac{(0)p+(-1)}{(p^2+1)^1}\right)dp$$

$$= \int \left(\frac{1}{p^2} - \frac{1}{p^2+(1)^2}\right)dp = \left[\frac{p^{-1}}{-1}\right] - \left[\frac{1}{1}\tan^{-1}\left(\frac{p}{1}\right)\right] + C$$

$$= \frac{-1}{p} - \tan^{-1}(p) + C = \underline{\underline{\frac{-1}{\sinh t} - \tan^{-1}(\sinh t) + C}}$$