21200 {09} section 7.2

2) $\int \cos^6 y \sin^3 y \, dy = \int \cos^6 y \sin^2 y (\sin y \, dy)$

$p = \cos y$

$dp = -\sin y \, dy$

$-1 dp = \sin y \, dy$

$= \int \cos^6 y (1 - \cos^2 y)(\sin y \, dy)$

$= \int (\cos^6 y - \cos^8 y)(\sin y \, dy)$

$= \int (p^6 - p^8)(-1 \, dp)$

$= -1 \left\{ \left[\frac{p^7}{7}\right] - \left[\frac{p^9}{9}\right] \right\} + C$

$= \frac{-1}{7} \cos^7 y + \frac{1}{9} \cos^9 y + C$

$= \frac{1}{9} \cos^9 y - \frac{1}{7} \cos^7 y + C$

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4) $\int_0^{\frac{\pi}{4}} \sin^5 x \, dx$

$\int \sin^5 x \, dx = \int \sin^4 x (\sin x \, dx) = \int (\sin^2 x)^2 (\sin x \, dx)$

$p = \cos x$

$dp = -\sin x \, dx$

$-1 dp = \sin x \, dx$

$= \int (1 - \cos^2 x)^2 (\sin x \, dx) = \int (1 - 2\cos^2 x + \cos^4 x)(\sin x \, dx)$

$= \int (1 - 2p^2 + p^4)(-1 \, dp) = -1 \left\{ [p] - 2\left[\frac{p^3}{3}\right] + \left[\frac{p^5}{5}\right] \right\} + C$

$= -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$

$\int_0^{\frac{\pi}{4}} \sin^5 x \, dx = \left[-\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C \right]_0^{\frac{\pi}{4}}$

$= \left[-\cos\left(\frac{\pi}{4}\right) + \frac{2}{3} \cos^3\left(\frac{\pi}{4}\right) - \frac{1}{5} \cos^5\left(\frac{\pi}{4}\right) + C \right] - \left[-\cos(0) + \frac{2}{3} \cos^3(0) - \frac{1}{5} \cos^5(0) + C \right]$

$= \left[-\left(\frac{1}{\sqrt{2}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3 - \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5 \right] - \left[-(1) + \frac{2}{3}(1)^3 - \frac{1}{5}(1)^5 \right] = \left[\frac{-1}{\sqrt{2}} + \frac{1}{3\sqrt{2}} - \frac{1}{20\sqrt{2}} \right] - \left[-1 + \frac{2}{3} - \frac{1}{5} \right]$

$= \left[\frac{-60}{60\sqrt{2}} + \frac{20}{60\sqrt{2}} - \frac{3}{60\sqrt{2}} \right] - \left[\frac{-15}{15} + \frac{10}{15} - \frac{3}{15} \right] = \left[\frac{-43}{60\sqrt{2}} \right] - \left[\frac{-8}{15} \right] = \frac{8}{15} - \frac{43}{60\sqrt{2}}$

6) $\int \cos^3\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)dt = \int \cos^2\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)\left(\cos\left(\frac{t}{2}\right)dt\right)$

$p = \sin\left(\frac{t}{2}\right)$

$dp = \frac{1}{2}\cos\left(\frac{t}{2}\right)dt$

$2\,dp = \cos\left(\frac{t}{2}\right)dt$

$= \int \left(1-\sin^2\left(\frac{t}{2}\right)\right)\sin^2\left(\frac{t}{2}\right)\left(\cos\left(\frac{t}{2}\right)dt\right)$

$= \int \left(\sin^2\left(\frac{t}{2}\right) - \sin^4\left(\frac{t}{2}\right)\right)\left(\cos\left(\frac{t}{2}\right)dt\right)$

$= \int (p^2 - p^4)(2\,dp) = 2\left\{\left[\frac{p^3}{3}\right] - \left[\frac{p^5}{5}\right]\right\} + C$

$$= \frac{2}{3}\sin^3\left(\frac{t}{2}\right) - \frac{2}{5}\sin^5\left(\frac{t}{2}\right) + C$$

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8) $\int_0^{\frac{\pi}{4}} \sin^2(2\theta)\,d\theta$

* reduction of power identity needed (modified double angle formula)

$\int \sin^2(2\theta)\,d\theta = \int \left(\frac{1-\cos(2(2\theta))}{2}\right)d\theta = \int \left(\frac{1}{2} - \frac{1}{2}\cos(4\theta)\right)d\theta$

$= \int \frac{1}{2}\,d\theta - \frac{1}{2}\int \cos(4\theta)\,d\theta$

substitution

$= \frac{1}{2}[\theta] - \frac{1}{2}\left[\frac{1}{4}\sin(4\theta)\right] + C$

$= \frac{1}{2}\theta - \frac{1}{8}\sin(4\theta) + C$

$\int_0^{\frac{\pi}{4}} \sin^2(2\theta)\,d\theta = \left[\frac{1}{2}\theta - \frac{1}{8}\sin(4\theta) + C\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{8}\sin\left(4\left(\frac{\pi}{4}\right)\right) + C\right] - \left[\frac{1}{2}(0) - \frac{1}{8}\sin(4(0)) + C\right]$

$= \left[\frac{\pi}{8} - \frac{1}{8}\sin(\pi)\right] - \left[0 - \frac{1}{8}\sin(0)\right] = \left[\frac{\pi}{8} - \frac{1}{8}(0)\right] - \left[-\frac{1}{8}(0)\right]$

$$= \frac{\pi}{8}$$

7.2/3

10) $\int_0^{\pi} \sin^2 t \cos^4 t \, dt$ * reduction of power formula needed

$$\int \sin^2 t \cos^4 t \, dt = \int \sin^2 t (\cos^2 t)^2 \, dt$$

$$= \int \left(\frac{1-\cos(2t)}{2}\right)\left(\frac{1+\cos(2t)}{2}\right)^2 dt$$

$$= \int \left(\frac{1-\cos(2t)}{2}\right)\left(\frac{1+\cos(2t)}{2}\right)\left(\frac{1+\cos(2t)}{2}\right) dt$$

$$= \frac{1}{8}\int \left(1-\cos^2(2t)\right)\left(1+\cos(2t)\right) dt$$

$$= \frac{1}{8}\int \left(1+\cos(2t)-\cos^2(2t)-\cos^3(2t)\right) dt$$

$$= \frac{1}{8}\int \left(1+\cos(2t)-\left(\frac{1+\cos(2(2t))}{2}\right)-\cos^2(2t)\cos(2t)\right) dt$$

$$= \frac{1}{8}\int dt + \frac{1}{8}\int \cos(2t)\,dt - \frac{1}{16}\int dt - \frac{1}{16}\int \cos(4t)\,dt - \frac{1}{8}\int (1-\sin^2(2t))(\cos(2t)\,dt)$$

$$= \frac{1}{16}\int dt + \frac{1}{8}\int \cos(\underbrace{2t}_{p_1 = 2t})\,dt - \frac{1}{16}\int \cos(\underbrace{4t}_{p_2 = 4t})\,dt - \frac{1}{8}\int (1-\underbrace{\sin^2(2t)}_{p_3 = \sin(2t)})(\cos(2t)\,dt)$$

$$= \frac{1}{16}[t] + \frac{1}{8}\left[\frac{1}{2}\sin(2t)\right] - \frac{1}{16}\left[\frac{1}{4}\sin(4t)\right] - \frac{1}{8}\left\{\left[\frac{1}{2}\sin(2t)\right] - \left[\frac{1}{2}\left(\frac{1}{3}\right)\sin^3(2t)\right]\right\} + C$$

$$= \frac{1}{16}t + \frac{1}{16}\sin(2t) - \frac{1}{64}\sin(4t) - \frac{1}{16}\sin(2t) + \frac{1}{48}\sin^3(2t) + C$$

$$= \frac{1}{16}t - \frac{1}{64}\sin(4t) + \frac{1}{48}\sin^3(2t) + C$$

10) continued...

$$\int_0^{\pi} \sin^2 t \cos^4 t \, dt = \left[\frac{1}{16}t - \frac{1}{64}\sin(4t) + \frac{1}{48}\sin^3(2t) + C\right]_0^{\pi}$$

$$= \left[\frac{1}{16}(\pi) - \frac{1}{64}\sin(4(\pi)) + \frac{1}{48}\sin^3(2(\pi)) + C\right] - \left[\frac{1}{16}(0) - \frac{1}{64}\sin(4(0)) + \frac{1}{48}\sin^3(2(0)) + C\right]$$

$$= \left[\frac{\pi}{16} - \frac{1}{64}(0) + \frac{1}{48}(0)^3\right] - \left[0 - \frac{1}{64}(0) + \frac{1}{48}(0)^3\right] = \underline{\underline{\frac{\pi}{16}}}$$

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12) $\int_0^{\frac{\pi}{2}} (2 - \sin\theta)^2 \, d\theta$ * reduction of power formula needed

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$$\int (2 - \sin\theta)^2 d\theta = \int (4 - 4\sin\theta + \sin^2\theta) d\theta$$

$$= \int \left(4 - 4\sin\theta + \left(\overset{*}{\frac{1 - \cos(2\theta)}{2}}\right)\right) d\theta = \int \left(4 - 4\sin\theta + \frac{1}{2} - \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \int \left(\frac{9}{2} - 4\sin\theta - \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$\phi = 2\theta$$

$$= \frac{9}{2}[\theta] - 4[-\cos\theta] - \frac{1}{2}\left[\frac{1}{2}\sin(2\theta)\right] + C = \frac{9}{2}\theta + 4\cos\theta - \frac{1}{4}\sin(2\theta) + C$$

$$\int_0^{\frac{\pi}{2}} (2 - \sin\theta)^2 d\theta = \left[\frac{9}{2}\theta + 4\cos\theta - \frac{1}{4}\sin(2\theta) + C\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{9}{2}\left(\frac{\pi}{2}\right) + 4\cos\left(\frac{\pi}{2}\right) - \frac{1}{4}\sin\left(2\left(\frac{\pi}{2}\right)\right) + C\right] - \left[\frac{9}{2}(0) + 4\cos(0) - \frac{1}{4}\sin(2(0)) + C\right]$$

$$= \left[\frac{9\pi}{4} + 4(0) - \frac{1}{4}(0)\right] - \left[0 + 4(1) - \frac{1}{4}(0)\right] = \left[\frac{9\pi}{4}\right] - [4]$$

$$= \underline{\underline{\frac{9\pi}{4} - 4 = \frac{9\pi - 16}{4}}}$$

7.2/5

14) $\int \left(1+\sqrt[3]{\sin t}\right)\cos^3 t\,dt = \int \left(1-\sqrt[3]{\sin t}\right)\cos^2 t\,(\cos t\,dt)$

$p = \sin t$

$dp = \cos t\,dt$

$= \int \left(1-\sqrt[3]{\sin t}\right)(1-\sin^2 t)(\cos t\,dt)$

$= \int \left(1-\sqrt[3]{p}\right)(1-p^2)(dp) = \int \left(1-p^{\frac{1}{3}}\right)(1-p^2)\,dp$

$= \int \left(1-p^{\frac{1}{3}}-p^2+p^{\frac{7}{3}}\right)dp = [p] - \left[\frac{p^{\frac{4}{3}}}{\frac{4}{3}}\right] - \left[\frac{p^3}{3}\right] + \left[\frac{p^{\frac{10}{3}}}{\frac{10}{3}}\right] + C$

$= \sin t - \frac{3}{4}\sin^{4/3} t - \frac{1}{3}\sin^3 t + \frac{3}{10}\sin^{\frac{10}{3}} t + C$

$= \sin t - \frac{3}{4}\left(\sqrt[3]{\sin t}\right)^4 - \frac{1}{3}\sin^3 t + \frac{3}{10}\left(\sqrt[3]{\sin t}\right)^{10} + C$

---

16) $\int \csc^5\theta\cos^3\theta\,d\theta = \int \left(\frac{1}{\sin^5\theta}\right)\cos^2\theta\,(\cos\theta\,d\theta)$

$p = \sin\theta$

$dp = \cos\theta\,d\theta$

$= \int \left(\frac{1}{\sin^5\theta}\right)(1-\sin^2\theta)(\cos\theta\,d\theta)$

$= \int \left(\frac{1}{\sin^5\theta} - \frac{\sin^2\theta}{\sin^5\theta}\right)(\cos\theta\,d\theta) = \int \left(\frac{1}{\sin^5\theta} - \frac{1}{\sin^3\theta}\right)(\cos\theta\,d\theta)$

$= \int \left(\frac{1}{p^5} - \frac{1}{p^3}\right)dp = \int \left(p^{-5} - p^{-3}\right)dp$

$= \left[\frac{p^{-4}}{-4}\right] - \left[\frac{p^{-2}}{-2}\right] + C = \frac{-1}{4\sin^4\theta} + \frac{1}{2\sin^2\theta} + C$

$= \frac{1}{2\sin^2\theta} - \frac{1}{4\sin^4\theta} + C = \frac{1}{2}\csc^2\theta - \frac{1}{4}\csc^4\theta + C$

18) $\int \tan^2 x \cos^3 x \, dx = \int \left(\frac{\sin^2 x}{\cos^2 x}\right) \cos^2 x \, (\cos x \, dx)$

$p = \sin x$

$dp = \cos x \, dx$

$= \int \sin^2 x \, (\cos x \, dx) = \int p^2 \, dp = \left[\frac{p^3}{3}\right] + C$

$= \frac{1}{3} \sin^3 x + C$

\# use double angle formula

20) $\int \sin x \cos\left(\frac{1}{2}x\right) dx = \int \sin\left(2\left(\frac{1}{2}x\right)\right) \cos\left(\frac{1}{2}x\right) dx$

$p = \cos\left(\frac{1}{2}x\right)$

$dp = \frac{-1}{2} \sin\left(\frac{1}{2}x\right) dx$

$-2dp = \sin\left(\frac{1}{2}x\right) dx$

$= \int \left(2 \sin\left(\frac{1}{2}x\right) \cos\left(\frac{1}{2}x\right)\right) \cos\left(\frac{1}{2}x\right) dx$

$= 2 \int \cos^2\left(\frac{1}{2}x\right) \left(\sin\left(\frac{1}{2}x\right) dx\right)$

$= 2 \int p^2 (-2\,dp) = -4 \left[\frac{p^3}{3}\right] + C$

$= \frac{-4}{3} \cos^3\left(\frac{1}{2}x\right) + C$

22) $\int \tan^2 \theta \sec^4 \theta \, d\theta = \int \tan^2 \theta \sec^2 \theta \, (\sec^2 \theta \, d\theta)$

$p = \tan \theta$

$dp = \sec^2 \theta \, d\theta$

$= \int \tan^2 \theta \, (1 + \tan^2 \theta)(\sec^2 \theta \, d\theta)$

$= \int (\tan^2 \theta + \tan^4 \theta)(\sec^2 \theta \, d\theta)$

$= \int (p^2 + p^4) \, dp = \left[\frac{p^3}{3}\right] + \left[\frac{p^5}{5}\right] + C$

$= \frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta + C$

24) $\int (\tan^2 x + \tan^4 x)\,dx = \int \tan^2 x(1+\tan^2 x)\,dx$

$p = \tan x$

$dp = \sec^2 x\,dx$

$= \int \tan^2 x \sec^2\theta\,dx = \int \tan^2 x(\sec^2 x\,dx)$

$= \int p^2\,dp = \left[\frac{p^3}{3}\right] + C = \underline{\underline{\frac{1}{3}\tan^3 x + C}}$

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26) $\int_0^{\frac{\pi}{4}} \sec^6\theta \tan^6\theta\,d\theta$

$\int \sec^6\theta\tan^6\theta\,d\theta = \int \sec^4\theta\tan^6\theta(\sec^2\theta\,d\theta)$

$p = \tan\theta$

$dp = \sec^2\theta\,d\theta$

$= \int (\sec^2\theta)^2\tan^6\theta(\sec^2\theta\,d\theta)$

$= \int (1+\tan^2\theta)^2\tan^6\theta(\sec^2\theta\,d\theta)$

$= \int (1+2\tan^2\theta+\tan^4\theta)\tan^6\theta(\sec^2\theta\,d\theta)$

$= \int (\tan^6\theta + 2\tan^8\theta + \tan^{10}\theta)(\sec^2\theta\,d\theta)$

$= \int (p^6 + 2p^8 + p^{10})\,dp = \left[\frac{p^7}{7}\right] + 2\left[\frac{p^9}{9}\right] + \left[\frac{p^{11}}{11}\right] + C$

$= \underline{\frac{1}{7}\tan^7\theta + \frac{2}{9}\tan^9\theta + \frac{1}{11}\tan^{11}\theta + C}$

$\int_0^{\frac{\pi}{4}} \sec^6\theta\tan^6\theta\,d\theta = \left[\frac{1}{7}\tan^7\theta + \frac{2}{9}\tan^9\theta + \frac{1}{11}\tan^{11}\theta + C\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{1}{7}\tan^7\left(\frac{\pi}{4}\right) + \frac{2}{9}\tan^9\left(\frac{\pi}{4}\right) + \frac{1}{11}\tan^{11}\left(\frac{\pi}{4}\right) + C\right] - \left[\frac{1}{7}\tan^7(0) + \frac{2}{9}\tan^9(0) + \frac{1}{11}\tan^{11}(0) + C\right]$

$= \left[\frac{1}{7}(1)^7 + \frac{2}{9}(1)^9 + \frac{1}{11}(1)^{11}\right] - \left[\frac{1}{7}(0)^7 + \frac{2}{9}(0)^9 + \frac{1}{11}(0)^{11}\right]$

$= \left[\frac{1}{7} + \frac{2}{9} + \frac{1}{11}\right] - [0] = \frac{99}{693} + \frac{154}{693} + \frac{63}{693} = \underline{\underline{\frac{316}{693}}}$

7.2/8

28) $\int \tan^5 x \sec^3 x \, dx = \int \tan^4 x \sec^2 x (\sec x \tan x \, dx)$

$p = \sec x$

$dp = \sec x \tan x \, dx$

$= \int (\tan^2 x)^2 \sec^2 x (\sec x \tan x \, dx)$

$= \int (\sec^2 x - 1)^2 \sec^2 x (\sec x \tan x \, dx)$

$= \int (\sec^4 x - 2\sec^2 x + 1) \sec^2 x (\sec x \tan x \, dx)$

$= \int (\sec^6 x - 2\sec^4 x + \sec^2 x)(\sec x \tan x \, dx)$

$= \int (p^6 - 2p^4 + p^2) \, dp = \left[\frac{p^7}{7}\right] - 2\left[\frac{p^5}{5}\right] + \left[\frac{p^3}{3}\right] + C$

$= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C$

---

30) $\int_0^{\frac{\pi}{4}} \tan^4 t \, dt$

$\int \tan^4 t \, dt = \int (\tan^2 t)^2 dt = \int (\sec^2 t - 1)^2 dt$

$= \int (\sec^4 t - 2\sec^2 t + 1) \, dt$

$= \int \sec^4 t \, dt - 2\int \sec^2 t \, dt + \int 1 \, dt$

$= \int \sec^2 t (\sec^2 t \, dt) - 2\int \sec^2 t \, dt + \int 1 \, dt$

$= \int (1 + \tan^2 t)(\sec^2 t \, dt) - 2\int \sec^2 t \, dt + \int 1 \, dt$

$= \int \sec^2 t \, dt + \int \tan^2 t (\sec^2 t \, dt) - 2\int \sec^2 t \, dt + \int 1 \, dt$

$=$

30) continued...

$$= \int \tan^2 t \,(\sec^2 t\, dt) - \int \sec^2 t\, dt + \int 1\, dt$$

$u = \tan t$

$$= \left[\tfrac{1}{3}\tan^3 t\right] - \left[\tan t\right] + \left[t\right] + C$$

$$\int_0^{\frac{\pi}{4}} \tan^4 t\, dt = \left[\tfrac{1}{3}\tan^3 t - \tan t + t + C\right]_0^{\frac{\pi}{4}}$$

$$= \left[\tfrac{1}{3}\tan^3\left(\tfrac{\pi}{4}\right) - \tan\left(\tfrac{\pi}{4}\right) + \left(\tfrac{\pi}{4}\right) + C\right] - \left[\tfrac{1}{3}\tan^3(0) - \tan(0) + (0) + C\right]$$

$$= \left[\tfrac{1}{3}(1)^3 - (1) + \tfrac{\pi}{4}\right] - \left[\tfrac{1}{3}(0) - (0) + 0\right] = \underline{\underline{\tfrac{\pi}{4} - \tfrac{2}{3}}}$$

32) $\int \tan^2 x \sec x\, dx = \int (\sec^2 x - 1)\sec x\, dx$

$$= \int (\sec^3 x - \sec x)\, dx = \int \sec^3 x\, dx - \int \sec x\, dx$$

use formula from Integration Notes

$$= \left[\tfrac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right)\right] - \left[\ln|\sec x + \tan x|\right] + C$$

$$= \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C$$

$$= \underline{\underline{\tfrac{1}{2}\sec x \tan x - \tfrac{1}{2}\ln|\sec x + \tan x| + C}}$$

34) $\int \frac{\tan x \sec^2 x}{\cos x}\, dx = \int \tan x \sec^2 x \left(\frac{1}{\cos x}\right) dx$

$p = \sec x$

$dp = \sec x \tan x \, dx$

$= \int \tan x \sec^2 x \, (\sec x)\, dx$

$= \int \sec^2 x \, (\sec x \tan x \, dx)$

$= \int p^2 \, dp = \left[\frac{p^3}{3}\right] + C = \underline{\underline{\frac{1}{3}\sec^3 x + C}}$

---

36) $\int \frac{\sin\theta + \tan\theta}{\cos^3\theta}\, d\theta = \int \left(\frac{\sin\theta}{\cos^3\theta} + \frac{\tan\theta}{\cos^3\theta}\right) d\theta$

$= \int \left(\frac{\sin\theta}{\cos^3\theta} + \frac{\left(\frac{\sin\theta}{\cos\theta}\right)}{\cos^3\theta}\right) d\theta = \int \left(\frac{\sin\theta}{\cos^3\theta} + \frac{\sin\theta}{\cos^4\theta}\right) d\theta$

$p = \cos\theta$

$dp = -\sin\theta \, d\theta$

$-1\, dp = \sin\theta \, d\theta$

$= \int \left(\frac{1}{\cos^3\theta} + \frac{1}{\cos^4\theta}\right)(\sin\theta \, d\theta)$

$= \int \left(\frac{1}{p^3} + \frac{1}{p^4}\right)(-1\, dp) = \int (p^{-3} + p^{-4})(-1\, dp)$

$= -1\left\{\left[\frac{p^{-2}}{-2}\right] + \left[\frac{p^{-3}}{-3}\right]\right\} + C = \frac{1}{2p^2} + \frac{1}{3p^3} + C$

$= \underline{\underline{\frac{1}{2\cos^2\theta} + \frac{1}{3\cos^3\theta} + C = \frac{1}{2}\sec^2\theta + \frac{1}{3}\sec^3\theta + C}}$

---

38) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^3 x \, dx$

$\int \cot^3 x \, dx = \int \cot^2 x \cot x \, dx = \int (\csc^2 x - 1)\cot x \, dx$

$= \int (\cot x \csc^2 x - \cot x)\, dx$

38) continued...

$$= \int \cot x \,(\csc^2 x\, dx) - \int \cot x\, dx$$

$p = \cot x$

$dp = -\csc^2 x\, dx$

$-1\, dp = \csc^2 x\, dx$

$$= \left[\frac{-1}{2}\cot^2 x\right] - \left[\ln|\sin x|\right] + C$$

$$= \frac{-1}{2}\cot^2 x - \ln|\sin x| + C$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^3 x\, dx = \left[\frac{-1}{2}\cot^2 x - \ln|\sin x| + C\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[\frac{-1}{2}\cot^2\left(\frac{\pi}{2}\right) - \ln\left|\sin\left(\frac{\pi}{2}\right)\right| + C\right] - \left[\frac{-1}{2}\cot^2\left(\frac{\pi}{4}\right) - \ln\left|\sin\left(\frac{\pi}{4}\right)\right| + C\right]$$

$$= \left[\frac{-1}{2}(0)^2 - \ln|(1)|\right] - \left[\frac{-1}{2}(1)^2 - \ln\left|\frac{1}{\sqrt{2}}\right|\right] = [0-0] - \left[\frac{-1}{2} - \ln\left|\frac{1}{\sqrt{2}}\right|\right]$$

$$= \frac{1}{2} + \ln\left|\frac{1}{\sqrt{2}}\right| = \frac{1}{2} - \ln|\sqrt{2}| = \frac{1}{2} - \ln\left(2^{\frac{1}{2}}\right) = \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1-\ln 2}{2}$$

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40) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^4\theta \cot^4\theta\, d\theta$

$$\int \csc^4\theta \cot^4\theta\, d\theta = \int \csc^2\theta \cot^4\theta\,(\csc^2\theta\, d\theta)$$

$p = \cot\theta$

$dp = -\csc^2\theta\, d\theta$

$-1\, dp = \csc^2\theta\, d\theta$

$$= \int (\cot^2\theta + 1)\cot^4\theta\,(\csc^2\theta\, d\theta)$$

$$= \int (\cot^6\theta + \cot^4\theta)(\csc^2\theta\, d\theta)$$

$$= \int (p^6 + p^8)(-1\, dp)$$

40) continued...

$$= -1\left\{\left[\frac{u^7}{7}\right] + \left[\frac{u^9}{9}\right]\right\} + C$$

$$= \frac{-1}{7}\cot^7\theta - \frac{1}{9}\cot^9\theta + C$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^4\theta\cot^4\theta\,d\theta = \left[\frac{-1}{7}\cot^7\theta - \frac{1}{9}\cot^9\theta + C\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[\frac{-1}{7}\cot^7\left(\frac{\pi}{2}\right) - \frac{1}{9}\cot^9\left(\frac{\pi}{2}\right) + C\right] - \left[\frac{-1}{7}\cot^7\left(\frac{\pi}{4}\right) - \frac{1}{9}\cot^9\left(\frac{\pi}{4}\right) + C\right]$$

$$= \left[\frac{-1}{7}(0)^7 - \frac{1}{9}(0)^9\right] - \left[\frac{-1}{7}(1)^7 - \frac{1}{9}(1)^9\right] = [0] - \left[\frac{-1}{7} - \frac{1}{9}\right] = \frac{9}{63} + \frac{7}{63} = \underline{\underline{\frac{16}{63}}}$$

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using formula in Integration Notes

42) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc^3 x\,dx = \left[\frac{1}{2}\left(-\csc x\cot x + \ln|\csc x - \cot x|\right) + C\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left[\frac{-1}{2}\csc\left(\frac{\pi}{3}\right)\cot\left(\frac{\pi}{3}\right) + \frac{1}{2}\ln\left|\csc\left(\frac{\pi}{3}\right) - \cot\left(\frac{\pi}{3}\right)\right| + C\right]$$

$$- \left[\frac{-1}{2}\csc\left(\frac{\pi}{6}\right)\cot\left(\frac{\pi}{6}\right) + \frac{1}{2}\ln\left|\csc\left(\frac{\pi}{6}\right) - \cot\left(\frac{\pi}{6}\right)\right| + C\right]$$

$$= \left[\frac{-1}{2}\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{2}\ln\left|\left(\frac{2}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{3}}\right)\right|\right] - \left[\frac{-1}{2}(2)(\sqrt{3}) + \frac{1}{2}\ln\left|2 - \sqrt{3}\right|\right]$$

$$= \left[\frac{-1}{3} + \frac{1}{2}\ln\left|\frac{1}{\sqrt{3}}\right|\right] - \left[-\sqrt{3} + \frac{1}{2}\ln\left|2 - \sqrt{3}\right|\right]$$

$$= \frac{-1}{3} + \frac{1}{2}\ln\left|\frac{1}{\sqrt{3}}\right| + \sqrt{3} - \frac{1}{2}\ln\left|2 - \sqrt{3}\right|$$

$$= \underline{\underline{\sqrt{3} - \frac{1}{3} + \frac{1}{2}\ln\left|\frac{1}{\sqrt{3}}\right| - \frac{1}{2}\ln\left|2 - \sqrt{3}\right|}}$$

44) we need to use addition formulas of Cosine

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$$
$$\frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) = \sin A \sin B$$

$$\int \sin 2\theta \sin 6\theta \, d\theta = \int \sin 6\theta \sin 2\theta \, d\theta$$
$$= \int \frac{1}{2}\left(\cos(6\theta - 2\theta) - \cos(6\theta + 2\theta)\right) d\theta$$
$$= \frac{1}{2}\int (\cos 4\theta - \cos 8\theta)\, d\theta$$
$$= \frac{1}{2}\int \underbrace{\cos 4\theta \, d\theta}_{\varphi_1 = 4\theta} - \frac{1}{2}\int \underbrace{\cos 8\theta \, d\theta}_{\varphi_2 = 8\theta}$$
$$= \frac{1}{2}\left[\frac{1}{4}\sin 4\theta\right] - \frac{1}{2}\left[\frac{1}{8}\sin 8\theta\right] + C$$
$$= \frac{1}{8}\sin 4\theta - \frac{1}{16}\sin 8\theta + C$$

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46) $\int t \cos^5(t^2)\, dt = \int \cos^5(t^2)(t\, dt)$

$\varphi = t^2$

$d\varphi = 2t\, dt$

$\frac{1}{2} d\varphi = t\, dt$

$$= \int \cos^5 \varphi \left(\frac{1}{2} d\varphi\right) = \frac{1}{2}\int \cos^4 \varphi (\cos\varphi \, d\varphi)$$
$$= \frac{1}{2}\int (\cos^2\varphi)^2 (\cos\varphi \, d\varphi)$$
$$= \frac{1}{2}\int (1 - \sin^2\varphi)^2 (\cos\varphi \, d\varphi)$$

46) continued...

$q = \sin p$

$dq = \cos p \, dp$

$$= \frac{1}{2}\int (1 - 2\sin^2 p + \sin^4 p)(\cos p \, dp)$$

$$= \frac{1}{2}\int (1 - 2q^2 + q^4)\, dq$$

$$= \frac{1}{2}\left\{ [q] - 2\left[\frac{q^3}{3}\right] + \left[\frac{q^5}{5}\right] \right\} + C$$

$$= \frac{1}{2}\sin p - \frac{1}{3}\sin^3 p + \frac{1}{10}\sin^5 p + C$$

$$= \frac{1}{2}\sin(t^2) - \frac{1}{3}\sin^3(t^2) + \frac{1}{10}\sin^5(t^2) + C$$

---

48) $\int \sec^2 y \cos^3(\tan y)\, dy = \int \cos^3(\tan y)(\sec^2 y \, dy)$

$p = \tan y$

$dp = \sec^2 y \, dy$

$$= \int \cos^3 p \, dp = \int \cos^2 p (\cos p \, dp)$$

$$= \int (1 - \sin^2 p)(\cos p \, dp)$$

$q = \sin p$

$dq = \cos p \, dp$

$$= \int (1 - q^2)\, dq = [q] - \left[\frac{q^3}{3}\right] + C$$

$$= \sin p - \frac{1}{3}\sin^3 p + C$$

$$= \sin(\tan y) - \frac{1}{3}\sin^3(\tan y) + C$$

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50) $\int_0^{\frac{\pi}{4}} \sqrt{1 - \cos 4\theta}\, d\theta$

$\sin^2 A = \frac{1 - \cos 2A}{2} \Rightarrow 2\sin^2 A = 1 - \cos 2A$

$$\int \sqrt{1 - \cos 4\theta}\, d\theta = \int \sqrt{2\sin^2(2\theta)}\, d\theta = \sqrt{2}\int \sin(2\theta)\, d\theta$$

$p = 2\theta$

$$= \sqrt{2}\left[\frac{-1}{2}\cos(2\theta)\right] + C$$

$$= \frac{-\sqrt{2}}{2}\cos(2\theta) + C = \frac{-1}{\sqrt{2}}\cos(2\theta) + C$$

50) continued...

$$\int_0^{\frac{\pi}{4}} \sqrt{1-\cos 4\theta}\, d\theta = \left[\frac{-1}{\sqrt{2}} \cos(2\theta) + C\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{-1}{\sqrt{2}} \cos\left(2\left(\frac{\pi}{4}\right)\right) + C\right] - \left[\frac{-1}{\sqrt{2}} \cos(2(0)) + C\right]$$

$$= \left[\frac{-1}{\sqrt{2}} \cos\left(\frac{\pi}{2}\right)\right] - \left[\frac{-1}{\sqrt{2}} \cos(0)\right] = \left[\frac{-1}{\sqrt{2}}(0)\right] - \left[\frac{-1}{\sqrt{2}}(1)\right] = \underline{\underline{\frac{1}{\sqrt{2}}}}$$

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52) $\int x \sec x \tan x \, dx = (x)(\sec x) - \int (\sec x)(dx)$

$u = x \qquad dv = \sec x \tan x \, dx$

$du = dx \qquad v = \sec x$

$$= x \sec x - \int \sec x \, dx$$

$$= x \sec x - [\ln|\sec x + \tan x|] + C$$

$$= \underline{\underline{x \sec x - \ln|\sec x + \tan x| + C}}$$

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54) $\int \sin^3 x \, dx = \int \sin^2 x (\sin x \, dx) = \int (1-\cos^2 x)(\sin x \, dx)$

$p = \cos x$

$dp = -\sin x \, dx$

$-1\, dp = \sin x \, dx$

$$= \int (1-p^2)(-1\, dp) = -1\left\{[p] - \left[\frac{p^3}{3}\right]\right\} + C_1$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C_1$$

$$\int x \sin^3 x \, dx = (x)\left(-\cos x + \frac{1}{3}\cos^3 x\right) - \int \left(-\cos x + \frac{1}{3}\cos^3 x\right)(dx)$$

$u = x \qquad dv = \sec^3 x \, dx$

$du = dx \qquad v = \left(-\cos x + \frac{1}{3}\cos^3 x\right)$

$$= -x\cos x + \frac{1}{3} x \cos^3 x + \int \cos x \, dx - \frac{1}{3}\int \cos^3 x \, dx$$

$$= -x\cos x + \frac{1}{3} x \cos^3 x + [\sin x] - \frac{1}{3}\int \cos^2 x (\cos x \, dx)$$

$$= -x\cos x + \frac{1}{3} x \cos^3 x + \sin x - \frac{1}{3}\int (1-\sin^2 x)(\cos x \, dx)$$

$$= -x\cos x + \frac{1}{3} x \cos^3 x + \sin x - \frac{1}{3}\left[\sin x - \frac{1}{3}\sin^3 x\right] + C$$

$$= \underline{\underline{-x\cos x + \frac{1}{3} x \cos^3 x + \frac{2}{3}\sin x + \frac{1}{9}\sin^3 x + C}}$$

$\int (1-\sin^2 x)(\cos x \, dx) = \int (1-q^2)\, dq$

$q = \sin x$

$dq = \cos x \, dx$

$$= [q] - \left[\frac{q^3}{3}\right] + C_2$$

$$= \sin x - \frac{1}{3}\sin^3 x + C_2$$

56) $\int \frac{1}{\sec\theta + 1}\, d\theta$

$$\frac{1}{\sec\theta+1} = \left(\frac{1}{\sec\theta+1}\right)\left(\frac{\sec\theta-1}{\sec\theta-1}\right) = \frac{\sec\theta-1}{\sec^2\theta-1} = \frac{\sec\theta-1}{\tan^2\theta} = \frac{\sec\theta}{\tan^2\theta} - \frac{1}{\tan^2\theta}$$

$$= \frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)} - \cot^2\theta = \frac{\cos\theta}{\sin^2\theta} - (\csc^2\theta - 1) = \frac{\cos\theta}{\sin^2\theta} - \csc^2\theta + 1$$

$$\int \frac{1}{\sec\theta+1}\, d\theta = \int \left(\frac{\cos\theta}{\sin^2\theta} - \csc^2\theta + 1\right) d\theta$$

$p = \sin\theta$
$dp = \cos\theta\, d\theta$

$$= \int \frac{1}{\sin^2\theta}(\cos\theta\, d\theta) - \int \csc^2\theta\, d\theta + \int 1\, d\theta$$

$$= \int \frac{1}{p^2}\, dp - [-\cot\theta] + [\theta] + C$$

$$= \int p^{-2}\, dp + \cot\theta + \theta + C$$

$$= \left[\frac{p^{-1}}{-1}\right] + \cot\theta + \theta + C$$

$$= \frac{-1}{\sin\theta} + \cot\theta + \theta + C$$

$$= -\csc\theta + \cot\theta + \theta + C = \cot\theta - \csc\theta + \theta + C$$