4.5

2) $y = \frac{-1}{2}x^2 + 3x - 1$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = \frac{-1}{2}[2x] + 3[1] - [0] = -x + 3$$

$$\frac{d^2y}{dx^2} = -1[1] + [0] = -1$$

$\frac{dy}{dx}$: Inc | $x=3$ | dec

local Max

$\frac{d^2y}{dx^2}$: C.D.

C.P.:

$$0 = \frac{dy}{dx} = -x + 3$$

$$0 = -x + 3$$

$$x = 3$$

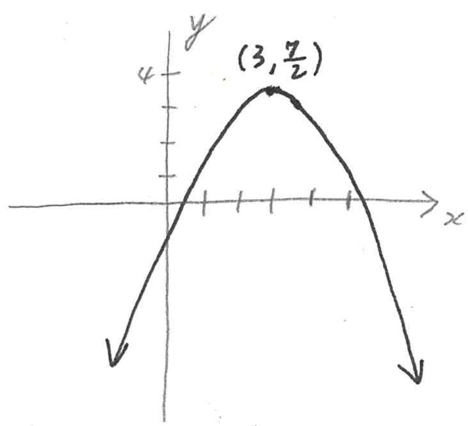
I.P.:

$$0 \neq \frac{d^2y}{dx^2} = -1$$

none

$\Rightarrow$ C.D. on $(-\infty, \infty)$

$$y\Big|_{x=3} = \frac{-1}{2}(3)^2 + 3(3) - 1 = \frac{-9}{2} + 9 - 1 = \frac{-9}{2} + \frac{18}{2} - \frac{2}{2} = \frac{7}{2} \quad \left(3, \frac{7}{2}\right)$$



4) $y = x^3 + 6x^2 + 9x$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = [3x^2] + 6[2x] + 9[1] = 3x^2 + 12x + 9$$

$$\frac{d^2y}{dx^2} = 3[2x] + 12[1] + [0] = 6x + 12$$

4) continued...

C.P.:

$0 = \frac{dy}{dx} = 3x^2 + 12x + 9$

$0 = 3(x^2 + 4x + 3)$

$0 = 3(x+3)(x+1)$

$x + 3 = 0 \;\big|\; x + 1 = 0$

$x = -3 \;\big|\; x = -1$

test at C.P. $x = -3$

$\frac{d^2y}{dx^2}\Big|_{x=-3} = 6(-3) + 12 < 0$ C.D.

local Max

test at C.P. $x = -1$

$\frac{d^2y}{dx^2}\Big|_{x=-1} = 6(-1) + 12 > 0$ C.U.

local min

coordinates:

$y\big|_{x=-3} = (-3)^3 + 6(-3)^2 + 9(-3)$

$= -27 + 54 - 27 = 0$

$(-3, 0)$

$y\big|_{x=-2} = (-2)^3 + 6(-2)^2 + 9(-2)$

$= -8 + 24 - 18 = -2$

$(-2, -2)$

$y\big|_{x=-1} = (-1)^3 + 6(-1)^2 + 9(-1)$

$= -1 + 6 - 9 = -4$

$(-1, -4)$

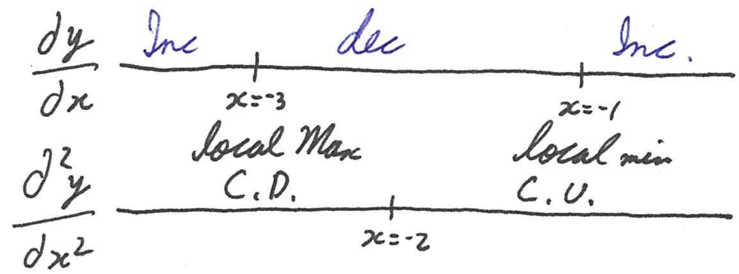
I.P.:

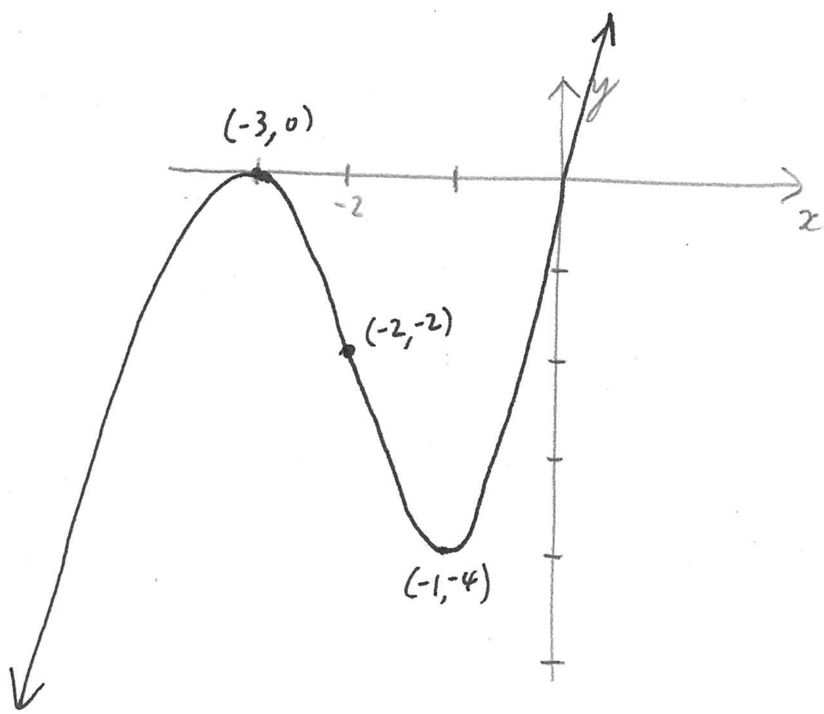
$0 = \frac{d^2y}{dx^2} = 6x + 12$

$0 = 6(x + 2)$

$x + 2 = 0$

$x = -2$





6) $y = 8x^2 - x^4$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = 8[2x] - [4x^3] = 16x - 4x^3$$

$$\frac{d^2y}{dx^2} = 16[1] - 4[3x^2] = 16 - 12x^2$$

C.P.:

$$0 = \frac{dy}{dx} = 16x - 4x^3$$

$$0 = 4x(4 - x^2)$$

$$0 = 4x(2+x)(2-x)$$

$4x = 0$, $x = 0$ | $2 + x = 0$, $x = -2$ | $2 - x = 0$, $2 = x$

I.P.:

$$0 = \frac{d^2y}{dx^2} = 16 - 12x^2$$

$$0 = 4(4 - 3x^2)$$

$$0 = 4(2 + \sqrt{3}\,x)(2 - \sqrt{3}\,x)$$

$2 + \sqrt{3}\,x = 0$, $x = \frac{-2}{\sqrt{3}}$ | $2 - \sqrt{3}\,x = 0$, $\frac{2}{\sqrt{3}} = x$

$\frac{dy}{dx}$: Inc | $x=-2$ local Max | dec | $x=0$ local min | Inc | $x=2$ local Max | dec

$\frac{d^2y}{dx^2}$: C.D. | $x = \frac{-2}{\sqrt{3}}$ | C.U. | $x = \frac{2}{\sqrt{3}}$ | C.D.

test at C.P. $x = -2$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = 16 - 12(-2)^2 = 16 - 12(4) < 0 \text{ C.D. local Max}$$

test at C.P. $x = 0$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 16 - 12(0)^2 = 16 > 0 \text{ C.U. local min}$$

test at C.P. $x = 2$

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 16 - 12(2)^2 = 16 - 12(4) < 0 \text{ C.D. local Max}$$

6) continued...

coordinates

$y|_{x=-2} = 8(-2)^2 - (-2)^4 = 32 - 16 = 16$

$(-2, 16)$

$y|_{x=\frac{-2}{\sqrt{3}}} = 8\left(\frac{-2}{\sqrt{3}}\right)^2 - \left(\frac{-2}{\sqrt{3}}\right)^4 = 8\left(\frac{4}{3}\right) - \frac{16}{9}$

$= \frac{32}{3} - \frac{16}{9} = \frac{96}{9} - \frac{16}{9} = \frac{80}{9}$

$\left(\frac{-2}{\sqrt{3}}, \frac{80}{9}\right)$

$y|_{x=0} = 8(0)^2 - (0)^4 = 0 - 0 = 0$

$(0, 0)$

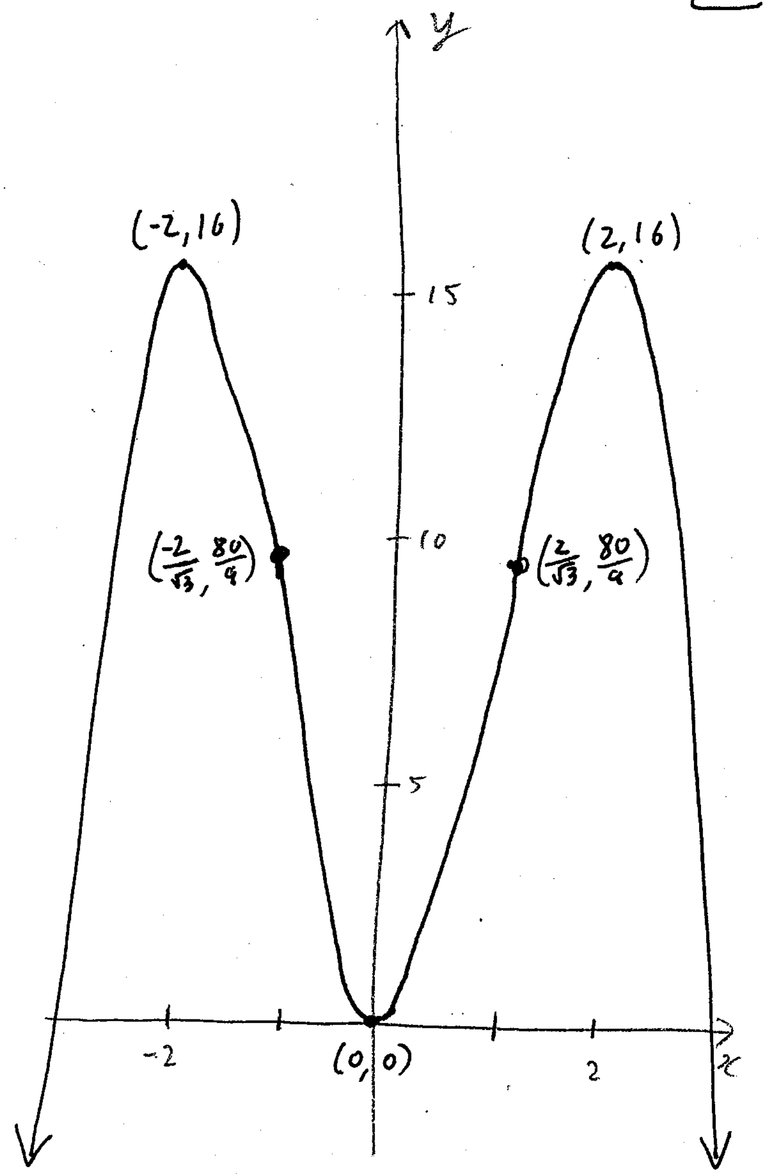
$y|_{x=\frac{2}{\sqrt{3}}} = 8\left(\frac{2}{\sqrt{3}}\right)^2 - \left(\frac{2}{\sqrt{3}}\right)^4 = 8\left(\frac{4}{3}\right) - \frac{16}{9}$

$= \frac{32}{3} - \frac{16}{9} = \frac{96}{9} - \frac{16}{9} = \frac{80}{9}$

$\left(\frac{2}{\sqrt{3}}, \frac{80}{9}\right)$

$y|_{x=2} = 8(2)^2 - (2)^4 = 32 - 16 = 16$

$(2, 16)$



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8) $y = x(x+2)^3$ domain: $(-\infty, \infty)$

$\frac{dy}{dx} = [1]\left((x+2)^3\right) + (x)\left[3(x+2)^2(1)\right]$

$= (x+2)^2\left\{[1](x+2) + (x)[3]\right\} = \left\{x + 2 + 3x\right\}(x+2)^2 = \left\{4x+2\right\}(x+2)^2$

$\frac{d^2y}{dx^2} = [4]\left((x+2)^2\right) + (4x+2)\left[2(x+2)^1(1)\right] = 2(x+2)\left\{[2](x+2) + (4x+2)[1]\right\}$

$= 2(x+2)\left\{2x + 4 + 4x + 2\right\} = 2(x+2)\left\{6x + 6\right\} = 2(x+2)\left\{6(x+1)\right\}$

$= 12(x+2)\left\{x+1\right\}$

8) continued...

C.P.:

$$0 = \frac{dy}{dx} = \{4x+2\}(x+2)^2$$

$$0 = 2\{2x+1\}(x+2)^2$$

$2x+1=0$ $\vdots$ $x = \frac{-1}{2}$ | $(x+2)^2 = 0$, $x+2=0$, $x=-2$

I.P.:

$$0 = \frac{d^2y}{dx^2} = 12(x+2)\{x+1\}$$

$$0 = 12(x+2)\{x+1\}$$

$x+2=0$, $x=-2$ | $x+1=0$, $x=-1$

full test at $x=-3$

$\frac{dy}{dx}$	dec	$x=-2$	dec.	$x=\frac{-1}{2}$ local min	Inc
$\frac{d^2y}{dx^2}$	C.U.	$x=-2$	C.D.	$x=-1$	C.U.

test at C.P. $x=\frac{-1}{2}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{-1}{2}} = 12\left(\left(\tfrac{-1}{2}\right)+2\right)\left\{\left(\tfrac{-1}{2}\right)+1\right\} = 12\left(\tfrac{3}{2}\right)\left\{\tfrac{1}{2}\right\} > 0 \quad \text{C.U. local min}$$

full test at $x=-3$

$$\left.\frac{dy}{dx}\right|_{x=-3} = \{4(-3)+2\}((-3)+2)^2 = \{-10\}(-1)^2 = \{-10\}(1) < 0 \quad \text{decreasing}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-3} = 12((-3)+2)\{(-3)+1\} = 12(-1)(-2) = 24 > 0 \quad \text{C.U.}$$

"this is IP with slope 0"

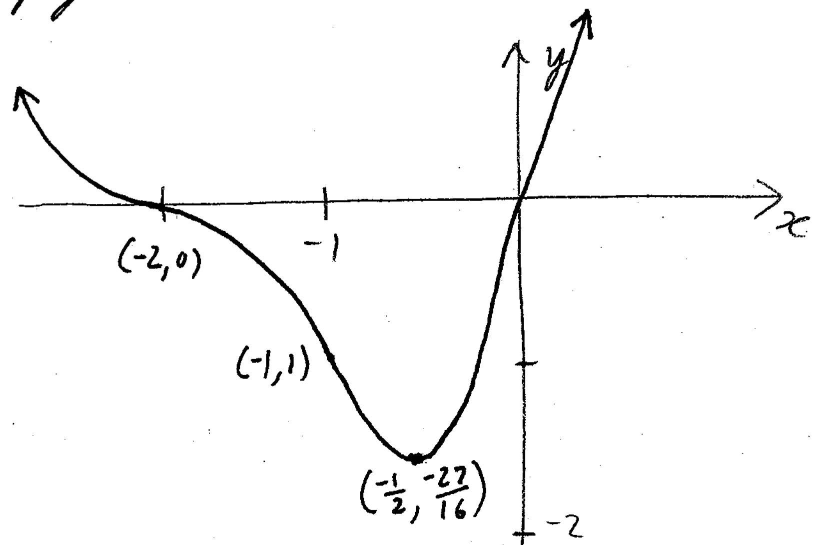
coordinates:

$$y|_{x=-2} = (-2)((-2)+2)^3 = (-2)(0)^3 = 0 \quad (-2,0)$$

$$y|_{x=-1} = (-1)((-1)+2)^3 = (-1)(1)^3 = -1 \quad (-1,-1)$$

$$y|_{x=\frac{-1}{2}} = \left(\tfrac{-1}{2}\right)\left(\left(\tfrac{-1}{2}\right)+2\right)^3 = \left(\tfrac{-1}{2}\right)\left(\tfrac{3}{2}\right)^3 = \frac{-27}{16} \quad \left(\tfrac{-1}{2}, \tfrac{-27}{16}\right)$$

8) continued (2nd page)...



10) $y = \dfrac{x^2-4}{x^2-2x}$

V.A.:

$0 = x^2 - 2x$

$0 = x(x-2)$

$x = 0 \mid x - 2 = 0$

$\quad\;\; x = 2$

$\Rightarrow$ domain: $(-\infty,0)\cup(0,2)\cup(2,\infty)$

we must test these results to verify if it is V.A. or missing point.

$\lim\limits_{x\to 0} \dfrac{x^2-4}{x^2-2x} = \dfrac{-4}{0} \Rightarrow x = 0$ is V.A.

$\lim\limits_{x\to 2} \dfrac{\overset{0}{x^2-4}}{\underset{0}{x^2-2x}} = \dfrac{0}{0} \Rightarrow x = 2$ is not V.A. it is a missing point

" $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-2x} = \lim\limits_{x\to 2} \dfrac{(x+2)(x-2)}{x(x-2)} = \lim\limits_{x\to 2} \dfrac{x+2}{x} = \dfrac{(2)+2}{(2)} = \dfrac{4}{2} = 2$ "

H.A.: $\lim\limits_{x\to\infty} \dfrac{x^2-4}{x^2-2x} = \lim\limits_{x\to\infty} \dfrac{\frac{x^2}{x^2} - \frac{4}{x^2}}{\frac{x^2}{x^2} - \frac{2x}{x^2}} = \lim\limits_{x\to\infty} \dfrac{1-\frac{4}{x^2}}{1-\frac{2}{x}} = \dfrac{1-0}{1-0} = 1$

$y = 1$ is H.A.

10) continued...

we should simplify our function in order to have easier derivatives

$$y = \frac{x^2-4}{x^2-2x} = \frac{(x+2)(x-2)}{x(x-2)} = \frac{x+2}{x}$$ where $x \neq 2$ and

domain: $(-\infty,0)\cup(0,2)\cup(2,\infty)$

now find derivatives on easier expression.

$$y = \frac{x+2}{x} = \frac{x}{x} + \frac{2}{x} = 1 + \frac{2}{x} = 1 + 2x^{-1}$$

$$\frac{dy}{dx} = [0] + 2[-1x^{-2}] = -2x^{-2} = \frac{-2}{x^2}$$

$$\frac{d^2y}{dx^2} = -2[-2x^{-3}] = \frac{4}{x^3}$$

C.P.:

$$0 = \frac{dy}{dx} = \frac{-2}{x^2}$$

none

full test at $x=-1$

I.P.:

$$0 = \frac{d^2y}{dx^2} = \frac{4}{x^3}$$

none

full test at $x=1$

"should be same as $x=1$"

full test at $x=3$

$\frac{dy}{dx}$ — dec) 0 (dec) 2 (dec

$\frac{d^2y}{dx^2}$ — C.D) 0 (C.U.) 2 (C.U.

10) continued (2nd page)...

full test at $x=-1$

$$\left.\frac{dy}{dx}\right|_{x=-1} = \frac{-2}{(-1)^2} = \frac{-2}{1} = -2 < 0 \text{ decreasing} \qquad y|_{x=-1} = \frac{(-1)+2}{(-1)} = \frac{1}{-1} = -1$$

$(-1,-1)$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = \frac{4}{(-1)^3} = \frac{4}{-1} = -4 < 0 \text{ C.D.}$$

full test at $x=1$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{-2}{(1)^2} = \frac{-2}{1} = -2 < 0 \text{ decreasing} \qquad y|_{x=1} = \frac{(1)+2}{(1)} = \frac{3}{1} = 3$$

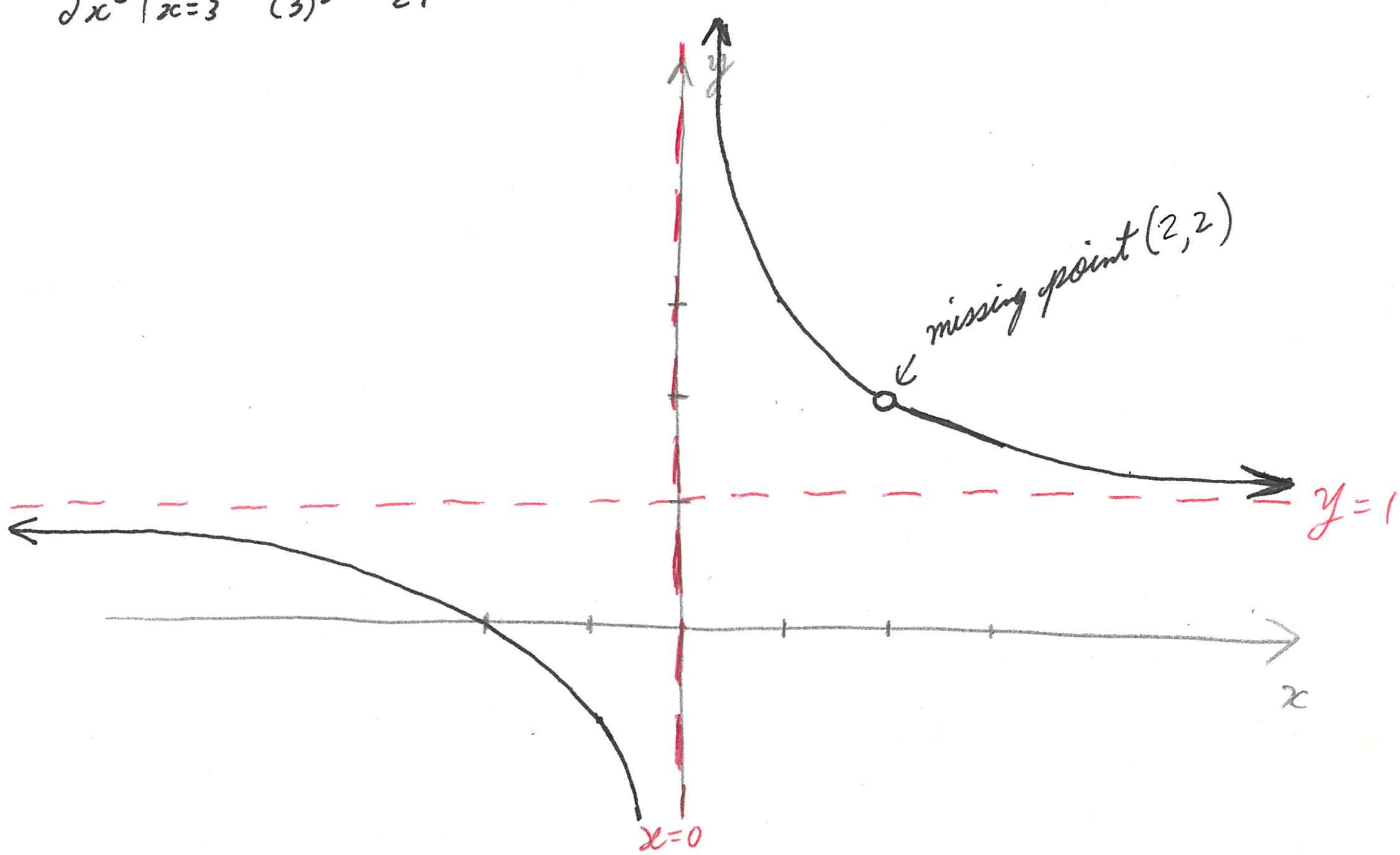
$(1,3)$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{4}{(1)^3} = \frac{4}{1} = 4 > 0 \text{ C.U.}$$

full test at $x=3$

$$\left.\frac{dy}{dx}\right|_{x=3} = \frac{-2}{(3)^2} = \frac{-2}{9} < 0 \text{ decreasing}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=3} = \frac{4}{(3)^3} = \frac{4}{27} > 0 \text{ C.U.}$$



12) $y = \dfrac{x^2}{x^2+9}$

V.A.:

$0 = x^2+9$ $\Longrightarrow$ domain: $(-\infty, \infty)$ and no V.A.

no solution

H.A.: $\lim\limits_{x\to\infty} \dfrac{x^2}{x^2+9} = \lim\limits_{x\to\infty} \dfrac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{9}{x^2}} = \lim\limits_{x\to\infty} \dfrac{1}{1+\frac{9}{x^2}} = \dfrac{1}{1+0} = \dfrac{1}{1} = 1$

$y=1$ is H.A.

$$\frac{dy}{dx} = \frac{[2x](x^2+9)-(x^2)[2x]}{(x^2+9)^2} = \frac{2x^3+18x-2x^3}{(x^2+9)^2} = \frac{18x}{(x^2+9)^2}$$

$$\frac{d^2y}{dx^2} = \frac{[18]((x^2+9)^2)-(18x)[2(x^2+9)(2x)]}{((x^2+9)^2)^2}$$

$$= \frac{18(x^2+9)\{[1](x^2+9)-(x)[2(2x)]\}}{(x^2+9)^4}$$

$$= \frac{18\{(x^2+9)-4x^2\}}{(x^2+9)^3} = \frac{18\{9-3x^2\}}{(x^2+9)^3} = \frac{18\{3(3-x^2)\}}{(x^2+9)^3} = \frac{54(3-x^2)}{(x^2+9)^3}$$

C.P.:

$0 = \dfrac{dy}{dx} = \dfrac{18x}{(x^2+9)^2}$

$0 = 18x$

$x = 0$

I.P.:

$0 = \dfrac{d^2y}{dx^2} = \dfrac{54(3-x^2)}{(x^2+9)^3}$

$0 = 54(3-x^2)$

$0 = 54(\sqrt{3}+x)(\sqrt{3}-x)$

$\sqrt{3}+x=0$	$\sqrt{3}-x=0$
$x=-\sqrt{3}$	$\sqrt{3}=x$

12) continued...

$\frac{dy}{dx}$	dec	$x=0$ local min	Inc.	
$\frac{d^2y}{dx^2}$	C.D.	$x=-\sqrt{3}$	C.U.	$x=\sqrt{3}$	C.D.

test at C.P. $x=0$

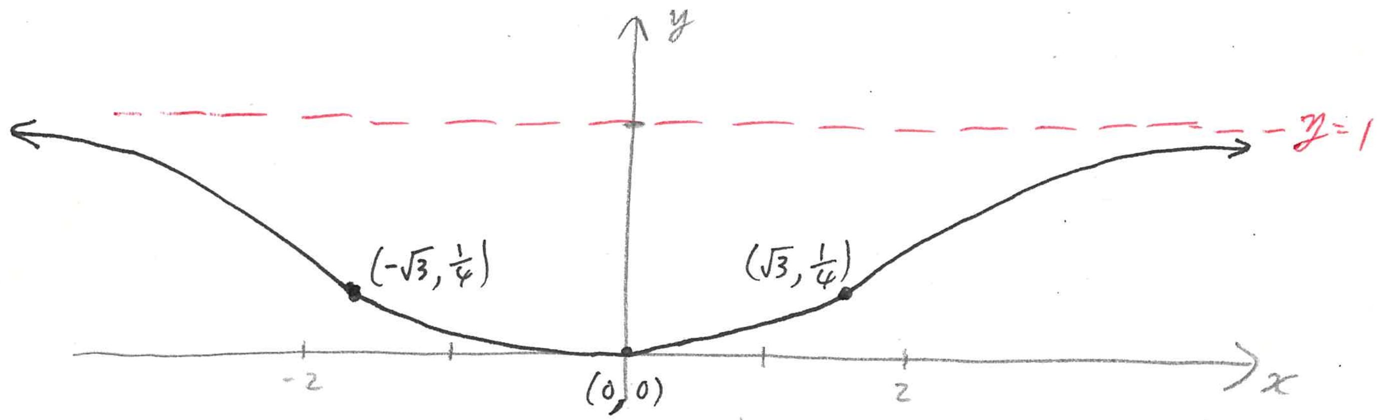
$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = \frac{54(3-(0)^2)}{((0)^2+9)^3} = \frac{54(3)}{(9)^3} > 0 \quad \text{C.U. local min}$$

coordinates:

$$y\Big|_{x=-\sqrt{3}} = \frac{(-\sqrt{3})^2}{(-\sqrt{3})^2+9} = \frac{3}{3+9} = \frac{3}{12} = \frac{1}{4} \quad \left(-\sqrt{3}, \tfrac{1}{4}\right)$$

$$y\Big|_{x=0} = \frac{(0)^2}{(0)^2+9} = \frac{0}{9} = 0 \quad (0,0)$$

$$y\Big|_{x=\sqrt{3}} = \frac{(\sqrt{3})^2}{(\sqrt{3})^2+9} = \frac{3}{3+9} = \frac{3}{12} = \frac{1}{4} \quad \left(\sqrt{3}, \tfrac{1}{4}\right)$$



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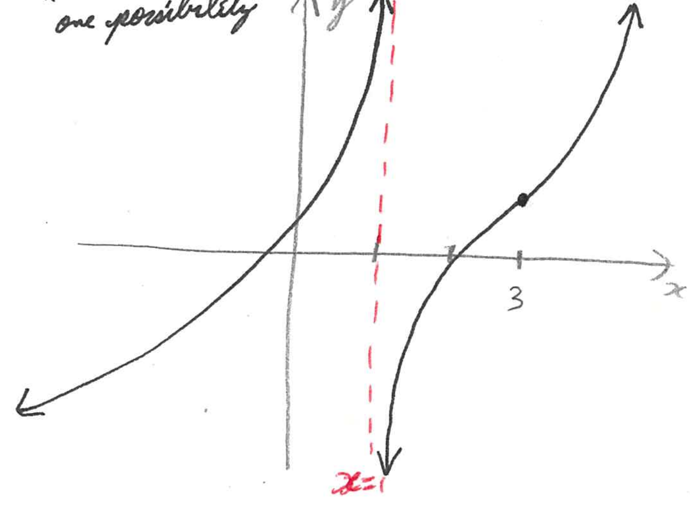
22) V.A.: $x=1$

$\frac{df}{dx}$	$f'(x)>0$ Inc	$f'(x)>0$ Inc	
$\frac{d^2f}{dx^2}$	$f''(x)>0$ C.U.	$f''(x)<0$ C.D.	$x=3$	$f''(x)>0$ C.U.

"one possibility"

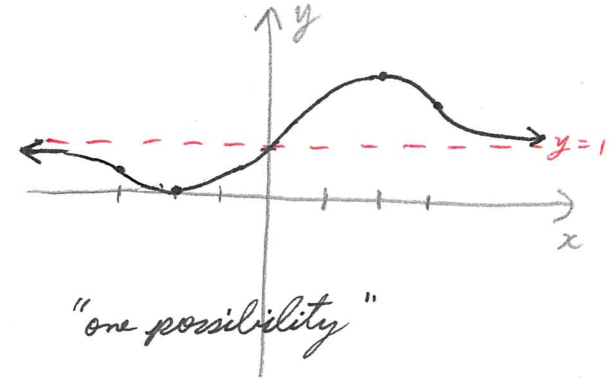


24) $\lim_{x\to\infty} f(x)=1$, $\lim_{x\to-\infty} f(x)=1 \Rightarrow$ H.A.: $y=1$

$f'(2)=0 \Rightarrow$ C.P. at $x=2$

$\frac{df}{dx}$: $f'(x)<0$ dec | $x=-2$ | $f'(x)>0$ Inc | $x=2$ | $f'(x)<0$ dec

$\frac{d^2f}{dx^2}$: C.D. | $x=-3$ | C.U. | $x=0$ | $f''(x)<0$ C.D. | $x=3$ | $f''(x)>0$ C.U.



"one possibility"

condition for $\frac{d^2f}{dx^2}$ in the interval $x<0$ is not given and I added them in order to make a continuous graph

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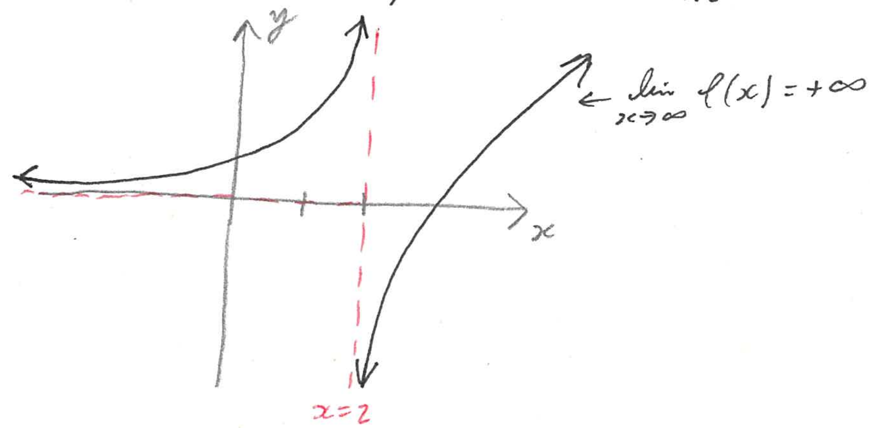
26) $\frac{df}{dx}$: $f'(x)>0$ Inc) 2 ($f'(x)>0$ Inc

$\frac{d^2f}{dx^2}$: C.U.) 2 (C.D.

$\lim_{x\to-\infty} f(x)=0$ H.A. to the left $y=0$

$\left.\begin{array}{l}\lim_{x\to2^-} f(x)=+\infty\\ \lim_{x\to2^+} f(x)=-\infty\end{array}\right\}$ V.A. at $x=2$



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28) $\frac{df}{dx}$: $f'(x)<0$ dec

$\frac{d^2f}{dx^2}$: $f''(x)>0$ C.U.