section 5.6

2) polynomial functions are simpler and does not contain any asymptotes.
rational function has/have asymptote(s).
if there is only 1, then it is a horizontal or slanted asymtote type.
rational function may not have a vertical asymptote.

---

4) when a rational function with no vertical asymptote will have a denominator that is not equal 0 for all real numbers.

Given $R(x) = \frac{p(x)}{q(x)}$

if degree of $p(x)$ > degree of $q(x)$, then $R(x)$ has slanted or higher order asymptotes.
if degree of $p(x)$ = degree of $q(x)$, then $R(x)$ has a horizontal asymptote and see technique shown in this section.
if degree of $p(x)$ < degree of $q(x)$, then $R(x)$ has a horizontal asymptote $y=0$.

6) $f(x) = \frac{x-1}{x+2}$

V.A.? $x+2=0$
$x=-2$ $\rightarrow$ domain: $\underline{(-\infty,-2) \cup (-2,\infty)}$

---

8) $f(x) = \frac{x^2+4}{x^2-2x-8}$

V.A.? $x^2-2x-8=0$

$(x+2)(x-4)=0$ $\rightarrow$ domain: $\underline{(-\infty,-2) \cup (-2,4) \cup (4,\infty)}$

$x+2=0$ | $x-4=0$

$x=-2$ | $x=4$

---

10) $f(x) = \frac{4}{x-1}$

H.A.? degree(4) < degree $(x-1)$ $\quad$ $\underline{\text{H.A.: } y=0}$

V.A.? $x-1=0$
$x=1$ $\rightarrow$ $\underline{\text{V.A.: } x=1}$ $\rightarrow$ domain: $\underline{(-\infty,1) \cup (1,\infty)}$

---

12) $f(x) = \frac{x}{x^2-9}$

H.A.? degree(x) < degree(x^2-9) $\quad$ $\underline{\text{H.A.: } y=0}$

V.A.? $x^2-9=0$

$(x+3)(x-3)=0$ $\rightarrow$ $\underline{\text{V.A.: } x=3 \text{ and } x=3}$

$x+3=0$ | $x-3=0$

$x=-3$ | $x=3$

domain: $\underline{(-\infty,-3) \cup (-3,3) \cup (3,\infty)}$

14) $f(x) = \dfrac{3+x}{x^3-27}$

H.A.? degree$(3+x) <$ degree(x^3-27) $\qquad$ H.A.: $y=0$

V.A.? $x^3-27=0$

$x^3-(3)^3$

$(x-3)\underbrace{(x^2+3x+9)}_{\text{not factorable}}=0$

$x-3=0$

$x=3$

V.A.: $x=3$

domain: $(-\infty,3)\cup(3,\infty)$

---

16) $f(x) = \dfrac{x^2-1}{x^3+9x^2+14x}$

H.A.? degree$(x^2-1) <$ degree(x^3+9x^2+14x) $\qquad$ H.A.: $y=0$

V.A.? $x^3+9x^2+14x=0$

$x(x^2+9x+14)=0$

$x\ (x+7)(x+2)=0$

$x+7=0$ | $x+2=0$ | $x=0$

$x=-7$ | $x=-2$ |

V.A.: $x=-7,\ x=-2,\ x=0$

domain: $(-\infty,-7)\cup(-7,-2)\cup(-2,0)\cup(0,\infty)$

---

18) $f(x) = \dfrac{x-4}{x-6}$

H.A.? degree$(x-4)=$ degree$(x-6)$

$\dfrac{x-4}{x-6} = \dfrac{\frac{x}{x}-\frac{4}{x}}{\frac{x}{x}-\frac{6}{x}} = \dfrac{1-\frac{4}{x}}{1-\frac{6}{x}}$ as $x\to\infty$ $\dfrac{1-(0)}{1-(0)} = \dfrac{1}{1} = 1$ $\qquad$ H.A.: $y=1$

V.A.? $x-6=0$

$x=6$ $\rightarrow$ V.A.: $x=6$ $\qquad$ domain: $(-\infty,6)\cup(6,\infty)$

20) $f(x) = \frac{x+5}{x^2+4}$

y-int: $f(0) = \frac{(0)+5}{(0)^2+4} = \frac{5}{4}$ $\quad \underline{\underline{(0, \frac{5}{4})}}$

x-int: $0 = f(x) = \frac{x+5}{x^2+4}$

$0 = x+5$ $\quad \underline{\underline{(-5, 0)}}$

$-5 = x$

---

22) $f(x) = \frac{x^2+8x+7}{x^2+11x+30}$

y-int: $f(0) = \frac{(0)^2+8(0)+7}{(0)^2+11(0)+30} = \frac{7}{30}$ $\quad \underline{\underline{(0, \frac{7}{30})}}$

x-int: $0 = f(x) = \frac{x^2+8x+7}{x^2+11x+30}$

$0 = x^2+8x+7$

$0 = (x+7)(x+1)$

$x+7=0$	$x+1=0$
$x=-7$	$x=-1$
$\underline{\underline{(-7, 0)}}$	$\underline{\underline{(-1, 0)}}$

---

24) $f(x) = \frac{94-2x^2}{3x^2-12}$

y-int: $f(0) = \frac{94-2(0)^2}{3(0)^2-12} = \frac{94}{-12} = \frac{-47}{6}$ $\quad \underline{\underline{(0, \frac{-47}{6})}}$

x-int: $0 = f(x) = \frac{94-2x^2}{3x^2-12}$

$0 = 94-2x^2$

$2x^2-94=0$

$2(x^2-47)=0$

$2(x^2-(\sqrt{47})^2)=0$

$2(x+\sqrt{47})(x-\sqrt{47})=0$

$x+\sqrt{47}=0$	$x-\sqrt{47}=0$
$x=-\sqrt{47}$ $\quad \underline{\underline{(-\sqrt{47}, 0)}}$	$x=\sqrt{47}$ $\quad \underline{\underline{(\sqrt{47}, 0)}}$

26) $f(x) = \frac{2x}{x-6}$

H.A.? degree $(2x)$ = degree $(x-6)$

$$\frac{2x}{x-6} = \frac{\frac{2x}{x}}{\frac{x}{x}-\frac{6}{x}} = \frac{2}{1-\frac{6}{x}} \text{ as } x \to \infty \quad \frac{2}{1-(0)} = \frac{2}{1} = 2 \quad \text{H.A. } y=1$$

end behavior: $\underline{\text{as } x \to \pm\infty,\ f(x) = 1}$

V.A.? $x-6=0$ V.A.: $x=6$ and $(x-6)$ has odd power

$x = 6$

test at $x=7$, $f(7) = \frac{2(7)}{(7)-6} = \frac{14}{1} = 14 > 1 \to$ on right side of V.A. $x=6$, $f(x) > 1$

local behavior: $\underline{\text{as } x \to 6^+,\ f(x) \to +\infty}$

since V.A. $x=6$ comes from $(x-6)$ with odd power,

$\underline{\text{as } x \to 6^-,\ f(x) \to -\infty}$

---

28) $f(x) = \frac{x^2-4x+3}{x^2-4x-5}$

H.A.? degree (x^2-4x+3) = degree (x^2-4x-5)

$$\frac{x^2-4x+3}{x^2-4x-5} = \frac{\frac{x^2}{x^2}-\frac{4x}{x^2}+\frac{3}{x^2}}{\frac{x^2}{x^2}-\frac{4x}{x^2}-\frac{5}{x^2}} = \frac{1-\frac{4}{x}+\frac{3}{x^2}}{1-\frac{4}{x}+\frac{5}{x^2}} \text{ as } x \to \infty \quad \frac{1-(0)+(0)}{1-(0)-(0)} = \frac{1}{1} = 1$$

H.A.: $y=1$

end behavior: $\underline{\text{as } x \to \pm\infty,\ f(x) = 1}$

28) continued...

V.A.? $x^2-4x-5=0$

$(x+1)(x-5)=0$ V.A.: $x=-1$ and $x=5$

$x+1=0$ | $x-5=0$ both $(x+1)$ and $(x-5)$ have odd power

$x=-1$ | $x=5$

test at $x=6$: $f(6)=\dfrac{(6)^2-4(6)+3}{(6)^2-4(6)-5}=\dfrac{36-24+3}{36-24-5}=\dfrac{15}{7}>1$

on right side of V.A. $x=5$, $f(x)>1$

---

test at $x=-2$: $f(-2)=\dfrac{(-2)^2-4(-2)+3}{(-2)^2-4(-2)-5}=\dfrac{4+8+3}{4+8-5}=\dfrac{15}{7}>1$

on left side of V.A. $x=-1$, $f(x)>1$

---

local behavior: as $x\to 5^+$, $f(x)\to+\infty$

since V.A. $x=5$ comes from $(x-5)$ with odd power,

as $x\to 5^-$, $f(x)\to-\infty$

as $x\to -1^-$, $f(x)\to+\infty$

since V.A. $x=-1$ comes from $(x+1)$ with odd power

as $x\to -1^+$, $f(x)\to-\infty$

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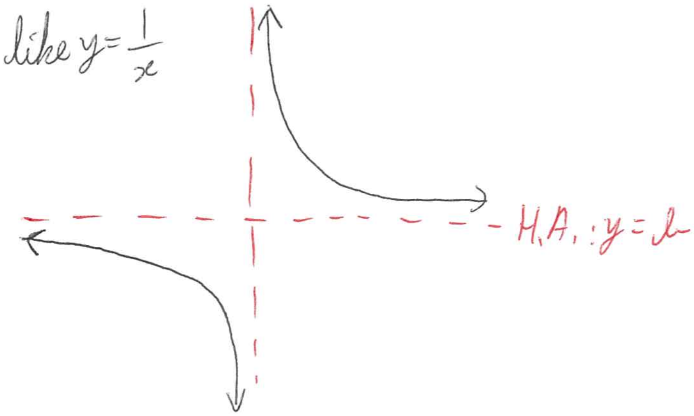
exercises 30-38 are skipped

like $y = \frac{1}{x}$



V.A.: $x = a$

$(x-a) = 0$

or

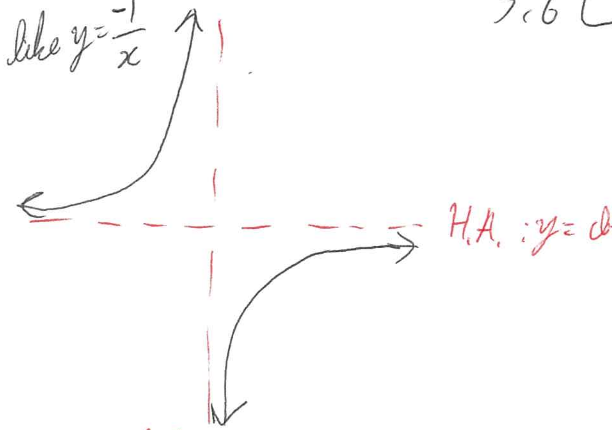
$(x-a)^3 = 0$

or

$(x-a)^n = 0$

n - odd

like $y = \frac{-1}{x}$



V.A.: $x = a$

$(x-a) = 0$

or

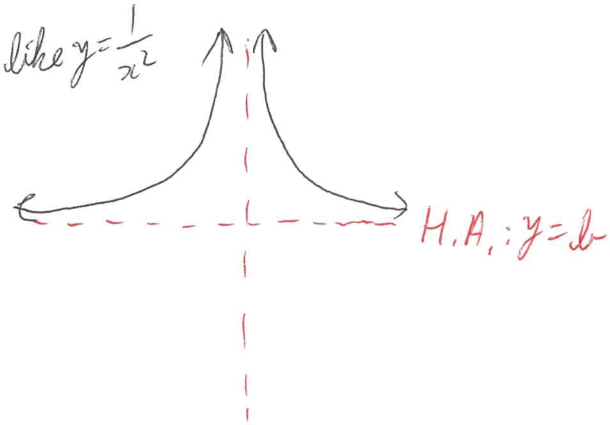
$(x-a)^3 = 0$

or

$(x-a)^n = 0$

n - odd

like $y = \frac{1}{x^2}$



V.A.: $x = a$

$(x-a)^2 = 0$

or

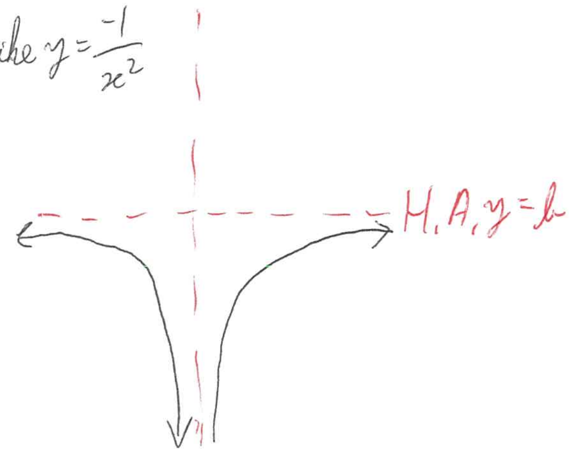
$(x-a)^4 = 0$

or

$(x-a)^n = 0$

n - even

like $y = \frac{-1}{x^2}$



V.A.; $x = a$

$(x-a)^2 = 0$

or

$(x-a)^4 = 0$

or

$(x-a)^n = 0$

n - even

Always use test point on the region containg $+\infty$ or $-\infty$ to see if the graph is above or below H.A. $y = b$ (99% accuracy)

40) $q(x) = \dfrac{x-5}{3x-1}$

H.A.? degree $(x-5)$ = degree $(3x-1)$

$$\frac{x-5}{3x-1} = \frac{\frac{x}{x} - \frac{5}{x}}{\frac{3x}{x} - \frac{1}{x}} = \frac{1 - \frac{5}{x}}{3 - \frac{1}{x}} \quad \text{as } x \to \infty \quad \frac{1-(0)}{3-(0)} = \frac{1}{3} \quad \text{H.A.: } y = \frac{1}{3}$$

V.A.? $3x-1=0$ $\quad (3x-1)$ odd power

$3x = 1$ $\quad$ V.A.: $x = \frac{1}{3}$ $\quad$ domain: $(-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$

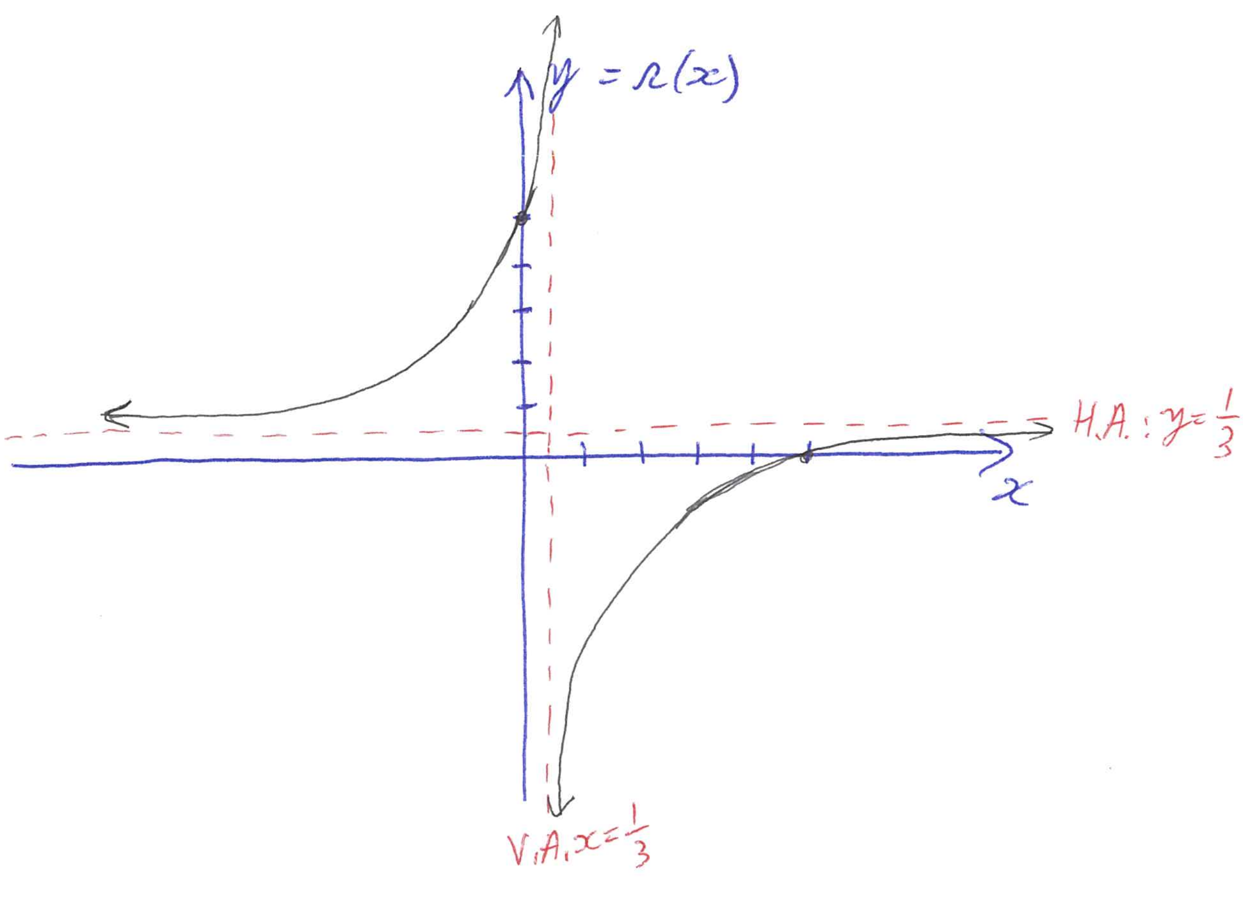
$x = \frac{1}{3}$

y-int: $q(0) = \dfrac{(0)-5}{3(0)-1} = \dfrac{-5}{-1} = 5$ $\quad (0, 5)$

x-int: $0 = q(x) = \dfrac{x-5}{3x-1}$

$0 = x - 5$ $\quad (5, 0)$

$5 = x$



42) $r(x) = \frac{5}{(x+1)^2}$

H.A.? degree $(5) <$ degree $((x+1)^2) \rightarrow$ H.A.: $y = 0$

V.A.? $(x+1)^2 = 0$ even power

$x+1=0$

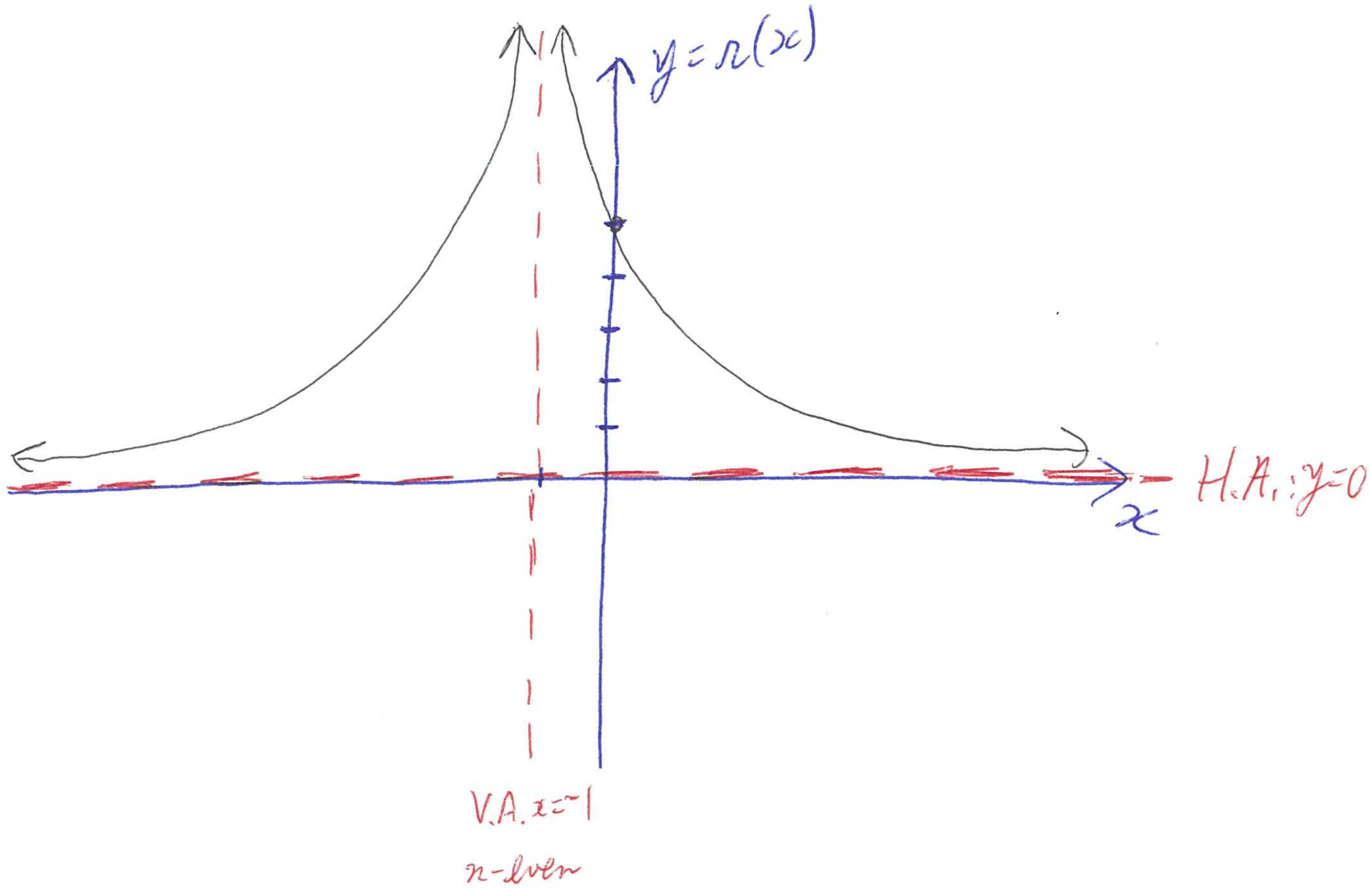
$x=-1$ V.A.: $x=-1$

domain: $(-\infty,-1) \cup (-1,\infty)$

y-int: $r(0) = \frac{5}{((0)+1)^2} = \frac{5}{(1)^2} = 5$ $(0,5)$

x-int: $0 = r(x) = \frac{5}{(x+1)^2}$

$0 \neq 5 \rightarrow$ no x-intercept



44) $g(x) = \dfrac{2x^2+7x-15}{3x^2-14x+15}$

H.A.? degree$(2x^2+7x-15)$ = degree$(3x^2-14x+15)$

$$\frac{2x^2+7x-15}{3x^2-14x+15} = \frac{\frac{2x^2}{x^2}+\frac{7x}{x^2}-\frac{15}{x^2}}{\frac{3x^2}{x^2}-\frac{14x}{x^2}+\frac{15}{x^2}} = \frac{2+\frac{7}{x}-\frac{15}{x^2}}{3-\frac{14}{x}+\frac{15}{x^2}} \text{ as } x\to\infty \; \frac{2+(0)-(0)}{3-(0)+(0)} = \frac{2}{3}$$

H.A.: $y = \frac{2}{3}$

V.A.? $3x^2-14x+15=0$

$(3x-5)(x-3)=0$

$3x-5=0$	$x-3=0$
$3x=5$	$x=3$
$x=\frac{5}{3}$	

odd powers

V.A.: $x=\frac{5}{3}$, $x=3$

domain: $(-\infty,\frac{5}{3})\cup(\frac{5}{3},3)\cup(3,\infty)$

y-int: $g(0)=\dfrac{2(0)^2+7(0)-15}{3(0)^2-14(0)+15} = \dfrac{-15}{15} = -1$ $(0,-1)$

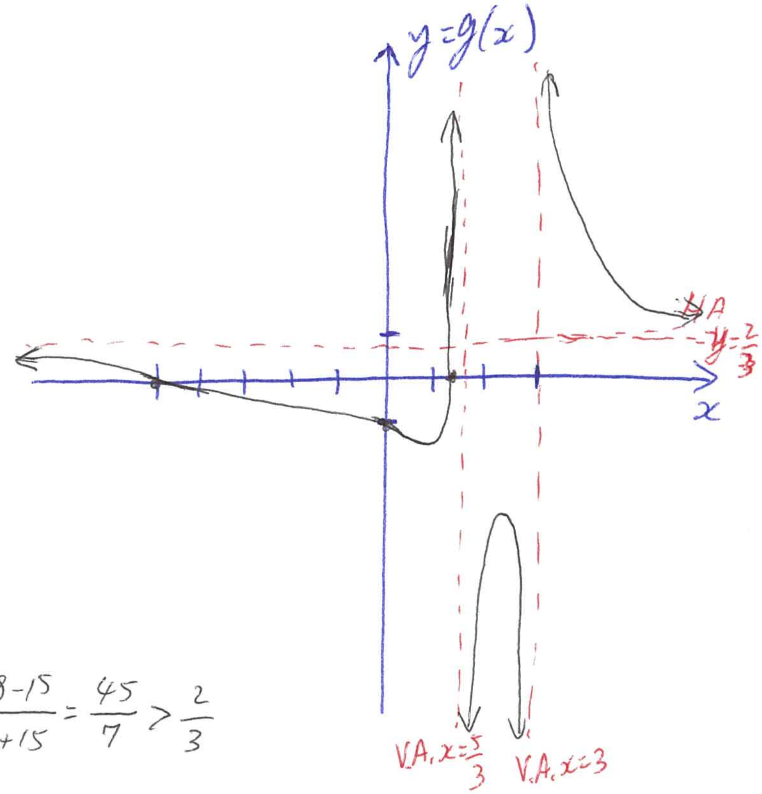
x-int: $0 = g(x) = \dfrac{2x^2+7x-15}{3x^2-14x+15}$

$0 = 2x^2+7x-15$

$0 = (x+5)(2x-3)$

odd powers (multiplicity +1)

$x+5=0$	$2x-3=0$
$x=-5$	$2x=3$
	$x=\frac{3}{2}$
$(-5,0)$	$(\frac{3}{2},0)$



test pt $x=4$: $g(4) = \dfrac{2(4)^2+7(4)-15}{3(4)^2-14(4)+15} = \dfrac{32+28-15}{48-56+15} = \dfrac{45}{7} > \dfrac{2}{3}$

46) $b(x) = \frac{x^2-x-6}{x^2-4}$

H.A.? degree(x^2-x-6) = degree(x^2-4)

$$\frac{x^2-x-6}{x^2-4} = \frac{\frac{x^2}{x^2}-\frac{x}{x^2}-\frac{6}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}} = \frac{1-\frac{1}{x}-\frac{6}{x^2}}{1-\frac{4}{x^2}} \text{ as } x\to\infty \ \frac{1-(0)-(0)}{1-(0)} = \frac{1}{1} = 1$$

H.A.: $y=1$

V.A.? $x^2-4=0$

$(x+2)(x-2)=0$

n - odd

$x+2=0$ | $x-2=0$

$x=-2$ | $x=2$

V.A.: $x=-2$ and $x=2$

discard same as x-int

domain: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

y-int: $b(0) = \frac{(0)^2-(0)-6}{(0)^2-4} = \frac{-6}{-4} = \frac{3}{2}$ $(0,\frac{3}{2})$

x-int: $0 = b(x) = \frac{x^2-x-6}{x^2-4}$

$0 = x^2-x-6$

$0 = (x+2)(x-3)$

n - odd

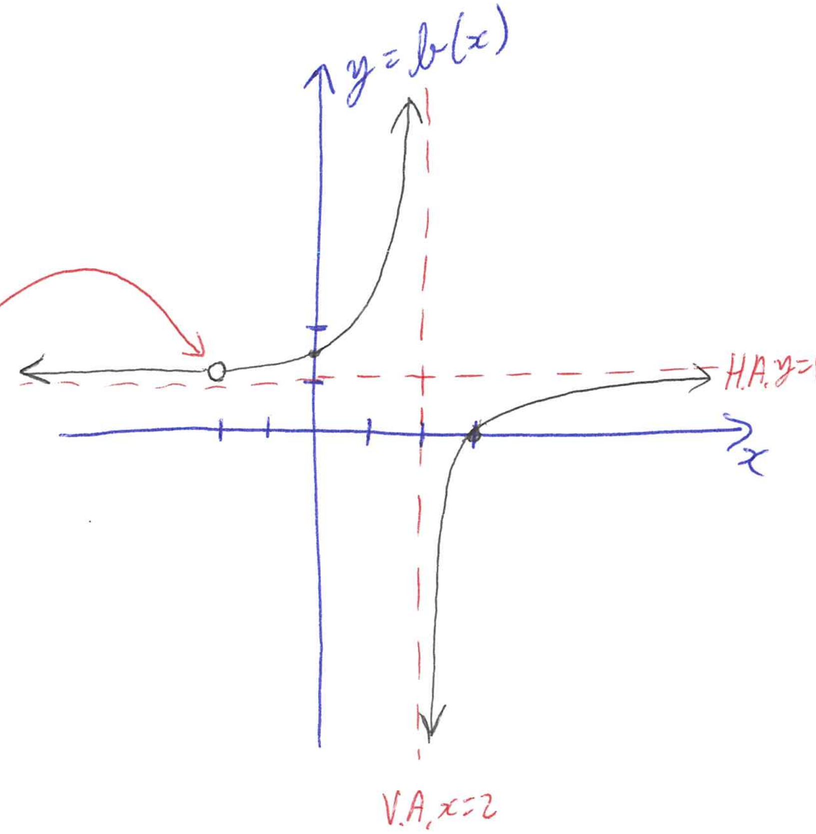
$x+2=0$ | $x-3=0$

$x=-2$ | $x=3$

$(-2,0)$ | $(3,0)$

discard same as V.A.

this is an empty plot (point)



ex. 47, 48 skipped (has slanted asymptotes)

50) $g(x) = \dfrac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

H.A.? degree (top) = degree (bottom)

$\dfrac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)} = \dfrac{\text{leading term: } x^3}{\text{leading term: } x^3}$ as $x \to \infty$ $\dfrac{1+0}{1+0} = \dfrac{1}{1} = 1$

H.A.: $y = 1$

V.A.? $(x-3)(x+1)(x+4) = 0$ odd powers

$x+4=0$	$x+1=0$	$x-3=0$
$x=-4$	$x=-1$	$x=3$

V.A.: $x=-4$, $x=-1$, $x=3$

domain: $(-\infty,-4)\cup(-4,-1)\cup(-1,3)\cup(3,\infty)$

y-int: $g(0) = \dfrac{((0)+2)^2((0)-5)}{((0)-3)((0)+1)((0)+4)} = \dfrac{(4)(-5)}{(-3)(1)(4)} = \dfrac{-5}{-3} = \dfrac{5}{3}$ $\quad (0, \frac{5}{3})$

x-int: $0 = g(x) = \dfrac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

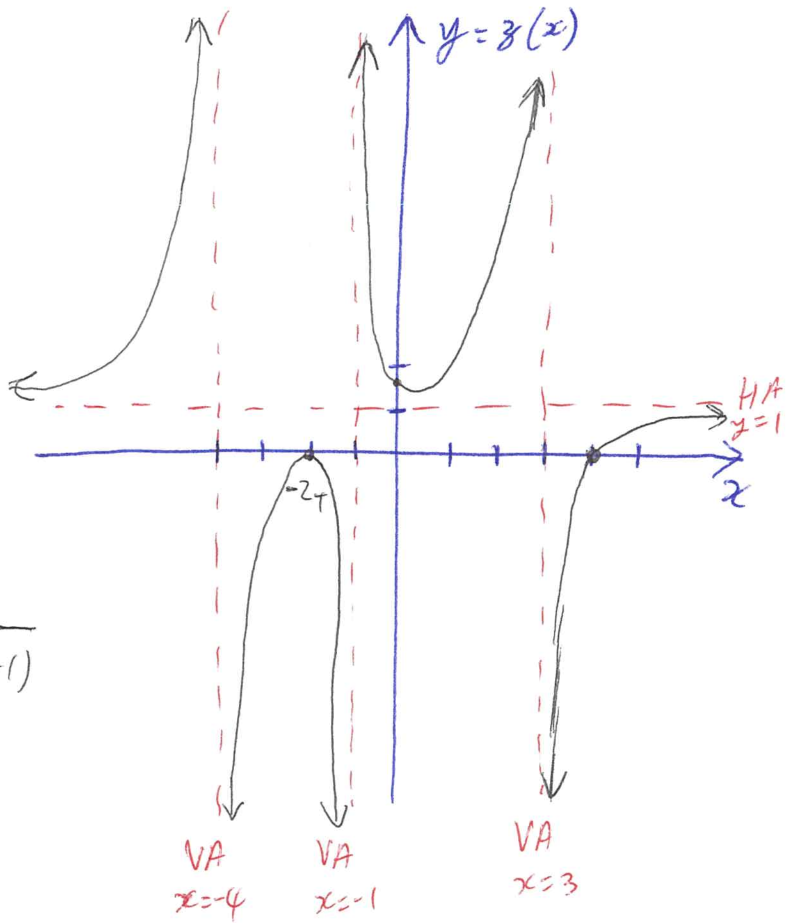
$0 = (x+2)^2(x-5)$

n-even	n-odd
$(x+2)^2=0$	$x-5=0$
$x+2=0$	$x=5$
$x=-2_T$	

test pt at $x=-5$

$g(-5) = \dfrac{((-5)+2)^2((-5)-5)}{((-5)-3)((-5)+1)((-5)+4)} = \dfrac{(-3)^2(-7)}{(-8)(-4)(-1)}$

$= \dfrac{-63}{-32} = \dfrac{63}{32} > 1$



52) V.A.: $x=-4 \rightarrow x+4=0$; $x=-1 \rightarrow x+1=0$

$(x+4)(x+1)$ on denominator

x-int: $(1,0) \rightarrow x=1 \rightarrow x-1=0$; $(5,0) \rightarrow x=5 \rightarrow x-5=0$

$(x-1)(x-5)$ on numerator

let $R(x) = a\dfrac{(x-1)(x-5)}{(x+4)(x+1)}$ y-int: $(0,7)$

$$(7) = R(0) = a\frac{((0)-1)((0)-5)}{((0)+4)((0)+1)}$$

we need to find a that fits y-int given.

$$7 = a\frac{5}{4}$$

$$a = \frac{7(4)}{5} = \frac{28}{5}$$

$$\underline{\underline{R(x) = \left(\frac{28}{5}\right)\left(\frac{(x-1)(x-5)}{(x+4)(x+1)}\right)}}$$

---

54) V.A.: $x=-3 \rightarrow x+3=0$; $x=6 \rightarrow x-6=0$

$(x+3)(x-6)$ on denominator

x-int: $(-2,0) \rightarrow x=-2 \rightarrow x+2=0$; $(1,0) \rightarrow x=1 \rightarrow x-1=0$

$(x+2)(x-1)$ on numerator

let $R(x) = a\dfrac{(x+2)(x-1)}{(x+3)(x-6)}$ H.A.: $y=-2$

$$= a\frac{(x^2+x-2)}{(x^2-3x-18)}$$

degree (x^2+x-2) = degree $(x^2-3x-18)$

54) continued...

$$a\frac{x^2+x-2}{x^2-3x-18} = a\frac{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{2}{x^2}}{\frac{x^2}{x^2}-\frac{3x}{x^2}-\frac{18}{x^2}} = a\frac{1+\frac{1}{x}-\frac{2}{x^2}}{1-\frac{3}{x}-\frac{18}{x^2}} \text{ as } x\to\infty \quad a\frac{1+(0)-2(0)}{1-(0)-(0)} = a$$

we need to find a that fits H.A. given H.A: $y=a$

$a = y = -2$

$$R(x) = -2\frac{(x+2)(x-1)}{(x+3)(x-6)}$$

---

56) V.A.: $x=3 \to x-3=0$ $(x-3)$ on denominator

double zero at $x=2 \to x=2_T \to x-2=0 \to (x-2)^2=0$

$(x-2)^2$ on numerator

let $R(x) = a\frac{(x-2)^2}{x-3}$ y-int: $(0,4)$

we need to find a that fits y-int given

$(4) = R(0) = a\frac{((0)-2)^2}{(0)-3}$

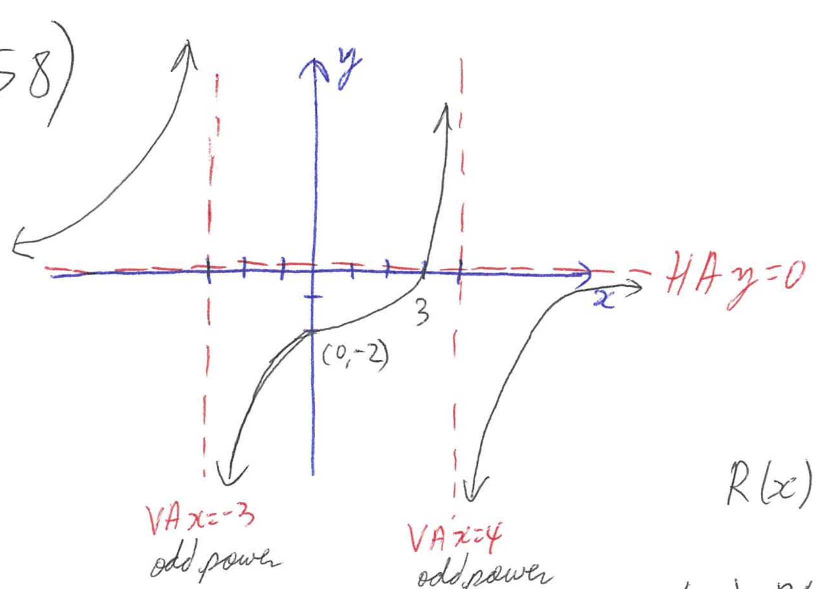
$4 = a\frac{(-2)^2}{-3}$

$4 = a\frac{4}{-3}$

$-3 = a$

$$R(x) = (-3)\frac{(x-2)^2}{(x-3)}$$

58)



V.A.
$x=-3 \rightarrow x+3=0$
$x=4 \rightarrow x-4=0$

$(x+3)(x-4)$ on denominator

x-int: $x=3 \rightarrow x-3=0$

$(x-3)$ on numerator

$R(x)=a\dfrac{(x-3)}{(x+3)(x-4)}$ y-int: $(0,-2)$

$(-2)=R(0)=a\dfrac{((0)-3)}{((0)+3)((0)-4)}$

$(-2)=a\dfrac{-3}{(3)(-4)}$

$(-2)=a\dfrac{1}{4} \rightarrow -8=a$

$$R(x)=(-8)\frac{(x-3)}{(x+3)(x-4)}$$

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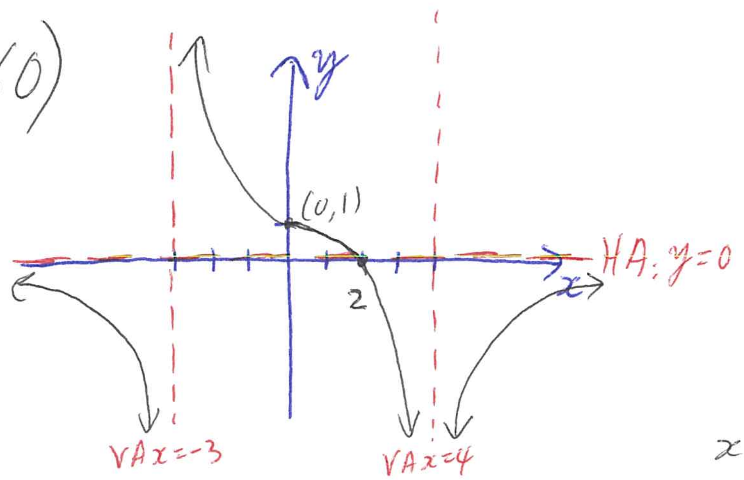
60)



V.A. $x=-3 \rightarrow x+3=0$

$x=4$ is n-even (points same direction)

$x-4=0 \rightarrow (x-4)^2=0$

$(x+3)(x-4)^2$ on denominator

x-int: $x=2 \rightarrow x-2=0$

$(x-2)$ on numerator

$R(x)=a\dfrac{(x-2)}{(x+3)(x-4)^2}$ y-int: $(0,1)$

$(1)=R(0)=a\dfrac{((0)-2)}{((0)+3)((0)-4)^2}$

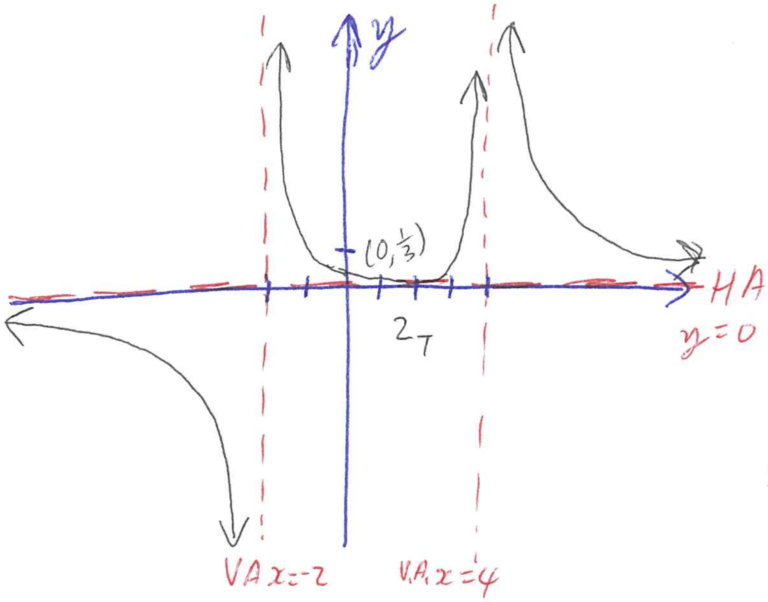
$1=a\dfrac{-2}{(3)(-4)^2}$

$1=a\dfrac{-1}{24} \Longrightarrow -24=a$

$$R(x)=(-24)\frac{(x-2)}{(x+3)(x-4)^2}$$

ex 61 and 62 have slanted asymptotes (skipped)

64)



V.A. $x=-2 \rightarrow x+2=0$

$x=4$ is n-even (points same direction)

$\downarrow$

$x-4=0 \rightarrow (x-4)^2=0$

$(x+2)(x-4)^2$ on denominator

x-int: $x=2_T$ is n-even

$\downarrow$

$x-2=0 \rightarrow (x-2)^2=0$

$(x-2)^2$ on numerator

$$R(x) = a\frac{(x-2)^2}{(x+2)(x-4)^2}$$

y-int: $(0, \frac{1}{3})$

$$\left(\frac{1}{3}\right) = R(0) = a\frac{((0)-2)^2}{((0)+2)((0)-4)^2}$$

$$\frac{1}{3} = a\frac{(-2)^2}{(2)(-4)^2}$$

$$\frac{1}{3} = a\frac{1}{8}$$

$$\frac{8}{3} = a$$

$$R(x) = \left(\frac{8}{3}\right)\frac{(x-2)^2}{(x+2)(x-4)^2}$$