Answer each question neatly on the line provided.

Name: ______________________________ ID: ______________

Skip one of the following 16 questions.

1. (5 points) Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ so that the $\mathfrak{B}-$matrix of the linear transformation $T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x}$ is $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$.

1. ______________

2. (5 points) Find k so that the system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 3x_1 + kx_2 + 4x_3 = 6 \\ x_1 + 2x_2 + kx_3 = 6 \end{cases}.$$

has a unique solution.

2. ______________

3. (5 points) Suppose a line L in $\mathbb{R}^2$ contains the vector $\vec{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Find the matrix A of the linear transformation $T(\vec{x}) = \text{proj}_L(\vec{x})$.

3. ______________

4. (5 points) Find the rank of the linear transformation $T(f) = f + 2f'$ from P_2 to P_2.

4. ______________

5. (5 points) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix})$ when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

5. ______________

6. (5 points) Find all vectors $\vec{x}$ in the kernel of $A = \begin{bmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

6. ______________

7. (5 points) Find a basis for the image of $A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$

7. ______________

8. (5 points) Find the matrix of the linear transformation $T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} M$ from the space of 2×2 upper triangular matrices $U^{2\times 2}$ to $U^{2\times 2}$ using the basis $\mathfrak{U} = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$ of $U^{2\times 2}$.

8. ______________

9. (5 points) Describe the kernel of A^2 geometrically when $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

9. ______________

10. (5 points) Find the QR factorization of $A = \begin{bmatrix} 1 & 6 \\ 1 & 4 \\ 1 & 6 \\ 1 & 4 \end{bmatrix}$.

10. ______________

11. (5 points) Two interacting populations of h and f can be modeled by the recursive equations

$$\begin{cases} h(t+1) = 6h(t) + 3f(t) \\ f(t+1) = 2h(t) + 7f(t) \end{cases}.$$

Find $f(10)$ if $h(0) = 2, f(0) = 2$.

11. ______________

12. (5 points) Find a diagonal matrix B so that there is an invertibel matrix S so that $B = S^{-1}AS$ when $A = \begin{bmatrix} .4 & .3 \\ .6 & .7 \end{bmatrix}$.

12. ______________

13. (5 points) Find the orthogonal projection of $\vec{x} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ onto the the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

13. ______________

14. (5 points) Find an orthogonal matrix S such that $S^T AS$ is diagonal when $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 7 \end{bmatrix}$.

14. ______________

15. (5 points) Find the matrix V in the singular value decomposition $A = U\Sigma V^T$ of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

15. ______________

16. (5 points) Let V be the span of $(\vec{v}_1 = \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -10 \\ 0 \end{bmatrix})$. Find the coordinates of $\vec{x} = \begin{bmatrix} 20 \\ 3 \end{bmatrix}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$.

16. ______________

Skip ONE True / False

1. (2 points) The matrix $\begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.

1. ______________

2. (2 points) If an $n \times n$ has n distinct eigenvalues then it must be symmetric.

2. ______________

3. (2 points) The quadratic form $q(x_1, x_2) = x_1^2 + \frac{1}{2}x_1x_2 + x_2^2$ is positive definite.

3. ______________

4. (2 points) If $\vec{v}$ is in the kernel of an $n \times n$ matrix A then $\vec{v}$ must be an eigenvector of A.

4. ______________

5. (2 points) $b - Ax^*$ is perpendicular to the kernel of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ when x^* is the least squares solution of the system $Ax = b$ for $b = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$.

5. ______________

6. (2 points) If all the entries of a 4×4 matrix A are 4 then the $\det A$ must be 4^4.

6. ______________

7. (2 points) Consider a 4×4 matrix A with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. If $\det(A) = 8$ then $\det \begin{bmatrix} \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \\ \vec{v}_4 \end{bmatrix} = 8$.

7. ______________

8. (2 points) If $\vec{u}, \vec{v}$, and $\vec{w}$ are nonzero vectors in $\mathbb{R}^2$ then $\vec{w}$ must be a linear combination of $\vec{v}$ and $\vec{u}$.

8. ______________

9. (2 points) If A is a 3×4 matrix of rank 3, then the system $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ must have infinitely many solutions.

9. ______________

10. (2 points) Matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible.

10. ______________