The City College Department of Mathematics
Fall 2025 Math 20212 Practice Final Exam

NAME: ________________________

YOUR INSTRUCTOR: ________________________________

CALCULATORS are NOT allowed.
Please show all your work. No credit will be given for answers without clear and complete solutions.
Use only the paper provided. You may write on the back if you need more space, but clearly indicate this on the front.
There are 10 problems for a total of 100 points.

Problem 1		Problem 6			
Problem 2		Problem 7			
Problem 3		Problem 8			
Problem 4		Problem 9			
Problem 5		Problem 10		**Total**	

1. Compute each of the following integrals (6 points each).

a. $\displaystyle\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$

b. $\displaystyle\int \sin^2 x \sin 2x\,dx$

c. $\displaystyle\int \frac{\sqrt{x^2-1}}{x^4}\,dx$

d. $\displaystyle\int (\arcsin x)^2\,dx$

---

a. $\displaystyle\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx.$

Perform partial fractions:

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Multiplying through and matching coefficients gives

$$A = 3, \qquad B = -2, \qquad C = 1.$$

Thus

$$\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx = \int_0^1 \left(\frac{3}{x+2} - \frac{2}{x+1} + \frac{1}{(x+1)^2}\right) dx.$$

Integrate:

$$\begin{aligned} &= \left[3\ln(x+2) - 2\ln(x+1) - \frac{1}{x+1}\right]_0^1 \\ &= \left(3\ln 3 - 2\ln 2 - \tfrac{1}{2}\right) - \left(3\ln 2 - 0 - 1\right) \\ &= 3\ln 3 - 5\ln 2 + \tfrac{1}{2}. \end{aligned}$$

b. $\displaystyle\int \sin^2 x \sin 2x\,dx.$

Use $\sin 2x = 2\sin x\cos x$. Then

$$\sin^2 x \sin 2x = 2\sin^3 x\cos x.$$

Let $u = \sin x$, $du = \cos x\,dx$. Then

$$\int 2\sin^3 x\cos x\,dx = 2\int u^3\,du = \frac{u^4}{2} + C = \frac{\sin^4 x}{2} + C.$$

c. $\displaystyle\int \frac{\sqrt{x^2-1}}{x^4}\,dx.$

Make the substitution $x=\sec\theta$ (so $\sqrt{x^2-1}=\tan\theta$), $dx=\sec\theta\tan\theta\,d\theta$. Then

$$\frac{\sqrt{x^2-1}}{x^4}\,dx=\frac{\tan\theta}{\sec^4\theta}\cdot\sec\theta\tan\theta\,d\theta=\tan^2\theta\cos^3\theta\,d\theta.$$

Use $\tan^2\theta=\sec^2\theta-1=\frac{1}{\cos^2\theta}-1$. Thus

$$\tan^2\theta\cos^3\theta=\cos\theta-\cos^3\theta.$$

So

$$\int(\cos\theta-\cos^3\theta)\,d\theta=\sin\theta-\left(\sin\theta-\tfrac{\sin^3\theta}{3}\right)+C=\frac{\sin^3\theta}{3}+C.$$

Return to x: $\sin\theta=\sqrt{x^2-1}/x$. Hence

$$\boxed{\int\frac{\sqrt{x^2-1}}{x^4}\,dx=\frac{(x^2-1)^{3/2}}{3x^3}+C.}$$

d. $\displaystyle\int(\arcsin x)^2\,dx.$

Integrate by parts: take $u=(\arcsin x)^2$, $dv=dx$. Then $du=2\arcsin x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$, $v=x$. So

$$\int(\arcsin x)^2\,dx=x(\arcsin x)^2-2\int\frac{x\arcsin x}{\sqrt{1-x^2}}\,dx.$$

Set $t=\arcsin x$ so $x=\sin t$, $dx=\cos t\,dt$. Then

$$\int\frac{x\arcsin x}{\sqrt{1-x^2}}\,dx=\int t\sin t\,dt=-t\cos t+\sin t+C.$$

Thus

$$\int(\arcsin x)^2\,dx=x(\arcsin x)^2+2\arcsin x\sqrt{1-x^2}-2x+C.$$

2. Evaluate the limits (4 points each).

a. $\lim\limits_{n\to\infty} \dfrac{(-1)^n \cos(\pi n)}{\sqrt{n}}$

b. $\lim\limits_{n\to\infty} n^2 e^{-n}$

---

a. Note that $\cos(\pi n) = (-1)^n$. Hence the numerator is $(1)^n(-1)^n = 1$. Therefore

$$\lim_{n\to\infty} \frac{(-1)^n \cos(\pi n)}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

b. Using L'Hôpital on the continuous analogue $\lim_{x\to\infty} x^2 e^{-x}$ we obtain

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

Hence, $\lim\limits_{n\to\infty} n^2 e^{-n} = 0$.

3. State for each series whether it converges absolutely, converges conditionally, or diverges. Name a test which supports your conclusion and justify why it applies, by showing a calculation or giving an explanation. (4 points each).

$$\text{(a)} \sum_{n=3}^{\infty} \frac{\ln n}{n^2} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2-1}}$$

---

a. Use the integral test on the series $\sum_{n=3}^{\infty} \frac{\ln n}{n^2}$.

Let $f(x) = \dfrac{\ln x}{x^2}$ for $x \geq 3$. Note that f is positive for $x \geq 3$. We check monotonicity:

$$f'(x) = \frac{1 \cdot x^2 - \ln x \cdot 2x}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}.$$

For $x \geq 3$ we have $\ln x \geq \ln 3 > 1/2$, so $1 - 2\ln x < 0$. Hence $f'(x) < 0$ for $x \geq 3$, so f is decreasing on $[3,\infty)$. Thus the hypotheses of the Integral Test are satisfied.

We evaluate the improper integral $\int_3^\infty \frac{\ln x}{x^2}\,dx$. Compute an antiderivative by integration by parts. Let $u = \ln x$, $dv = x^{-2}dx$. Then $du = \frac{1}{x}\,dx$ and $v = -\frac{1}{x}$. Thus

$$\int \frac{\ln x}{x^2}\,dx = uv - \int v\,du = -\frac{\ln x}{x} - \int \left(-\frac{1}{x} \cdot \frac{1}{x}\right) dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx.$$

Since $\int \frac{1}{x^2}\,dx = -\frac{1}{x}$, we obtain the antiderivative $F(x) = -\frac{\ln x}{x} - \frac{1}{x} = -\frac{\ln x + 1}{x} + C$.

Now take the improper integral:

$$\int_3^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \Big[F(t) - F(3)\Big] = \lim_{t\to\infty} \left(-\frac{\ln t + 1}{t}\right) - \left(-\frac{\ln 3 + 1}{3}\right).$$

Because $\dfrac{\ln t + 1}{t} \to 0$ as $t \to \infty$ by L'Hospital, the limit equals $\frac{\ln 3+1}{3}$. Thus the integral converges. By the Integral Test, the series $\sum_{n=3}^{\infty} \dfrac{\ln n}{n^2}$ converges. Since the terms are positive, this is absolute convergence as well.

$\boxed{\text{The series converges absolutely.}}$

b. We use the Ratio Test to determine convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$.

Use the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!/(n+1)!^2}{(2n)!/n!^2} = \frac{(2n+2)(2n+1)}{(n+1)^2} \to 4 > 1.$$

By the Ratio test the series diverges. $\boxed{\text{Diverges.}}$

c. We use Comparison Test to check for absolute convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2-1}}$.

For large n, $|a_n| \sim 1/n^2$. Indeed

$$\frac{1}{n\sqrt{n^2-1}} \leq \frac{1}{n\sqrt{n^2 - \frac{1}{2}n^2}} = \frac{\sqrt{2}}{n^2}.$$

Since $\sum \frac{\sqrt{2}}{n^2}$ converges, $\sum |a_n|$ converges by comparison. Hence the series converges absolutely (and therefore converges). $\boxed{\text{Converges absolutely.}}$

4. Evaluate each of the following integrals or show that it is divergent (5 points each).

a. $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^3}\,dx$ **b.** $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx.$

---

a.

$$\int_0^\infty \frac{x}{(x^2+1)^3}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+1)^3}\,dx.$$

To find the antiderivative we use the substitution $u = x^2+1$, $du = 2x\,dx$.

$$\int \frac{x}{(x^2+1)^3}\,dx = \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\left(-\frac{1}{2}u^{-2}\right)+C = -\frac{1}{4}u^{-2}+C = -\frac{1}{4(x^2+1)^2}+C.$$

Evaluate the limit:

$$\begin{aligned}\lim_{t\to\infty}\int_0^t \frac{x}{(x^2+1)^3}\,dx &= \lim_{t\to\infty}\left(-\frac{1}{4(t^2+1)^2}-\left(-\frac{1}{4(0^2+1)^2}\right)\right)\\ &= \lim_{t\to\infty}\left(\frac{1}{4}-\frac{1}{4(t^2+1)^2}\right) = \frac{1}{4}.\end{aligned}$$

Therefore $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^3}\,dx = \boxed{\tfrac{1}{4}}$.

b.

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{t\to(\pi/2)^-}\int_0^t \frac{\cos x}{\sqrt{\sin x}}\,dx.$$

To find the antiderivative substitute $u = \sin x$, $du = \cos x\,dx$. Then

$$\int \frac{\cos x}{\sqrt{\sin x}}\,dx = \int u^{-1/2}\,du = 2u^{1/2}+C = 2\sqrt{\sin x}+C.$$

Evaluate the limit:

$$\begin{aligned}\lim_{t\to(\pi/2)^-}\int_0^t \frac{\cos x}{\sqrt{\sin x}}\,dx &= \lim_{t\to(\pi/2)^-}\Big[2\sqrt{\sin x}\Big]_0^t\\ &= \lim_{t\to(\pi/2)^-}\left(2\sqrt{\sin t}-2\sqrt{\sin 0}\right) = 2.\end{aligned}$$

Therefore $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx = \boxed{2}$.

5. (6 points each)

a. Without attempting to evaluate it, determine whether the following integral converges or diverges. Justify your answer.

$$\int_1^\infty \frac{\arctan x}{\sqrt{e^x+2}}\,dx$$

b. Write out the form of the partial fraction decomposition of the following function. Do not attempt to determine the numerical values of the coefficients.

$$f(x) = \frac{12x-5}{(x-1)(x+2)^2(x^2+x+3)^2}$$

---

a. For all $x \geq 1$, $\arctan x \leq \pi/2$ and $\sqrt{e^x+2} \geq \sqrt{e^x} = e^{x/2}$. Hence for $x \geq 1$

$$0 \leq \frac{\arctan x}{\sqrt{e^x+2}} \leq \frac{\pi/2}{e^{x/2}}.$$

Since $\int_1^\infty e^{-x/2}\,dx$ converges, by Comparison Test the given integral converges.

$\boxed{\text{Converges (by comparison).}}$

b. Write the form of the partial fraction decomposition of

$$f(x) = \frac{12x-5}{(x-1)(x+2)^2(x^2+x+3)^2}.$$

Because $x-1$ is linear distinct, $(x+2)$ is linear repeated of multiplicity 2, and x^2+x+3 is an irreducible quadratic (squared), the decomposition has the form

$$\frac{12x-5}{(x-1)(x+2)^2(x^2+x+3)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{Dx+E}{x^2+x+3} + \frac{Fx+G}{(x^2+x+3)^2},$$

where A,B,C,D,E,F,G are constants to be determined.

6. (5 points) Find the radius of convergence and determine the exact interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$.

We rewrite the series as a power series centered at $a = \frac{1}{2}$. Since $2x - 1 = 2(x - \frac{1}{2})$, we have

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{2^n}{5^n\sqrt{n}}(x - \tfrac{1}{2})^n.$$

To determine the Radius of Convergence we compute

$$L = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \frac{2^{n+1}/(5^{n+1}\sqrt{n+1})}{2^n/(5^n\sqrt{n})} = \lim_{n\to\infty} \frac{2}{5}\sqrt{\frac{n}{n+1}} = \frac{2}{5}.$$

Therefore the radius of convergence is $R = \frac{1}{L} = \frac{1}{2/5} = \frac{5}{2}$.

Interval of Convergence

The inequality $|x - \frac{1}{2}| < \frac{5}{2}$ is equivalent to $-2 < x < 3$.

Endpoint $x = -2$

Then $2x - 1 = -5$, so the series becomes

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

This is an alternating series with terms decreasing to 0, so it converges by the Alternating Series Test. Thus the series converges at $x = -2$.

Endpoint $x = 3$

Then $2x - 1 = 5$, so the series becomes

$$\sum_{n=1}^{\infty} \frac{5^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

a divergent p-series with $p = \frac{1}{2}$. Thus the series diverges at $x = 3$.

$$\boxed{R = \tfrac{5}{2}, \qquad \text{interval of convergence } [-2, 3).}$$

7. For $f(x) = \arctan(2x)$

a. find the first three terms of its Taylor series centered at $a = \dfrac{\sqrt{3}}{2}$;

b. use the geometric series to find its Taylor series centered at $a = 0$.

(4 points each)

---

a. The Taylor series of f centered at a is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.
First compute derivatives of $f(x) = \arctan(2x)$:

$$f'(x) = \frac{2}{1+4x^2}, \qquad f''(x) = -\frac{16x}{(1+4x^2)^2}.$$

Evaluate at $a = \dfrac{\sqrt{3}}{2}$. Note

$$2a = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \quad \Rightarrow \quad f(a) = \arctan(\sqrt{3}) = \frac{\pi}{3}.$$

Also

$$1 + 4a^2 = 1 + 4\left(\frac{3}{4}\right) = 1 + 3 = 4,$$

so

$$f'(a) = \frac{2}{1+4a^2} = \frac{2}{4} = \frac{1}{2},$$

and

$$f''(a) = -\frac{16a}{(1+4a^2)^2} = -\frac{16 \cdot \frac{\sqrt{3}}{2}}{4^2} = -\frac{8\sqrt{3}}{16} = -\frac{\sqrt{3}}{2}.$$

Thus

$$\frac{f''(a)}{2} = -\frac{\sqrt{3}}{4}.$$

Therefore the Taylor expansion (first three terms) centered at $a = \dfrac{\sqrt{3}}{2}$ is

$$\boxed{\arctan(2x) = \frac{\pi}{3} + \frac{1}{2}\left(x - \frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\sqrt{3}}{2}\right)^2 + \cdots .}$$

b. Start from the geometric series (valid for $|r| < 1$):

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n.$$

Put $r = -4x^2$. For $|-4x^2| < 1$, i.e. $|x| < \frac{1}{2}$, we get

$$\frac{1}{1+4x^2} = \sum_{n=0}^{\infty} (-4x^2)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}.$$

Differentiate/integrate carefully to reach $\arctan(2x)$. Note

$$\frac{d}{dx}\big(\arctan(2x)\big) = \frac{2}{1+4x^2}.$$

Thus

$$\arctan(2x) = \int \frac{2}{1+4x^2}\,dx.$$

Use the series for $1/(1+4x^2)$ and integrate termwise (valid for $|x| < \frac{1}{2}$):

$$\begin{aligned}\arctan(2x) &= 2\int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}\,dx = 2\sum_{n=0}^{\infty} (-1)^n 4^n \int x^{2n}\,dx \\ &= 2\sum_{n=0}^{\infty} (-1)^n 4^n \frac{x^{2n+1}}{2n+1} + C.\end{aligned}$$

Choose the constant C so that both sides vanish at $x = 0$; since $\arctan(0) = 0$ the constant is 0. Hence

$$\boxed{\arctan(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{2\cdot 4^n}{2n+1}\,x^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{2n+1}\,x^{2n+1}, \qquad |x| < \tfrac{1}{2}.}$$

8. (6 points) Draw a sketch of the conic whose equation is

$$x^2 + 4y^2 - 6x + 5 = 0.$$

Identify which sort of conic it is. On your sketch, show and label whichever of the following are present: vertices, asymptotes, and foci.

Complete the square in x:

$$x^2 - 6x + 9 + 4y^2 = -5 + 9 \quad \Rightarrow \quad (x-3)^2 + 4y^2 = 4.$$

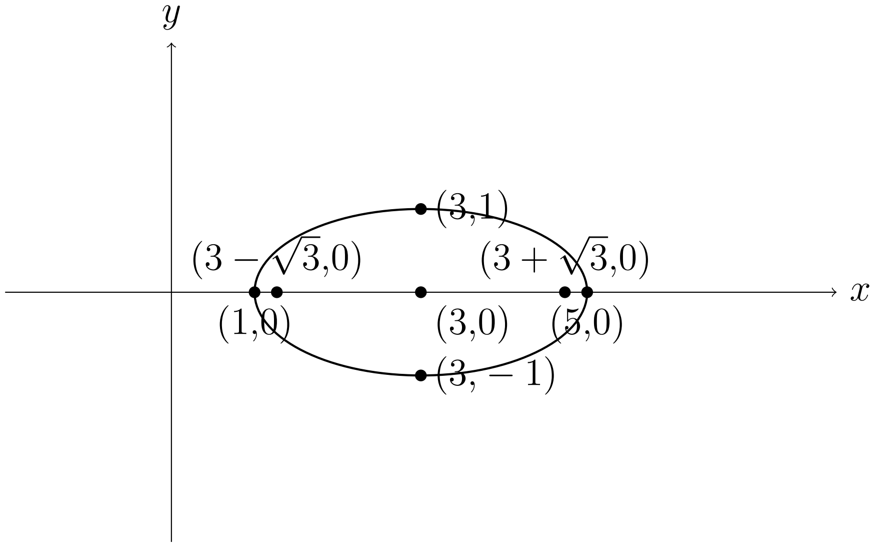
Divide by 4:

$$\frac{(x-3)^2}{4} + \frac{y^2}{1} = 1.$$

This is an *ellipse* centered at (3,0) with axes $a = 2$ along the x-direction and $b = 1$ along the y-direction.

- Vertices along the major axis: $(3 \pm 2{,}0) = (1{,}0)$ and $(5{,}0)$.
- Vertices along the other axis: $(3, \pm 1)$.
- Foci: $c = \sqrt{a^2 - b^2} = \sqrt{4-1} = \sqrt{3}$, so foci at $(3 \pm \sqrt{3}{,}0)$.

Sketch:



9.(5 points each)

a. A radioactive substance loses half of its mass every 8 years. Assuming exponential decay, after how many years will the substance be reduced to one-third of its original mass?

b. Find the solution of the differential equation which satisfies the given initial condition

$$x + 3y^2\sqrt{x^2+1}\frac{dy}{dx} = 0, \quad y(0) = 1$$

---

a. We model the radioactive decay using

$$y(t) = Ce^{kt},$$

where $k < 0$ is the decay constant. We are told the substance loses half its mass every 8 years. Thus,

$$\frac{1}{2}C = Ce^{k\cdot 8}.$$

Cancel C and take the natural logarithm:

$$\ln\left(\frac{1}{2}\right) = 8k \quad \Longrightarrow \quad k = \frac{\ln(1/2)}{8} = -\frac{\ln 2}{8}.$$

We now use this value of k to find the time when the mass is one-third. We now solve

$$\frac{1}{3}C = Ce^{kt}.$$

Cancel C and take the natural logarithm again:

$$\ln\left(\frac{1}{3}\right) = kt.$$

Substitute $k = -\frac{\ln 2}{8}$:

$$t = \frac{\ln(1/3)}{-\frac{\ln 2}{8}} = 8\,\frac{\ln 3}{\ln 2}.$$

Answer:

$$\boxed{t = 8\,\frac{\ln 3}{\ln 2}\text{ years.}}$$

b. Rearrange to separate variables:

$$3y^2\sqrt{x^2+1}\,\frac{dy}{dx} = -x \quad \Rightarrow \quad 3y^2\,dy = -\frac{x}{\sqrt{x^2+1}}\,dx.$$

Integrate both sides.

$$\text{Left-hand side:} \quad \int 3y^2\,dy = 3\cdot\frac{y^3}{3} + C = y^3 + C.$$

For the right-hand side we use substitution $u = x^2+1, \qquad du = 2x\,dx.$. Then

$$\int -\frac{x}{\sqrt{x^2+1}}\,dx = \int -\frac{1}{2}u^{-1/2}\,du = -\frac{1}{2}\cdot\frac{u^{1/2}}{1/2} + C = -u^{1/2} + C = -\sqrt{x^2+1} + C.$$

Thus the integrated equation is

$$y^3 = -\sqrt{x^2+1} + C.$$

Use initial condition $y(0) = 1$: $1^3 = -\sqrt{0+1} + C = -1 + C$ so $C = 2$. Hence

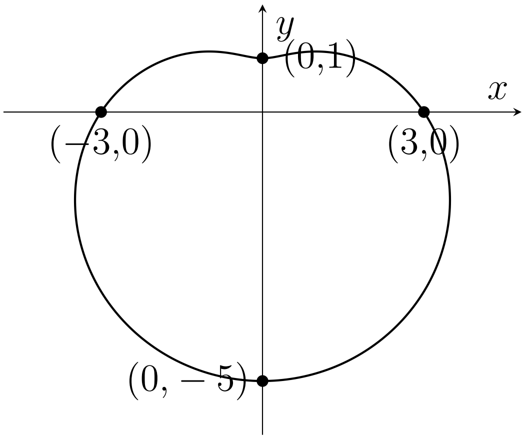
$$\boxed{y^3 = 2 - \sqrt{x^2+1} \quad \Rightarrow \quad y = \left(2-\sqrt{x^2+1}\right)^{1/3}.}$$

10. (6 points) Sketch the curve given by the equation $r = 3 - 2\sin\theta$ in polar coordinates, labeling the x and y intercepts, and compute the area it encloses.

We find the intercepts:

- At $\theta = 0$: $r = 3$ gives $x = 3, y = 0$.
- At $\theta = \pi$: $r = 3$ gives $x = -3, y = 0$.
- At $\theta = \pi/2$: $r = 3 - 2 = 1$ gives $x = 0, y = 1$.
- At $\theta = 3\pi/2$: $r = 3 - 2(-1) = 5$ gives $x = 0, y = -5$.

Sketch:



Area enclosed:

$$\begin{aligned}
A = \frac{1}{2}\int_0^{2\pi} r^2\,d\theta &= \frac{1}{2}\int_0^{2\pi}\left(9 - 12\sin\theta + 4\sin^2\theta\right)d\theta \\
&= \frac{1}{2}\left[\int_0^{2\pi} 9\,d\theta - \int_0^{2\pi} 12\sin\theta\,d\theta + \int_0^{2\pi} 4\sin^2\theta\,d\theta\right] \\
&= \frac{1}{2}\left[9\cdot(2\pi) - 12\left(-\cos\theta\Big|_0^{2\pi}\right) + 4\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta\right] \\
&= \frac{1}{2}\left[18\pi - 12(0) + 4\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta\right] \\
&= 11\pi
\end{aligned}$$