Math 20300
Calculus III

Lesson 35
Approximations Using Taylor Polynomials

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Approximations Using Taylor Polynomials

In Lesson 34, we saw $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Then we can use Taylor polynomials to approximate e^x:

$$e^x \approx T_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}$$

Ex. Use the 10^{th} degree Maclaurin polynomial for $f(x) = e^x$ to approximate e.

$$e^x \approx T_{10}(x) = \sum_{n=0}^{10} \frac{x^n}{n!}$$

so $e \approx T_{10}(1) = \sum_{n=0}^{10} \frac{1}{n!} \approx 2.718281801$

but we don't know how accurate this is.

Ex. Use the Maclaurin series for $f(x) = e^x$ to get a decimal approximation for e accurate to within 10^{-10}.

We know $R_N(x) = \dfrac{f^{(N+1)}(z)}{(N+1)!} x^{N+1}$ and we are approximating at $x = 1$. $f^{(n)}(x) = e^x \ \forall x$

$$|R_N(x)| = \left| \frac{e^z x^{N+1}}{(N+1)!} \right| = \frac{e^z}{(N+1)!} |x|^{N+1}$$

We know $\underset{\substack{\uparrow \\ \text{center} \\ \text{of} \\ \text{series}}}{0} < z < \underset{\substack{\uparrow \\ \text{our } x\text{-value}}}{1}$ and $e^z < e < 3$

need N such that

$$|R_N(1)| = \frac{e^z (1)^{N+1}}{(N+1)!} < \frac{3}{(N+1)!} < 10^{-10}$$

(use trial + error)

$$\text{for } N = 13, \quad \frac{3}{(N+1)!} = \frac{3}{(14)!} \approx 3.44 \times 10^{-11}$$

$$= .344 \times 10^{-10}$$

$$< 10^{-10}$$

$\therefore$ We approximate

$$e \approx \sum_{n=0}^{13} \frac{1}{n!} \approx 2.718281828446759 .$$

(we know $|e - 2.718281828446759| < 10^{-10}$).

We also saw

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \forall x \in \mathbb{R}$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \left(x - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} \qquad \forall x \in \mathbb{R}$$

Ex. Find an approximation of $\cos(1.5)$ accurate to 10^{-5}.

using $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, we have

$$R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!} x^{N+1} \quad \text{with} \quad 0 < z < \underset{\underset{x}{\uparrow}}{1.5}$$

$$R_N(1.5) = \frac{f^{(N+1)}(z)}{(N+1)!} (1.5)^{N+1}$$

$$|R_N(1.5)| = \frac{|f^{(N+1)}(z)|}{(N+1)!} (1.5)^{N+1} \le \frac{(1.5)^{N+1}}{(N+1)!} \overset{\text{need}}{<} 10^{-5}$$

trial & error, $N = 10$.

means we should use the $\underbrace{10^{th} \text{ degree}}$ poly.

then $\cos(1.5) \approx \sum_{n=0}^{5} \frac{(-1)^n (1.5)^{2n}}{(2n)!} \approx 0.0707369341$

But notice, using $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$

we get $R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!}(x-\frac{\pi}{2})^{N+1}$ with $\underset{\underset{x}{\uparrow}}{1.5} < z < \underset{\underset{\text{center}}{\uparrow}}{\frac{\pi}{2}}$

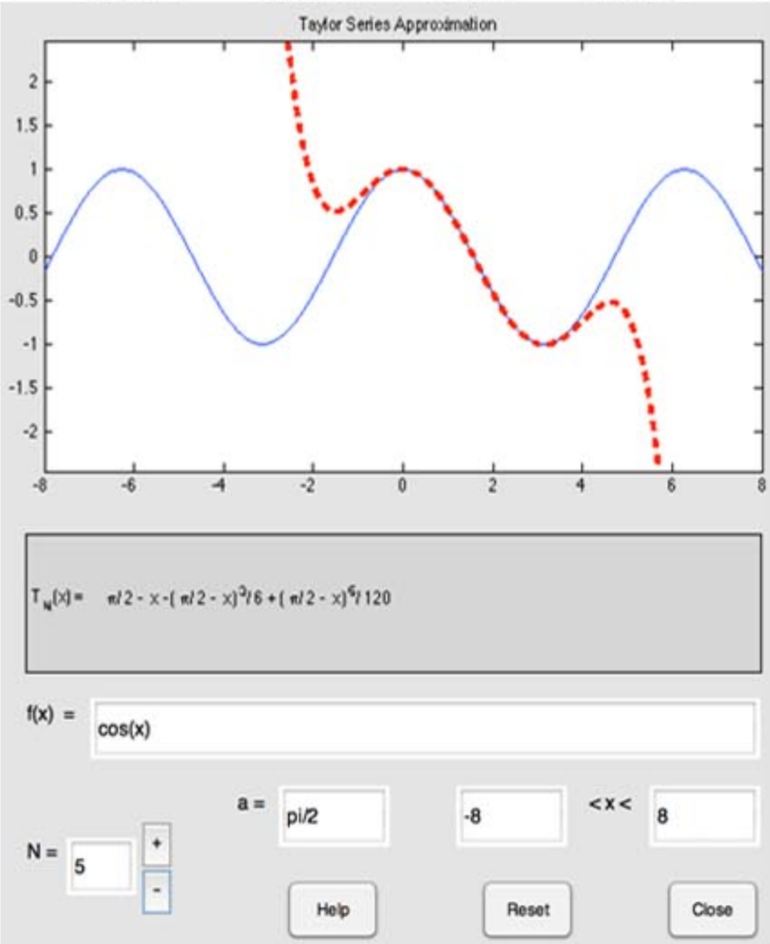
so $R_N(1.5) = \frac{f^{(N+1)}(z)}{(N+1)!}(1.5-\frac{\pi}{2})^{N+1}$

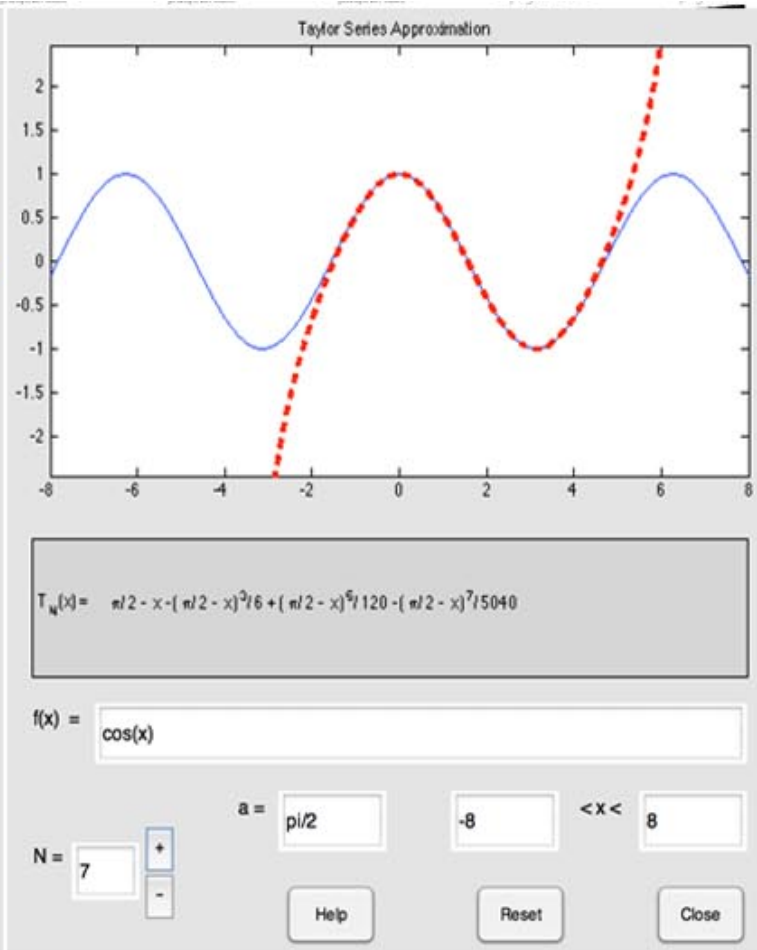
& $|R_N(1.5)| = \frac{|f^{(N+1)}(z)|}{(N+1)!}|1.5-\frac{\pi}{2}|^{N+1} \le \frac{|1.5-\frac{\pi}{2}|^{N+1}}{(N+1)!}$ need $< 10^{-5}$

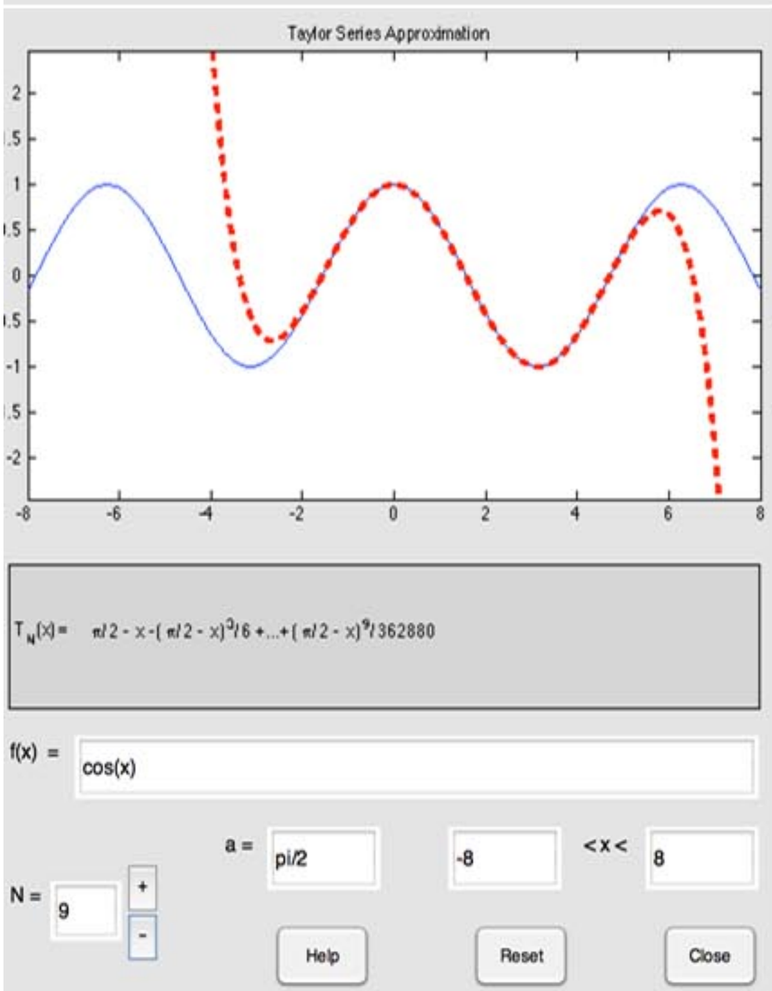
trial & error, $N = 3$. means third degree polynomial

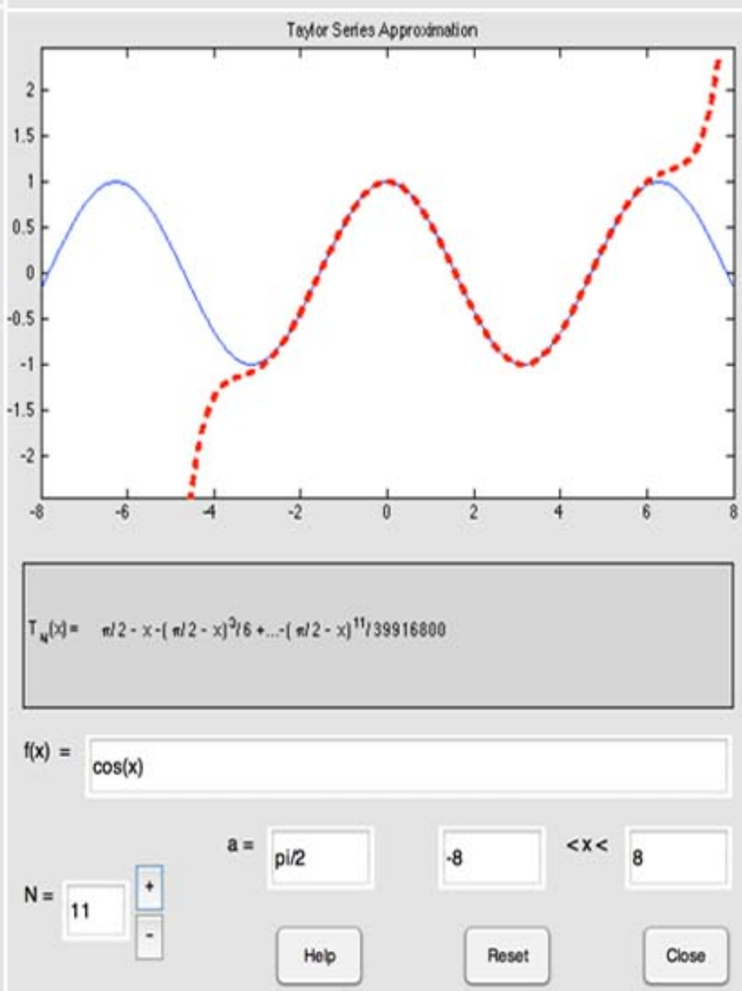
and $\cos(1.5) \approx \sum_{n=0}^{1} \frac{(-1)^{n+1}(1.5-\frac{\pi}{2})^{2n+1}}{(2n+1)!} \approx 0.0707371868$

We saw in lesson 34 these comparisons of Taylor polynomials to the actual function $\cos(x)$ centered at $\frac{\pi}{2}$


Taylor Series Approximation
T_N(x) = π/2 - x -(π/2 - x)^3/6 + (π/2 - x)^5/120
f(x) =
cos(x)
a =
pi/2
-8
< x <
8
N =
5
+
-
Help
Reset
Close


Taylor Series Approximation
T_N(x) = π/2 - x -(π/2 - x)^3/6 + (π/2 - x)^5/120 -(π/2 - x)^7/5040
f(x) =
cos(x)
a =
pi/2
-8
< x <
8
N =
7
+
-
Help
Reset
Close


Taylor Series Approximation
T_N(x) = π/2 - x -(π/2 - x)^3/6 +...+ (π/2 - x)^9/362880
f(x) =
cos(x)
a =
pi/2
-8
< x <
8
N =
9
+
-
Help
Reset
Close


Taylor Series Approximation
T_N(x) = π/2 - x -(π/2 - x)^3/6 +...-(π/2 - x)^11/39916800
f(x) =
cos(x)
a =
pi/2
-8
< x <
8
N =
11
+
-
Help
Reset
Close

Notice that each polynomial has an interval around $\frac{\pi}{2}$ for which the approximation is close.

Ex. Estimate the range of x-values for which the 9^{th} degree Taylor polynomial for $\cos x$ centered at $\frac{\pi}{2}$ is accurate to within 10^{-5}.

We need $|R_9(x)| < 10^{-5}$

We know for each x, $R_9(x) = \dfrac{f^{(10)}(z)}{10!}\left(x-\frac{\pi}{2}\right)^{10}$ for some z between x & $\frac{\pi}{2}$.

$$|R_9(x)| = \frac{|f^{(10)}(z)|}{10!}\left|x-\tfrac{\pi}{2}\right|^{10} \leq \frac{\left|x-\frac{\pi}{2}\right|^{10}}{10!} \overset{\text{need}}{<} 10^{-5}$$

so $\left|x-\frac{\pi}{2}\right|^{10} < 10^{-5}\cdot 10!$

$$\left|x-\frac{\pi}{2}\right| < \sqrt[10]{10^{-5}\cdot 10!} \approx 1.4321$$

radius around $\frac{\pi}{2}$

for which the approximation

is close enough

so $\quad \frac{\pi}{2} - 1.4321 < x < \frac{\pi}{2} + 1.4321$

$$.1387 < x < 3.0029$$