Math 20300
Calculus III

Lesson 33
Representing Functions as Power Series

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Representing Functions as Power Series.

We know that the power series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

for $|x| < 1$, geometric series.

So we can say that for $f(x) = \frac{1}{1-x}$,

f has the power series representation

$$f(x) = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1.$$

What other functions can be expressed as power series?

Any function of the form

$$f(x) = \frac{a}{1-r(x)}$$

$$= \sum_{n=0}^{\infty} a\,(r(x))^n \quad \text{for} \quad |r(x)| < 1 .$$

Ex. $$f(x) = \frac{2}{1+x^4} = \frac{2}{1-\underbrace{(-x^4)}_{r(x)}} = \sum_{n=0}^{\infty} 2\left(-x^4\right)^n$$

$$= \sum_{n=0}^{\infty} 2\,(-1)^n\, x^{4n} = 2 - 2x^4 + 2x^8 - 2x^{12} + \cdots$$

$$\text{for} \quad |-x^4| < 1 \;\Rightarrow\; |x| < 1 \quad \begin{pmatrix}\text{diverges} \\ |x| \geq 1\end{pmatrix}$$

Ex. $$f(x) = \frac{1}{4+x^2} = \frac{1}{4\left(1+\frac{x^2}{4}\right)} = \frac{\frac{1}{4}}{1+\frac{x^2}{4}} = \frac{\frac{1}{4}}{1-\left(-\frac{x^2}{4}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{4}\left(-\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty} \frac{1}{4}\,(-1)^n\,\frac{x^{2n}}{4^n}$$

$$= \sum_{n=0}^{\infty} (-1)^n\,\frac{x^{2n}}{4^{n+1}}$$

for $\left|-\frac{x^2}{4}\right|<1 \Rightarrow |x^2|<4 \Rightarrow x^2<4$

$\Rightarrow -2<x<2$

or $|x|<2$.

(diverges $\left|-\frac{x^2}{4}\right| \geq 1$, ie $|x| \geq 2$).

We can get more functions with power series representations by the following Theorem:

Theorem: if the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has radius of convergence $r>0$ (not $=0$, possibly ∞)

then the function $f(x)$ defined by

$$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

is differentiable on the interval $(c-r, c+r)$ and:

1) $f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$

$$= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

NOTE: this says

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n(x-c)^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(a_n(x-c)^n\right)$$

which is <u>not</u> trivial. This is switching the order of limits (derivative is a limit), which can not always be done.

Ex. $\lim\limits_{m\to\infty} \lim\limits_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{m}} = \lim\limits_{m\to\infty} 0^{\frac{1}{m}}$

$$= \lim_{m\to\infty} 0 = 0$$

but $\lim\limits_{n\to\infty} \lim\limits_{m\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{m}} = \lim\limits_{n\to\infty} \left(\frac{1}{n}\right)^0$

$$= \lim_{n\to\infty} 1 = 1.$$

and

2) $$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$$

NOTE: This says

$$\int \sum_{n=0}^{\infty} a_n (x-c)^n \,dx = \sum_{n=0}^{\infty} \int a_n (x-c)^n \,dx$$

again, not trivial.

Also note: for the interval of convergence

of the series $\sum_{n=0}^{\infty} \frac{d}{dx}\left(a_n (x-c)^n\right)$ or

of $\sum_{n=0}^{\infty} \int a_n (x-c)^n \,dx$,

need to check the endpoints of the interval

of convergence. The center and radius

are the same as the original series, but the behavior at the endpoints of the interval could change after taking the derivative or antiderivative.

Ex. We know $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$.

interval of convergence

Let's examine $\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n)$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) \qquad\qquad = \sum_{n=1}^{\infty} n x^{n-1}.$$

$$= \frac{d}{dx}\left((1-x)^{-1}\right) = -(1-x)^{-2}(-1)$$

$$= \frac{1}{(1-x)^2}$$

$\therefore \quad \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$ we know we have convergence for $-1 < x < 1$, but (in general) we have to check endpts here, it can't converge for $x=1$ because $\frac{1}{(1-1)^2}$

does not exist

$$x=1: \sum_{n=1}^{\infty} n(1)^{n-1} = \sum_{n=1}^{\infty} n \qquad \lim_{n\to\infty} n \neq 0 \quad \text{diverges}$$

$$x=-1: \sum_{n=1}^{\infty} n(-1)^{n-1} \qquad \lim_{n\to\infty} n(-1)^{n-1} \neq 0 \quad \text{diverges}$$

$$\therefore \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{for } -1 < x < 1 \text{ or } |x| < 1 \text{ or } (-1, 1)$$

interval of convergence

$$\text{Also, } \int \sum_{n=0}^{\infty} x^n \, dx = \sum_{n=0}^{\infty} \int x^n \, dx = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

"

$$\int \frac{1}{1-x} dx = -\int \frac{1}{u} du$$

$u = 1-x$

$du = -dx$

$$= -\ln|1-x| + C_2$$

$$= -\ln(1-x) + C_2$$

$$\therefore \quad -\ln(1-x) = C_1 - C_2 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\ln(1-x) = \underbrace{C_2 - C_1}_{C} - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\ln(1-x) = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$ at least for $|x| < 1$

Check $x = \pm 1$.

(since $\ln(1-1)$ DNE, can't converge at $x=1$)

$x=1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ harmonic series, diverges.

$x=-1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ alternating harmonic, converges.

$\displaystyle\lim_{n\to\infty} \frac{1}{n+1} = 0$, $\displaystyle\frac{1}{n+2} < \frac{1}{n+1}$

and $\displaystyle\frac{1}{n+1} > 0$

$\therefore$ converges by alternating series test.

$$\therefore \quad \ln(1-x) = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad \text{for } -1 \le x < 1$$

interval of convergence

but we still need to find C.

use $x=0$ ($x=c$, in general).

$$\ln(1-x) = C - \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \right)$$

$$\ln(1-0) = C - (0 + 0 + 0 + \cdots)$$

$$0 = C$$

$$\therefore \quad \ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad \text{for } -1 \le x < 1.$$

Ex. We know

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for $|-x^2| < 1 \Rightarrow |x| < 1$

interval of convergence

then $\arctan x = \int \frac{1}{1+x^2}\,dx =$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\, dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

at least for $|x| < 1$, check endpts:

$x = 1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges

like alternating harmonic

$x = -1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{\overbrace{2n+1}^{\text{odd}}}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges.

$$\therefore \arctan x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

for $-1 \le x \le 1$

now to find C:

$$\arctan x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

let $x = 0$

$$\underbrace{\arctan(0)}_{0} = C + 0 - 0 + 0 \cdots$$

$$C = 0$$

$$\therefore \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

for $-1 \le x \le 1$

Ex. Find a power series representation for $f(x) = x \arctan x$.

We know $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

for $-1 \le x \le 1$

then $x \arctan x = x\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\right)$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+1}.$$

same interval of convergence $-1 \leq x \leq 1$.

Ex. Find a power series representation of

$$f(x) = \frac{\arctan x}{x}.$$

We know $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

for $-1 \leq x \leq 1$

then $\frac{\arctan x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}$$

but here, $x \neq 0$. so for $[-1,0) \cup (0,1]$.