Math 20300
Calculus III

Lesson 32
Power Series

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Power Series

A <u>power series</u> is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

like a polynomial, but infinitely many terms.

For each value for x, this is a series of numbers and we can discuss its convergence.

So $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a function whose domain is the set of x-values for which the series converges.

Ex. if $a_i = a \ \forall i$ (all coefficients are the same)

then $\sum_{n=0}^{\infty} a x^n$ geometric series $r = x$

so $\sum_{n=0}^{\infty} a x^n$ converges to $\frac{a}{1-x}$ for $|x|<1$

and diverges for $|x| \geq 1$.

In general, we'll have to use the ratio test and check endpoints (when $L=1$) for each power series

Same example above (all a_i are the same)

$\sum_{n=0}^{\infty} a x^n$ Ratio test for $x \neq 0$ $\lim_{n\to\infty} \left|\frac{\cancel{a} x^{n+1}}{\cancel{a} \cancel{x^n}}\right| =$

but for $x=0$, converges to a.

$= \lim_{n\to\infty} |x| = \underbrace{|x|}_{L}$

Ratio test says if $L<1$ converges absolutely

$L>1$ diverges

$L=1$ no conclusion

so here $|x|<1 \Rightarrow$ series converges absolutely

$|x|>1 \Rightarrow$ series diverges

and $L=1$: $x=1$ $\displaystyle\sum_{n=0}^{\infty} a\cdot 1^n = \sum_{n=0}^{\infty} a$ diverges

$x=-1$ $\displaystyle\sum_{n=0}^{\infty} a(-1)^n = a-a+a-a\cdots$

radius of convergence = 1
center $x=0$

diverges

$\therefore$ $|x|\geq 1$ diverges.

interval of convergence $\rightarrow$ $|x|<1$ converges. $(-1, 1)$
a b
radius $\frac{b-a}{2}$

More general version:

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

is a <u>power series centered at $x=c$</u>.

Notice $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ from above is a

power series centered at $x=0$.

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Theorem: For the domain / convergence of $\sum_{n=0}^{\infty} a_n(x-c)^n$, one of the following must be true:

1) The series converges absolutely only at $x=c$, and diverges for all other x-values

radius of convergence $= 0$ interval: $x=c$ (not an interval)

2) The series converges absolutely $\forall x \in \mathbb{R}$

radius of convergence $= \infty$ interval: $\mathbb{R}$

3) There is a positive number $\underset{\text{radius}}{r}$ such that for $\underbrace{x \in (c-r, c+r)}$ ie $|x-c| < r$

the series converges absolutely.

for $x < c-r$ and $x > c+r$ ie $|x-c| > r$

the series diverges.

(nothing said about $x = c-r$, $x = c+r$ ie $|x-c| = r$

must check those.)

Here, r is the radius of convergence and $(c-r, c+r)$ is the interval of convergence.

Ex. $\sum_{n=0}^{\infty} n!(x-2)^n$ Find the radius and interval of convergence.

Ratio test: for $x \neq 2$
$$\lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}\,(x-2)^{\cancel{n}+1}}{\cancel{n!}\,\cancel{(x-2)^n}} \right| =$$

$$= \lim_{n\to\infty} |(n+1)(x-2)| = \infty = L > 1$$

$\therefore$ The series diverges for $x \neq 2$.

For $x=2$, get $\underbrace{0!}_{1}\underbrace{(x-2)^0}_{1} + 1!(x-2)^1 + 2!(x-2)^2 + \cdots$

then plug $x=2$ $= 1 + 0 + 0 \cdots = 1$.

$\therefore$ The series converges for $x=2$ only.

(power series always converge at their center)

radius of convergence $= 0$, interval of convergence $x=2$. (not really an interval)

Ex. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n4^n}$ find the x-values for which the series converges.

Ratio test: $\lim_{n\to\infty} \left| \frac{(x-1)^{\cancel{n+1}}}{(n+1)\,4^{\cancel{n+1}}} \cdot \frac{n\,\cancel{4^n}}{\cancel{(x-1)^n}} \right| =$

$$= \lim_{n\to\infty} \left| \frac{(x-1)}{4} \cdot \frac{\cancel{n}^{\,1}}{(\cancel{n}+1)} \right| = \left| \frac{x-1}{4} \right| = L$$

if $\left|\frac{x-1}{4}\right| < 1$, converges absolutely

$\left|\frac{x-1}{4}\right| > 1$, diverges

$\left|\frac{x-1}{4}\right| = 1$ have to check.

$\left|\frac{x-1}{4}\right| < 1$

$\frac{|x-1|}{4} < 1$

$|x-1| < 4$

$-4 < x-1 < 4$

$+1 \quad\ +1 \quad +1$

$-3 < x < 5$ absolute convergence

divergence for $x < -3$ and $x > 5$,

but what about $x=-3$ and $x=5$?

this is when $\left|\frac{x-1}{4}\right|=1$

$$x=-3: \quad \sum_{n=1}^{\infty} \frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{n\,4^n} = \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}_{\text{alternating harmonic}}$$

alternating series test

$$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad \frac{1}{n+1} < \frac{1}{n} \ \forall n$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges.}$$

$$x=5: \quad \sum_{n=1}^{\infty} \frac{(4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ harmonic, diverges.}$$

so $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n4^n}$ converges for $x \in [-3, 5)$ and diverges otherwise.

radius of convergence $=4$

interval of convergence $[-3,5)$

Ex. $\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$ find the x-values for which the series converges

notice $\frac{(2x)^n}{n!} = \frac{2^n x^n}{n!}$ $\quad a_n = \frac{2^n}{n!}$

Ratio test: $\lim_{n\to\infty} \left| \frac{(2x)^{n+1}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{(2x)^n}} \right| =$

$$= \lim_{n\to\infty} \left| \frac{2x}{n+1} \right| = 0 \quad \forall x$$

$$\uparrow$$

$$L < 1 \quad \forall x$$

$\therefore$ the series converges absolutely $\forall x \in \mathbb{R}$

radius of convergence : ∞

interval of convergence : $\mathbb{R}$

We'll see in Lesson 34 that this series converges to e^{2x} for all x.

Ex. $\sum_{n=0}^{\infty} (3x+1)^n$ find the interval + radius of convergence.

$$(3x+1)^n = \left(3\left(x+\tfrac{1}{3}\right)\right)^n = 3^n \left(x+\tfrac{1}{3}\right)^n$$

center at $x = -\frac{1}{3}$

Can use Ratio test, but more info if we recognize this as a geometric series

$r = 3x+1 \quad a = 1$

converges to $\dfrac{1}{1-(3x-1)}$ for $|3x+1| < 1$

$\dfrac{1}{2-3x}$

$$-1 < 3x+1 < 1$$
$$-2 < 3x < 0$$
$$-\frac{2}{3} < x < 0$$

diverges for $|3x+1| \geq 1$ $\quad x \leq -\frac{2}{3}$, $x \geq 0$

interval of convergence: $\left(-\frac{2}{3}, 0\right)$

radius of convergence: $\frac{1}{3}$

Ex. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ find the x-values for which the series converges

Ratio test: $\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right|$

$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} x^{\cancel{2n+2}}}{\underset{(2n+2)(2n+1)\cancel{(2n)!}}{(2n+2)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{(-1)^n} \cancel{x^{2n}}} \right|$

$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = \underset{= L < 1}{0} \quad \forall x.$

$\therefore$ series converges absolutely $\forall x$.

In lesson 34, we'll see that this series converges to $\cos(x)$ $\forall x$.