Math 20300
Calculus III

Lesson 31
Series Convergence Tests

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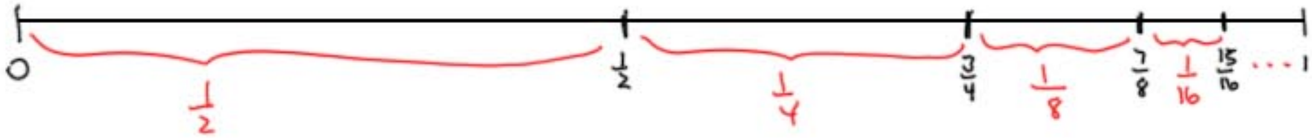
Series Convergence Tests

lesson 30

Convergence Test:	When it can be used:	Conclusions:
Geometric Series	$\sum_{k=0}^{\infty} ar^k$ or $\sum_{n=1}^{\infty} ar^{n-1}$	For $\lvert r\rvert < 1$, converges to $\frac{a}{1-r}$ For $\lvert r\rvert \geq 1$, diverges
A Test for Divergence	All series	If $\lim_{k\to\infty} a_k \neq 0$, the series diverges.
The Integral Test	$\sum_{n=1}^{\infty} a_n$ where $a_n = f(n)$ for f continuous and decreasing, and $f(x) \geq 0$	$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both converge or both diverge
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges for $p > 1$, diverges for $p \leq 1$.
The Comparison Test	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ where $0 \leq a_n \leq b_n$	If $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ diverges
The Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ where $a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge
The Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ where $a_n > 0$ for all n	If $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} \leq a_n$ for all n, then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.
Absolute Convergence	Any series with some positive and some negative terms	If $\sum_{n=1}^{\infty} \lvert a_n\rvert$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely. (This implies convergence.)
The Ratio Test	Any series (especially those with exponentials or factorials)	For $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L$, if $L < 1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely, if $L > 1$ or the limit is ∞, $\sum_{n=1}^{\infty} a_n$ diverges, if $L = 1$, no conclusion.
The Root Test	Any series (especially those with exponentials)	For $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L$, if $L < 1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely, if $L > 1$ or the limit is ∞, $\sum_{n=1}^{\infty} a_n$ diverges, if $L = 1$, no conclusion.

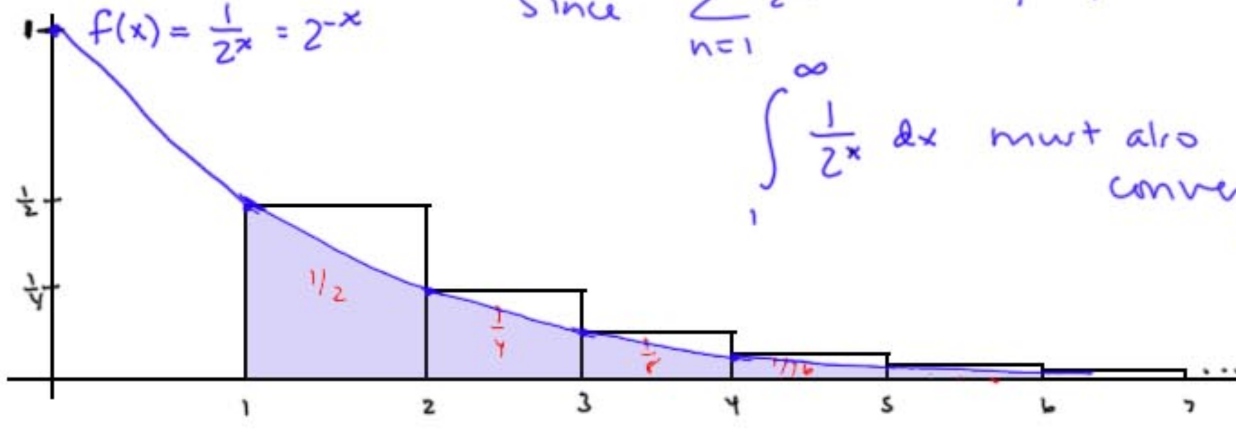
We saw in lesson 30:



$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$

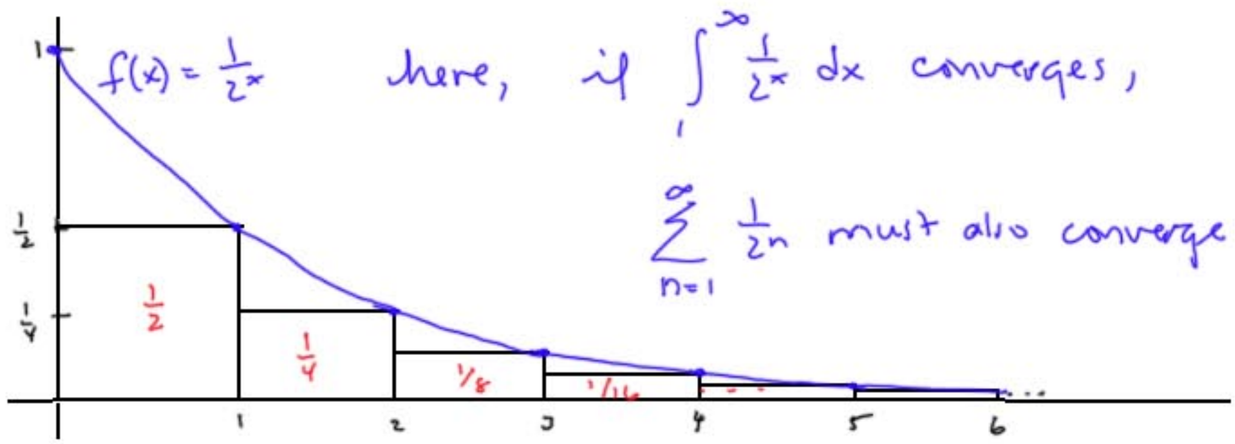
$$a_n=\frac{1}{2^n} \qquad \sum_{n=1}^{\infty}\frac{1}{2^n}=1$$

Consider this as a sum of areas:



since $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges,

$\int_1^{\infty}\frac{1}{2^x}\,dx$ must also converge



here, if $\int_1^{\infty}\frac{1}{2^x}\,dx$ converges,

$\sum_{n=1}^{\infty}\frac{1}{2^n}$ must also converge

Recall: $\int_1^{\infty} \frac{1}{2^x}\,dx = \lim_{b\to\infty} \int_1^b 2^{-x}\,dx =$

$$= \lim_{b\to\infty} -\frac{1}{\ln 2} 2^{-x}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{\ln 2}\left(2^{-b} - 2^{-1}\right)$$

$$= \lim_{b\to\infty} -\frac{1}{\ln 2}\left(\frac{1}{2^b} - \frac{1}{2}\right)$$

$$= -\frac{1}{\ln 2}\left(-\frac{1}{2}\right) = \frac{1}{2\ln 2}.$$

The Integral Test	$\sum_{n=1}^{\infty} a_n$ where $a_n = f(n)$ for f continuous and decreasing, for $f(x) \geq 0$	$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both converge or both diverge

Ex. Does $\sum_{n=1}^{\infty} e^{-n}$ converge or diverge?

Consider $f(x) = e^{-x}$ continuous, decreasing, ≥ 0.

$$\int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b =$$

$$= \lim_{b\to\infty} -\left(e^{-b} - e^{-1}\right) = \lim_{b\to\infty} -\left(\cancelto{0}{\frac{1}{e^b}} - \frac{1}{e}\right) = \frac{1}{e}$$

$\therefore \int_1^\infty e^{-x}\,dx$ converges and $\therefore \sum_{n=1}^{\infty} e^{-n}$ converges.

Recall from improper integrals (Calc II) we know $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent for $p>1$
divergent for $p \le 1$

$\therefore$

p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges for $p>1$, diverges for $p \le 1$.

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

Consider $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$. We know $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges

notice $\frac{1}{n^3+n} < \frac{1}{n^3} \quad \forall n \ge 1$

So for any partial sum :

$$S_N = \sum_{n=1}^{N} \frac{1}{n^3+n} < \sum_{n=1}^{N} \frac{1}{n^3} < \sum_{n=1}^{\infty} \frac{1}{n^3} = S$$

So S_N is a sequence that is monotonically increasing, and bounded (above by S, below by S_1)

$$\therefore S_N \text{ converges } \therefore \sum_{n=1}^{\infty} \frac{1}{n^3+n} \text{ converges.}$$

The Comparison Test	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ where $0 \le a_n \le b_n$	If $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ diverges

Ex. $\sum_{k=1}^{\infty} \frac{3^k+1}{2^k}$

$$\underset{\substack{\uparrow \\ b_n}}{\frac{3^k+1}{2^k}} > \underset{\substack{\uparrow \\ a_n}}{\frac{3^k}{2^k}} \quad + \quad \underset{\substack{\uparrow \\ a_n}}{\sum_{k=1}^{\infty} \left(\frac{3}{2}\right)^k} \text{ diverges}$$

$$\therefore \sum_{k=1}^{\infty} \frac{3^k+1}{2^k} \text{ diverges.}$$

Now, what about $\sum_{k=1}^{\infty} \frac{3^k-1}{2^k}$? $\quad \frac{3^k-1}{2^k} < \frac{3^k}{2^k}$

so comparison test doesn't apply.

but the behavior is so much like $\frac{3^k}{2^k}$, so

we think $\sum_{k=1}^{\infty} \frac{3^k - 1}{2^k}$ should diverge.

The Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ where $a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge

$$\lim_{k\to\infty} \frac{\frac{3^k - 1}{2^k}}{\frac{3^k}{2^k}} = \lim_{k\to\infty} \frac{3^k - 1}{\cancel{2^k}} \cdot \frac{\cancel{2^k}}{3^k} = \lim_{k\to\infty} \frac{3^k - 1}{3^k}$$

$$\overset{LH}{=} \lim_{k\to\infty} \frac{3^k \ln 3}{3^k \ln 3} = 1$$

$\therefore$ Since $\sum_{k=1}^{\infty} \left(\frac{3}{2}\right)^k$ diverges, $\sum_{k=1}^{\infty} \frac{3^k - 1}{2^k}$ diverges also.

The limit comparison test is also useful with rational expressions when the comparison test doesn't apply:

Ex. $\sum_{n=2}^{\infty} \frac{2n}{n^3 - 1}$ behaves like $\frac{2n}{n^3} = \frac{2}{n^2}$ converges (p-series)

but $\frac{2n}{n^3 - 1} > \frac{2n}{n^3}$ so the comparison theorem doesn't help.

Use limit comparison:

note $\frac{2n}{n^3-1} > 0$ and $\frac{2n}{n^3} > 0$ for $n \geq 2$ as defined,

and $\lim_{n\to\infty} \frac{\frac{2n}{n^3-1}}{\frac{2n}{n^3}} = \lim_{n\to\infty} \frac{2n}{n^3-1} \cdot \frac{n^3}{2n} = \lim_{n\to\infty} \frac{n^3}{n^3-1} = 1 > 0.$

Since $\sum_{n=2}^{\infty} \frac{2n}{n^3}$ converges, so does $\sum_{n=2}^{\infty} \frac{2n}{n^3-1}$.

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We know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

but what about $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

is the alternating sign enough for convergence?

The Alternating Series Test	$\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ where $a_n > 0$ for all n	If $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} \leq a_n$ for all n, then the series $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges.

Proof uses that the sequence of partial sums S_{2N} is monotonic and bounded, and that $\lim_{n\to\infty} a_n = 0$

to show S_{2N+1} converges to same limit.

So for $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ notice notation $a_n = \frac{1}{n} > 0 \ \forall n$

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n+1} < \frac{1}{n} \ \forall n$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges.}$$

Note: Can also use for $\sum_{n=1}^{\infty} (-1)^n a_n$ ...

** The Alternating Series Estimation Theorem:

If $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, $a_n > 0$ satisfies $\lim_{n\to\infty} a_n = 0$ and

$a_{n+1} \leq a_n \ \forall n$

(same conditions as alternating series test)

then $|S - S_n| \leq a_{n+1}$

$\nwarrow$ next term in series (without $\pm$)

$\uparrow$

difference between total sum and n^{th} partial sum

So to approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ with $|\text{error}| < .0005$

(accurate to 3 decimal places)

we look for $\underset{a_{n+1}}{\frac{1}{n+1}} < .0005 \Rightarrow \frac{1}{.0005} < n+1$

$$2000 < n+1$$
$$-1 \qquad -1$$
$$1999 < n$$

$\Rightarrow$ choosing $n = 2000$

$\sum_{n=1}^{2000} \frac{(-1)^{n+1}}{n}$ gives an approximation

to $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ with $|error| < .0005$

ie accurate to 3 decimal places.

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For a series like $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ notice that

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series)

and we know $\lim_{n\to\infty} \frac{1}{n^2} = 0$, $\frac{1}{(n+1)^2} < \frac{1}{n^2}\ \forall n$, $\frac{1}{n^2} > 0\ \forall n$

$\therefore$ by alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges

In This case, since $\sum_{n=1}^{\infty} |a_n|$ converges, we say

$\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u>.

Theorem: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent,

it is convergent.

Proof: Notice $0 \le a_n + \underbrace{|a_n|}_{\substack{\text{either} \\ a_n \text{ or } -a_n}} \le 2|a_n|$

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} 2|a_n|$ converges

$\therefore \sum_{n=1}^{\infty} (a_n + |a_n|)$ converges. And Then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n| \quad \text{converges.} \;//$$

So,

Absolute Convergence	Any series with some positive and some negative terms	If $\sum_{n=1}^{\infty} \lvert a_n \rvert$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely. (This implies convergence.)

For $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$

diverges. Since $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges,

$\sum_{n=1}^{\infty} a_n$ is called conditionally convergent.

The Ratio Test	Any series (especially those with exponentials or factorials)	For $\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = L$, if $L<1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely, if $L>1$ or the limit is ∞, $\sum_{n=1}^{\infty} a_n$ diverges, if $L=1$, no conclusion.

Ex. $\sum_{n=1}^{\infty} \frac{e^n}{(n+1)!}$ $\quad a_n = \frac{e^n}{(n+1)!} \quad a_{n+1} = \frac{e^{n+1}}{(n+1+1)!} = \frac{e^{n+1}}{(n+2)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{e^{\cancel{n+1}}}{(n+2)\cancel{!}} \cdot \frac{\cancel{(n+1)!}}{\cancel{e^n}} \right| =$$

$$= \lim_{n\to\infty} \left|\frac{e}{n+2}\right| = 0 \quad \therefore \sum_{n=1}^{\infty} \frac{e^n}{(n+1)!} \text{ converges.}$$

The Root Test	Any series (especially those with exponentials) $(\ )^n$	For $\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert}=L$, if $L<1$, $\sum_{n=1}^{\infty}a_n$ converges absolutely, if $L>1$ or the limit is ∞, $\sum_{n=1}^{\infty}a_n$ diverges, if $L=1$, no conclusion.

$$\sum_{k=4}^{\infty}\left(\frac{e^k}{k^2}\right)^k \qquad \lim_{k\to\infty}\sqrt[k]{\left(\frac{e^k}{k^2}\right)^k} = \lim_{k\to\infty}\left(\left(\frac{e^k}{k^2}\right)^k\right)^{1/k}$$

$$= \lim_{k\to\infty}\frac{e^k}{k^2} = \infty \quad \therefore \sum_{k=4}^{\infty}\left(\frac{e^k}{k^2}\right)^k \text{ diverges.}$$

(L'Hopital's Rule)

Ex. $\sum_{n=1}^{\infty}\frac{3^n}{n^2 2^n}$ even though has $\left(\frac{3}{2}\right)^n$,

$$\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert} = \lim_{n\to\infty}\left(\frac{3^n}{n^2 2^n}\right)^{1/n} \qquad \underbrace{(n^2)^{1/n}}_{\text{requires logarithmic limit.}}$$

better choice would be ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{\cancel{n+1}}}{(n+1)^2 2^{\cancel{n+1}}}\cdot\frac{n^2\cancel{2^n}}{\cancel{3^n}}\right| = \lim_{n\to\infty}\frac{3}{2}\cdot\frac{n^2}{(n+1)^2}$$

$$= \frac{3}{2} > 1 \qquad \therefore \sum_{n=1}^{\infty}\frac{3^n}{n^2 2^n} \text{ diverges.}$$