Math 20300
Calculus III

Lesson 28
Spherical Coordinates

Dr. A. Marchese, The City College of New York

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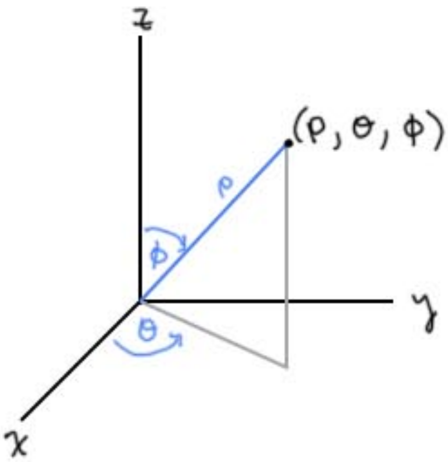
Spherical Coordinates

Another way to describe points in $\mathbb{R}^3$:



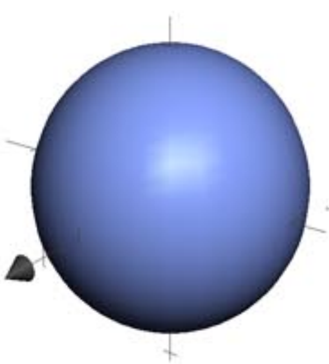
ρ = distance to the origin (like r in polar coords) $\rho \geq 0$

θ = angle from the positive x-axis (same as polar coords)

ϕ = angle from the positive z-axis $\quad 0 \leq \phi \leq \pi$

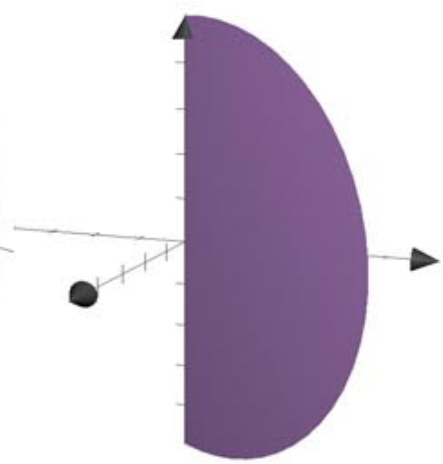
"azimuthal angle"

Some basic surfaces in spherical coordinates.



ρ = constant

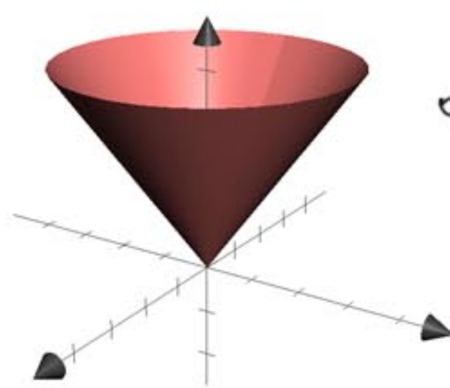
sphere



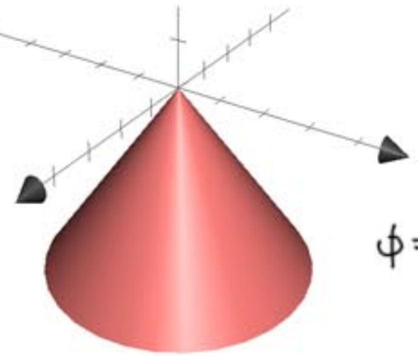
θ = constant

half plane

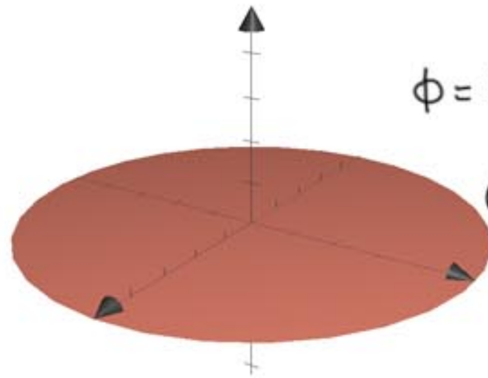
(different than polar coords because $\rho \geq 0$, but no restriction on r)



ϕ = constant $< \frac{\pi}{2}$



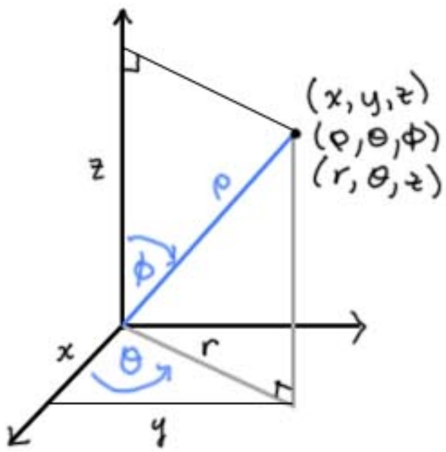
$\phi = \text{constant} > \frac{\pi}{2}$



$\phi = \frac{\pi}{2}$

(xy plane)

Conversion between rectangular, spherical, and cylindrical coordinates:

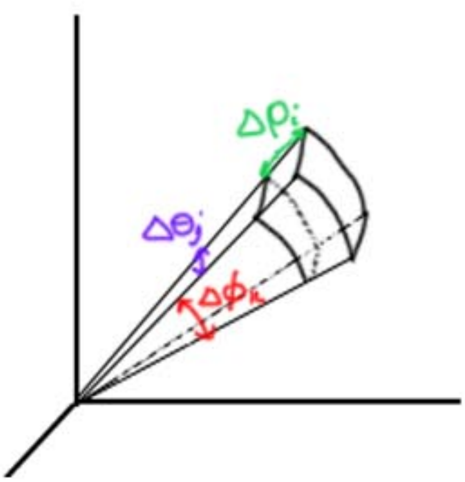


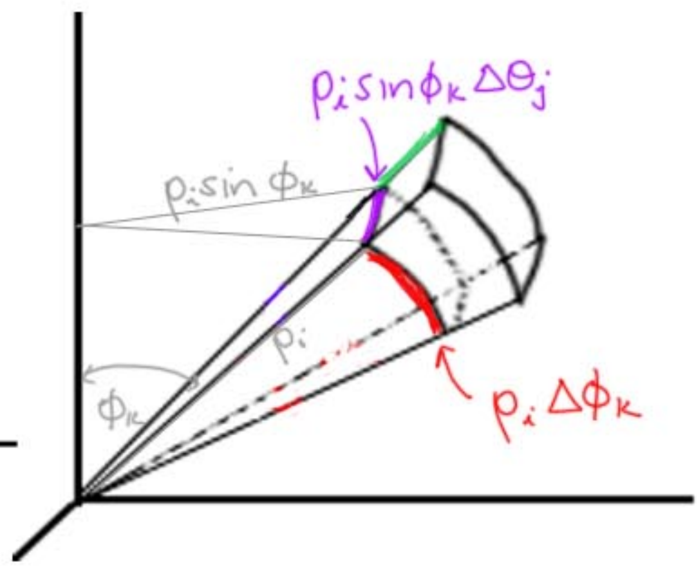
$r = \rho \sin\phi \qquad x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$





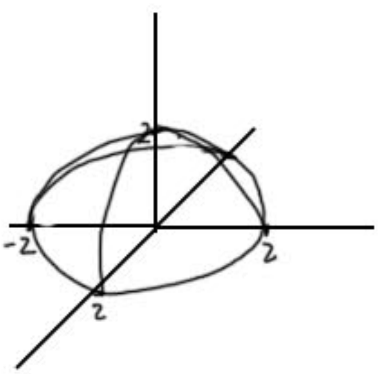
$S = r\theta$
arc length

spherical wedge volume $\approx \rho_i^2 \sin\phi_k \Delta\rho_i \Delta\theta_j \Delta\phi_k$

So that for the triple integral in spherical coordinates we have $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

Ex. Find the center of mass of a hemisphere of radius 2 with density proportional to distance from the origin squared.

ρ



$$\bar{x} = 0$$
$$\bar{y} = 0$$

$$\bar{z} = \frac{1}{\text{mass}(D)} \iiint_D z\,(\text{density}(x,y,z))\, dV$$

$$\text{mass}(D) = \iiint_D \text{density}(x,y,z)\, \underbrace{dV}_{\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$

here, density $d(\rho,\theta,\phi) = k\rho^2$

$$\text{so mass}(D) = \int_0^{\pi/2}\int_0^{2\pi}\int_0^{2} k\rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$0 \le \rho \le 2$$
$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \pi/2$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \frac{\rho^5}{5} k \sin\phi \Big|_0^2 \, d\theta \, d\phi =$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \frac{32}{5} k \sin\phi \, d\theta \, d\phi =$$

$$= \int_0^{\pi/2} \frac{32}{5} k \theta \sin\phi \Big|_0^{2\pi} d\phi = \int_0^{\pi/2} \frac{64\pi k}{5} \sin\phi \, d\phi$$

$$= -\frac{64\pi k}{5} \cos\phi \Big|_0^{\pi/2} = -\frac{64\pi k}{5}(0-1) = \frac{64\pi k}{5}$$

$= \text{mass}$

then $\bar{z} = \dfrac{5}{64\pi k} \displaystyle\int_0^{\pi/2} \int_0^{2\pi} \int_0^2 \underbrace{\rho\cos\phi}_{z} \, \underbrace{k\rho^2}_{\text{density}} \, \underbrace{\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}_{dV}$

$$= \frac{5}{64\pi k} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2} \rho^5 k \cos\phi \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{5}{64\pi \cancel{k}} \int_0^{\pi/2} \int_0^{2\pi} \frac{\rho^6}{6} \cancel{k} \cos\phi \sin\phi \Big|_0^2 \, d\theta \, d\phi$$

$$= \frac{5}{\cancel{64}\pi} \int_0^{\pi/2} \int_0^{2\pi} \frac{\cancel{64}}{6} \cos\phi \sin\phi \, d\theta \, d\phi$$

$$= \frac{5}{6\pi} \int_0^{\pi/2} \cos\phi \sin\phi \, \theta \Big|_0^{2\pi} \, d\phi$$

$$= \frac{5}{3\cancel{6}\cancel{\pi}} \cdot \cancel{2\pi} \int_0^{\pi/2} \cos\phi \sin\phi \, d\phi$$

$u = \sin\phi$

$du = \cos\phi \, d\phi$

$\phi = 0 \quad u = 0$

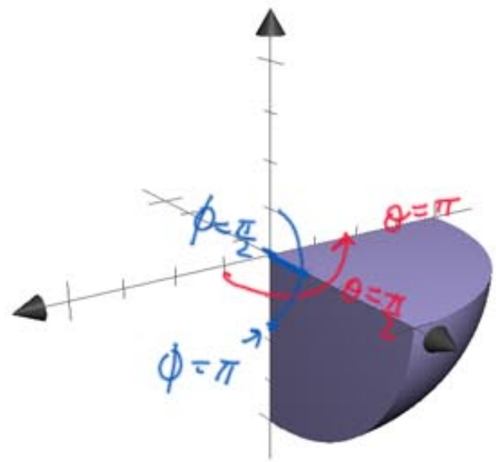
$\phi = \frac{\pi}{2} \quad u = 1$

$$= \frac{5}{3} \int_0^1 u \, du = \frac{5}{3} \frac{u^2}{2} \Big|_0^1 = \frac{5}{6} = \bar{z}.$$

Center of mass $(0, 0, \frac{5}{6})$.

Ex. Find the volume of the part of the solid (spherical) ball of radius 1 with

$x < 0,\ y > 0,\ z < 0$.



$0 \le \rho \le 1$

$\frac{\pi}{2} \le \theta \le \pi$

$\frac{\pi}{2} \le \phi \le \pi$

$$\int_{\frac{\pi}{2}}^{\pi} \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{1} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_{\frac{\pi}{2}}^{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{\rho^3}{3} \sin\phi \Big|_0^1 d\theta \, d\phi = \int_{\frac{\pi}{2}}^{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{1}{3} \sin\phi \, d\theta \, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \frac{1}{3} \theta \sin\phi \Big|_{\frac{\pi}{2}}^{\pi} d\phi = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{3} \cdot \frac{\pi}{2} \sin\phi \, d\phi =$$

$$= -\frac{\pi}{6}\cos\phi\Big|_{\pi/2}^{\pi} = -\frac{\pi}{6}(-1-0) = \frac{\pi}{6}.$$

(we know volume of a sphere $= \frac{4}{3}\pi r^3$, and this is

$\frac{1}{8}$ of the volume of a sphere with $r=1$,

$\frac{1}{8}\cdot\frac{4}{3}\pi = \frac{\pi}{6}$.)

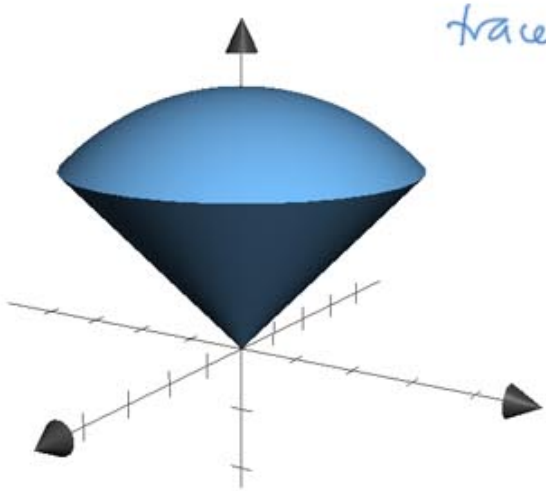
When should you choose spherical coordinates for your triple integral?

Only consider spherical coordinates when your domain is a sphere or part of a sphere.

Spherical coordinates still might not be the best choice, but that's when you consider it.

Ex. Find $\iiint_D (x^2+y^2)\,dV$ where D is the solid below the sphere

$x^2+y^2+z^2=4$ and above the cone $z=\sqrt{x^2+y^2}$



traces: $x=0 \quad z=\sqrt{y^2}=|y|$
$y=0 \quad z=|x|$

$0 \le \rho \le 2$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/4$

$x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$$z=\sqrt{x^2+y^2} \Rightarrow z^2 = x^2+y^2$$

$$\rho^2\cos^2\phi = \rho^2\sin^2\phi\,\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}$$

$$\cos^2\phi = \sin^2\phi$$

$$\tan^2\phi = 1$$

$$\tan\phi = \pm 1 \qquad 0 \le \phi \le \pi$$

$$\phi = \frac{\pi}{4}, \frac{3\pi}{4} \quad \text{but} \quad z \geq 0$$

$$\rho\cos\phi \geq 0$$

$$\cos\phi \geq 0$$

$$\therefore \phi = \frac{\pi}{4}.$$

So $\displaystyle\iiint_D (x^2+y^2)\,dV \qquad x^2+y^2 = r^2 = \rho^2\sin^2\phi$

Spherical: $\int_0^{\pi/4}\int_0^{2\pi}\int_0^{2} \rho^2 \sin^2\phi \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

$$= \int_0^{\pi/4}\int_0^{2\pi}\int_0^{2} \rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi .$$

$$= \dots = \frac{64\pi}{5} \int_0^{\pi/4} \sin^3\phi \, d\phi$$

$$= \frac{64\pi}{5} \int_0^{\pi/4} \sin\phi \, (\underbrace{\sin^2\phi}_{(1-\cos^2\phi)}) \, d\phi \quad \dots$$

Cylindrical: $x^2+y^2=r^2$

top: $z = \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$

bottom $z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$

for domain in $\mathbb{R}^2$, need curve of intersection:

$4-x^2-y^2 = x^2+y^2$

$4 = 2x^2+2y^2$

$x^2+y^2 = 2$

$r^2 = 2$ $\quad 0 \le r \le \sqrt{2}$ and $0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} r^2\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \Big(\underbrace{r^3\sqrt{4-r^2}}_{\text{u-sub}} - r^4\Big)\,dr\,d\theta \quad = \ldots$$

$$= -\frac{16\pi}{15}\left(5\sqrt{2}-8\right).$$