Math 20300
Calculus III

Lesson 26
Triple Integrals

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Triple Integrals

In lesson 24, we saw total mass of a thin plate. Now we want total mass of a solid, with density = $\frac{\text{mass}}{\text{volume}}$

for a thin plate

$$\iint_D \rho(x,y)\, dA$$

domain in $\mathbb{R}^2$

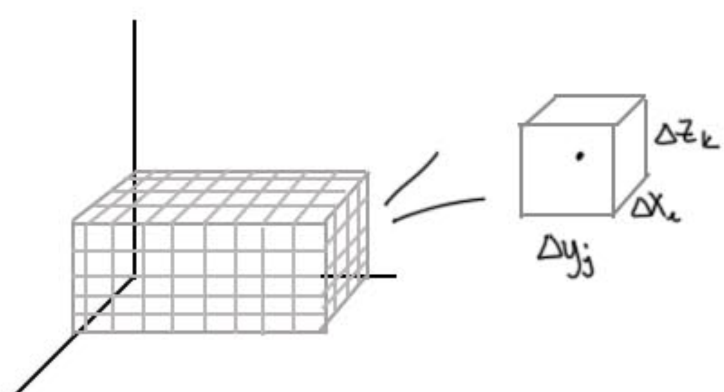
for a solid

$$\iiint_D \rho(x,y,z)\, dV$$

volume

domain in $\mathbb{R}^3$

Definition of Triple Integral:



(x_i^*, y_j^*, z_k^*) a point in this box

$D = \{(x,y,z): a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$

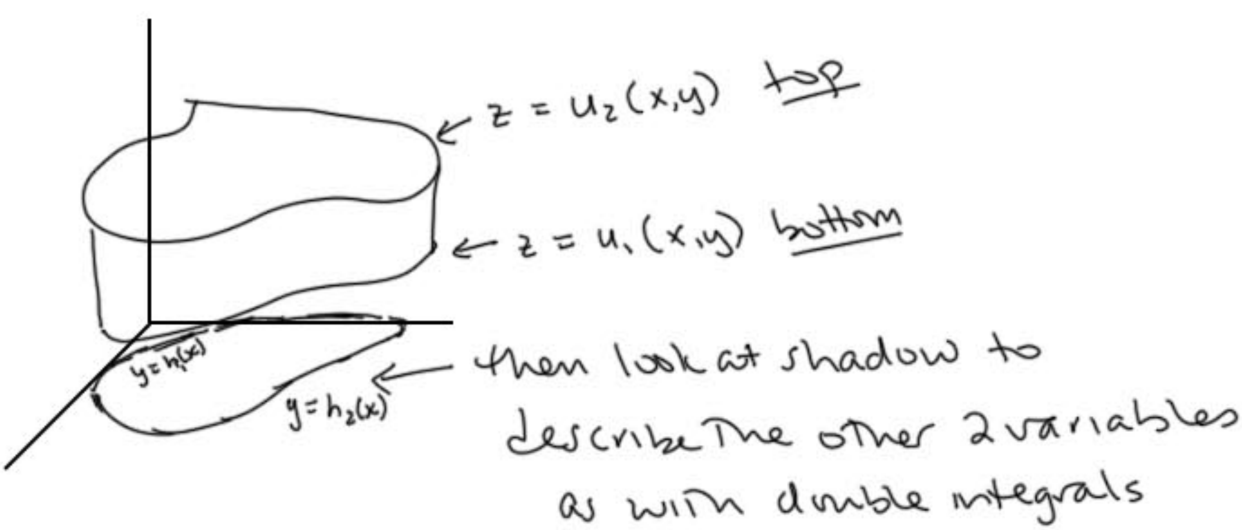
$$\iiint_D f(x,y,z)\, dV = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0 \\ \max \Delta z_k \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} f(x_i^*, y_j^*, z_k^*)\, \Delta x_i\, \Delta y_j\, \Delta z_k$$

and Fubini's Theorem says:

$$\underbrace{\iiint_D f(x,y,z)\,dV}_{\text{"triple integral"}} = \underbrace{\int_r^s \int_c^d \int_a^b f(x,y,z)}_{\text{"iterated integral"}}\; \underbrace{dx\, dy\, dz}_{\text{or any order, with appropriate bounds.}}$$

For integrating over general regions in $\mathbb{R}^3$, we have 3 types of domains:

Type 1, top to bottom region



$$\int_a^b \int_{h_1(x)}^{h_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

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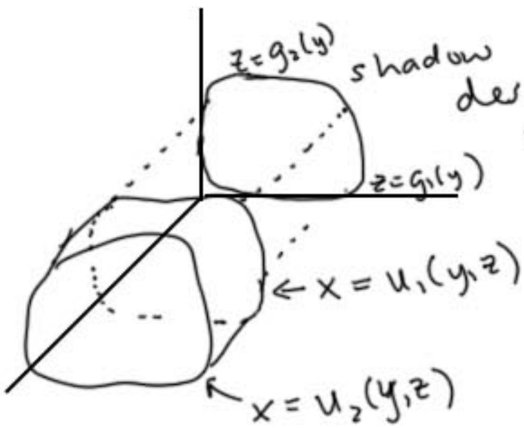
Type 2, front to back



describe as you would a double integral

$\leftarrow$ integrate first on x

$$\int_c^d \int_{g_1(y)}^{g_2(y)} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx\, dz\, dy$$

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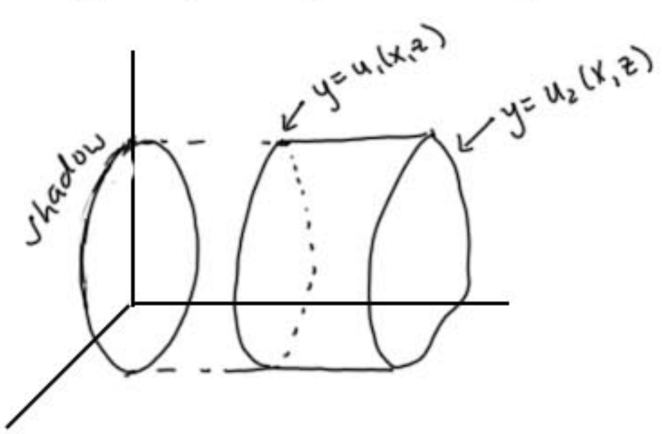
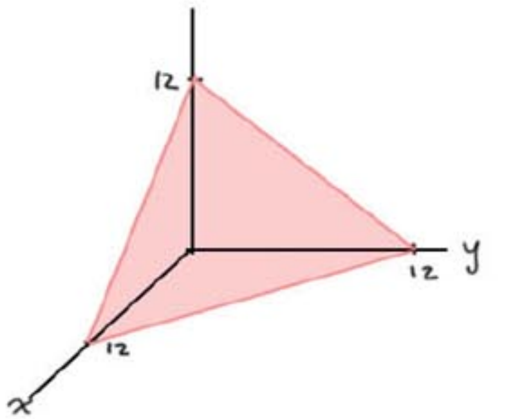
Type 3, right to left



$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy\, dz\, dx$$

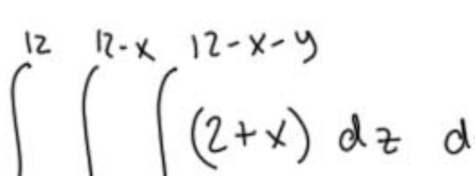
Ex. Find the mass of the solid bounded by the coordinate planes and the plane $x+y+z=12$ with density $\rho(x,y,z) = 2+x$



we can describe this solid as type 1, type 2, or type 3.

as Type 1, top to bottom, $0 \le z \le 12-x-y$

then look at the shadow in the xy plane



$0 \le y \le 12-x$ or $0 \le x \le 12-y$

$0 \le x \le 12$ $\quad$ $0 \le y \le 12$

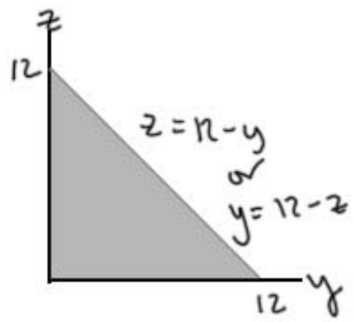
$$\int_0^{12}\int_0^{12-x}\int_0^{12-x-y} (2+x)\, dz\, dy\, dx \quad \text{or} \quad \int_0^{12}\int_0^{12-y}\int_0^{12-x-y} (2+x)\, dz\, dx$$

If we view as a Type 2, front to back:

$$x + y + z = 12 \implies x = 12 - y - z$$

$$0 \le x \le 12 - y - z$$

Then look at shadow in yz plane:



$$\begin{matrix} 0 \le z \le 12 - y \\ 0 \le y \le 12 \end{matrix} \quad \text{OR} \quad \begin{matrix} 0 \le y \le 12 - z \\ 0 \le z \le 12 \end{matrix}$$

$$\int_0^{12}\int_0^{12-y}\int_0^{12-y-z} (2+x)\,dx\,dz\,dy \quad \text{OR} \quad \int_0^{12}\int_0^{12-z}\int_0^{12-y-z} (2+x)\,dx\,dy\,dz$$

Type 3, right to left

$$0 \le y \le 12 - x - z \qquad \begin{matrix} 0 \le z \le 12 - x \\ 0 \le x \le 12 \end{matrix} \quad \text{OR} \quad \begin{matrix} 0 \le x \le 12 - z \\ 0 \le z \le 12 \end{matrix}$$

$$\int_0^{12}\int_0^{12-x}\int_0^{12-x-z} (2+x)\,dy\,dz\,dx \quad \text{OR} \quad \int_0^{12}\int_0^{12-z}\int_0^{12-x-z} (2+x)\,dy\,dx\,dz$$

all 6 integrals give same answer, we'll solve one:

$$\int_0^{12}\int_0^{12-z}\int_0^{12-x-z}(2+x)\,dy\,dx\,dz = \int_0^{12}\int_0^{12-z}(2+x)y\Big|_0^{12-x-z}dx\,dz$$

$$= \int_0^{12}\int_0^{12-z}(2+x)(12-x-z)\,dx\,dz = \int_0^{12}\int_0^{12-z}\left(24+10x-x^2-2z-xz\right)dx\,dz$$

$$24+12x-2x-x^2-2z-xz$$

$$=\int_0^{12}\left[24x+5x^2-\frac{x^3}{3}-2xz-\frac{x^2}{2}z\right]_0^{12-z}dz$$

$$=\int_0^{12}\left[24(12-z)+5(12-z)^2-\frac{(12-z)^3}{3}-2(12-z)z-\frac{(12-z)^2z}{2}\right]dz$$

$$-24z+2z^2 \qquad -\frac{(144-24z+z^2)z}{2}$$

$$\left[24\left(12z-\frac{z^2}{2}\right)-5\frac{(12-z)^3}{3}+\frac{(12-z)^4}{12}-12z^2+\frac{2z^3}{3}-\frac{72z^2}{2}+\frac{8z^3}{2}-\frac{z^4}{2}\right]_0^{12}$$

$$\left[24(144-72)-5(0)+(0)-12(144)+\frac{2(12)^3}{3}-\frac{72(144)}{2}+\frac{8(12)^3}{2}-\frac{12^4}{8}\right]$$

$$- \left[-5\frac{(12)^3}{3} + \frac{12^4}{12} \right] =$$

$$= 24(72) - 1728 + 2 \cdot \frac{1728}{3} - 5184 + 6912 - 2592$$

$$+ \frac{5(1728)}{3} - 1728 =$$

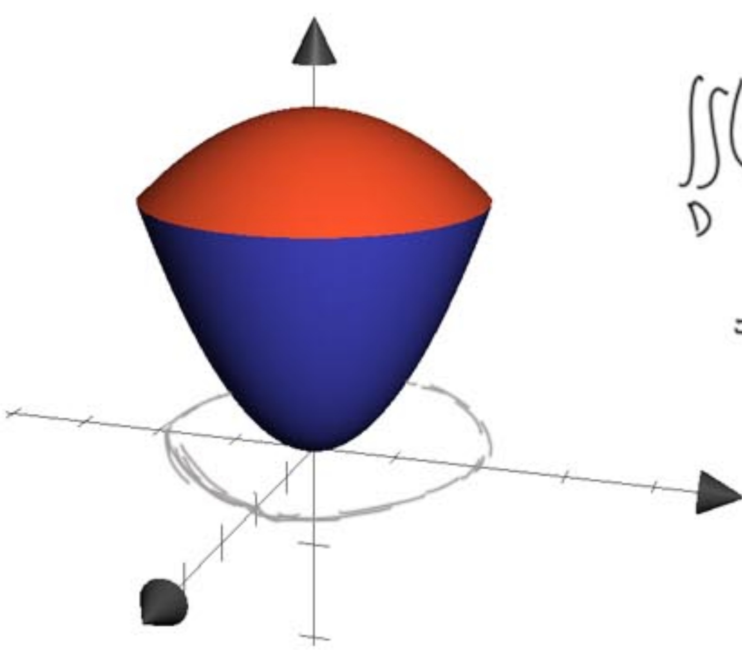
$$= 1728 - 1728 + 1152 - 5184 + 6912 - 2592 + 2880 - 1728$$

$$= 1440 \text{ total mass}$$

Ex. Find the volume of the solid between

$$z = 16 - x^2 - y^2 \quad \text{and} \quad z = 3x^2 + 3y^2$$

from lesson 23:



$$\iint_D \left((16 - x^2 - y^2) - (3x^2 + 3y^2) \right) dA$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} (16 - 4x^2 - 4y^2)\, dy\, dx$$

$$\int_0^{2\pi} \int_0^2 (16 - 4r^2)\, r\, dr\, d\theta$$

Can also use a triple integral of the function

$f(x,y,z) = 1$:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{3x^2+3y^2}^{16-x^2-y^2} 1 \; dz\, dy\, dx \quad \text{OR} \quad \int_0^{2\pi}\int_0^{2}\int_{3r^2}^{16-r^2} 1 \; dz \; r\, dr\, d\theta$$

Average value of $w = f(x,y,z)$ over domain D

is $\displaystyle \frac{1}{\text{Volume}(D)} \iiint_D f(x,y,z)\, dV$

and $\displaystyle \text{volume}(D) = \iiint_D dV$.

$$\text{Mass} = \iiint_D \rho(x,y,z)\, dV \qquad \rho(x,y,z) \text{ density}$$

Center of mass $\displaystyle \bar{x} = \frac{1}{\text{mass}(D)} \iiint_D x\, \rho(x,y,z)\, dV$

$$\bar{y} = \frac{1}{\text{mass}(D)} \iiint_D y\, \rho(x,y,z)\, dV$$

$$\bar{z} = \frac{1}{\text{mass}(D)} \iiint_D z\, \rho(x,y,z)\, dV$$

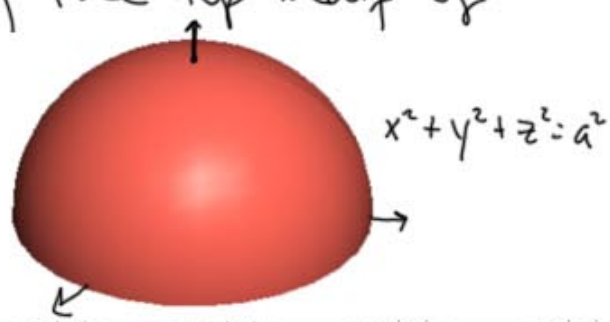
moments of inertia:

about x axis: $I_x = \iiint_D (y^2+z^2)\, dV$

about y axis: $I_y = \iiint_D (x^2+z^2)\, dV$

about z axis: $I_z = \iiint_D (x^2+y^2)\, dV$

Ex. Find the center of mass of the top half of the sphere of radius a.



we know $\bar{x} = \bar{y} = 0$

we can assume $\rho(x,y,z)=1$, so mass = volume

$$= \frac{1}{2}\,\frac{4}{3}\pi r^3$$

$$= \frac{2}{3}\pi a^3$$

$$\bar{z} = \frac{1}{\frac{2}{3}\pi a^3}\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} z\,dz\; r\,dr\,d\theta$$

$$= \frac{3}{2\pi a^3}\int_0^{2\pi}\int_0^{a} r\,\frac{z^2}{2}\Big|_0^{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$= \frac{3}{2\pi a^3}\int_0^{2\pi}\int_0^{a} \frac{r}{2}(a^2-r^2)\,dr\,d\theta$$

$$= \frac{3}{2\pi a^3}\int_0^{2\pi}\int_0^{a} \frac{1}{2}(a^2r-r^3)\,dr\,d\theta$$

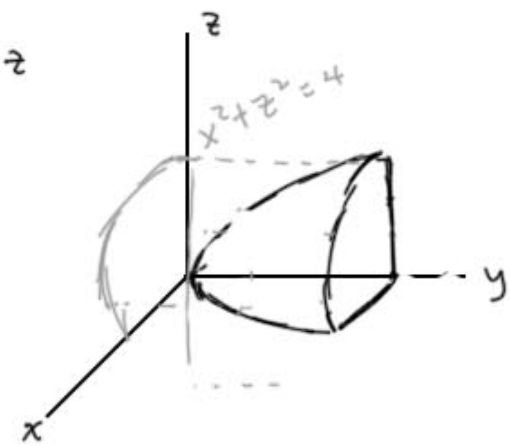
$$= \frac{3}{2\pi a^3}\int_0^{2\pi}\left[\frac{1}{2}a^2\frac{r^2}{2}-\frac{r^4}{8}\right]_0^a d\theta$$

$$= \frac{3}{2\pi a^3}\int_0^{2\pi}\left(\frac{1}{4}a^4-\frac{1}{8}a^4\right)d\theta = \frac{3}{2\pi a^3}\int_0^{2\pi}\frac{1}{8}a^4\,d\theta$$

$$= \frac{3a}{16\pi}\,\theta\Big|_0^{2\pi} = \frac{3a\cdot\cancel{2}\cancel{\pi}}{8\cancel{16}\cancel{\pi}} = \frac{3}{8}a = \bar{z}.$$

Ex. Sketch the solid whose volume is given:

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_{x^2+z^2}^{4} dy\, dx\, dz$$



$x^2+z^2 \le y \le 4$

$0 \le x \le \sqrt{4-z^2}$

$0 \le z \le 2$

$y = x^2+z^2$ $\quad x=0$
$\quad y = z^2$

$z=0 \quad y=x^2$

$y=k \quad x^2+z^2=k$

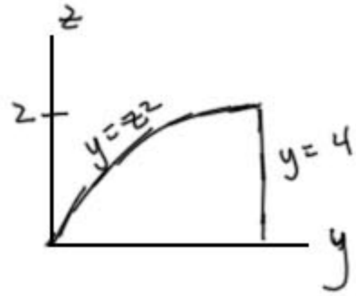
Ex. Now write an integral for the volume above with a different inner integral.

front to back region: front $\;y = x^2+z^2$

$x = \sqrt{y-z^2}$

back $\quad x=0$

now look at shadow:



$0 \le x \le \sqrt{y-z^2}$
$z^2 \le y \le 4$
$0 \le z \le 2$

$$\int_0^2 \int_{z^2}^4 \int_0^{\sqrt{y-z^2}} dx\, dy\, dz$$