Math 20300
Calculus III

Lesson 25
Surface Area

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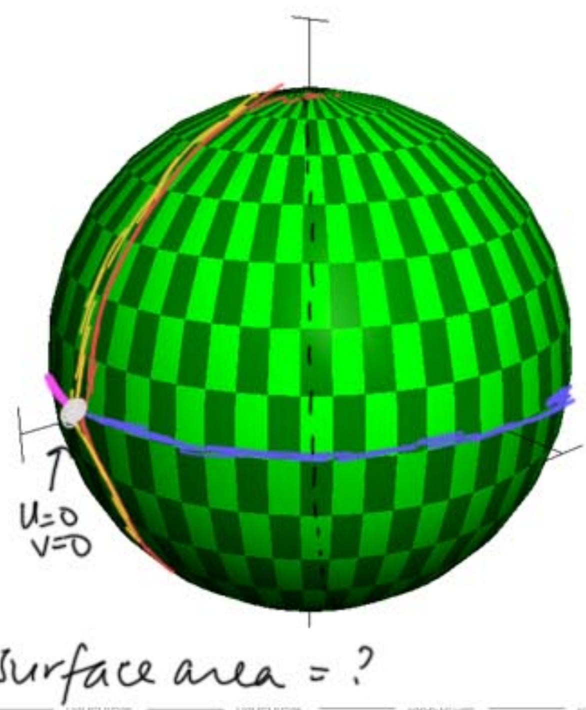
Surface Area

Recall: parametric surfaces

$$x = x(u,v)$$
$$y = y(u,v) \qquad \text{lesson 10}$$
$$z = z(u,v)$$

for example, The sphere of radius 2:

$$x(u,v) = 2\cos u \cos v \qquad 0 \le u \le 2\pi$$
$$y(u,v) = 2\cos u \sin v \qquad 0 \le v \le \pi$$
$$z(u,v) = 2\sin u$$



$$\text{area} = \pi(2\pi) = 2\pi^2$$

surface area = ?

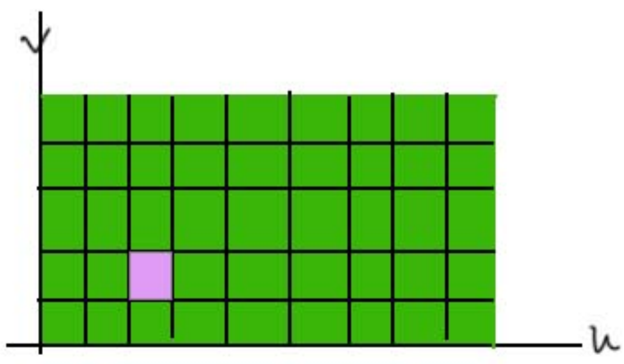
Consider the position vector function

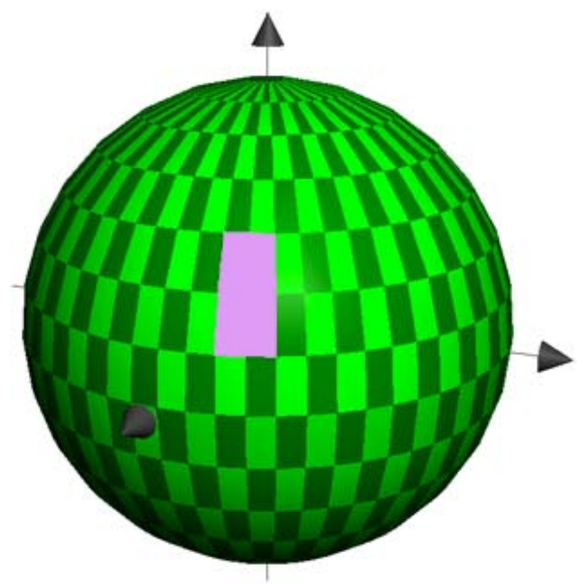
$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

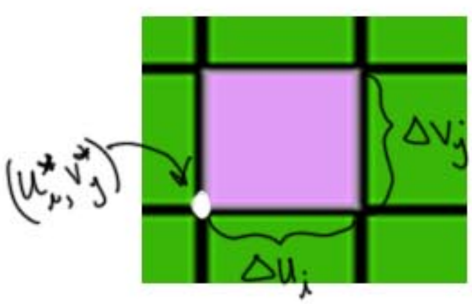
then $\displaystyle \vec{r}_u(u,v) = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$ and

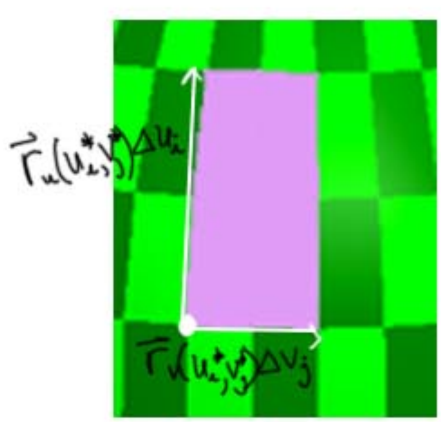
$$\vec{r}_v(u,v) = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$$

are tangent vectors to the surface.



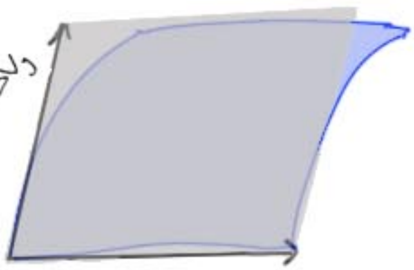






$\vec{r}_u(u_i^*, v_j^*)\,\Delta u_i$ and $\vec{r}_v(u_i^*, v_j^*)\,\Delta v_j$ are tangent vectors that approximate the curve length along the surface patch

In general



area of parallelogram is used to approximate surface area of patch :

$$\|\vec{r}_u(u_i^*, v_j^*)\Delta u_i \times \vec{r}_v(u_i^*, v_j^*)\Delta v_j\|$$

$$= \|\vec{r}_u(u_i^*, v_j^*) \times \vec{r}_v(u_i^*, v_j^*)\| \Delta u_i \Delta v_j$$

and the sum $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} \|\vec{r}_u(u_i^*, v_j^*) \times \vec{r}_v(u_i^*, v_j^*)\| \Delta u_i \Delta v_j$

approximates the surface area of the entire surface

taking limit as $\begin{matrix} \max \Delta u_i \\ \max \Delta v_j \end{matrix} \to 0$,

$$\text{Surface area of } S = \iint_D \|\vec{r}_u \times \vec{r}_v\|\, dA$$

where $\vec{r}(u,v)$ is a parametrization for the smooth surface S over domain D (covering S just once).

$\downarrow$

"smooth" surface means $\|\vec{r}_u \times \vec{r}_v\| \neq 0$ at any $(u,v) \in D$.

Ex. Find the surface area of the sphere above,

$$x(u,v) = 2\cos u \cos v \qquad 0 \le u \le 2\pi$$
$$y(u,v) = 2\cos u \sin v \qquad 0 \le v \le \pi$$
$$z(u,v) = 2\sin u$$

$$\vec{r}_u(u,v) = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$$

$$= -2\sin u\cos v\,\vec{i} - 2\sin u \sin v\,\vec{j} + 2\cos u\,\vec{k}$$

$$\vec{r}_v(u,v) = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$$

$$= -2\cos u \sin v\,\vec{i} + 2\cos u\cos v\,\vec{j} + 0\,\vec{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin u\cos v & -2\sin u \sin v & 2\cos u \\ -2\cos u\sin v & 2\cos u \cos v & 0 \end{vmatrix}$$

$$= -4\cos^2 u \cos v\, \vec{i} - 4\cos^2 u \sin v\, \vec{j} +$$

$$+ \left(-4\sin u \cos u \cos^2 v - 4 \sin u \cos u \sin^2 v\right) \vec{k}$$

$$= -4\cos^2 u \cos v\, \vec{i} - 4\cos^2 u \sin v\, \vec{j} - 4 \sin u \cos u\, \vec{k}$$

and $\|\vec{r}_u \times \vec{r}_v\| =$

$$= \sqrt{16\cos^4 u \cos^2 v + 16\cos^4 u \sin^2 v + 16 \sin^2 u \cos^2 u}$$

$$= 4\sqrt{\cos^4 u \underbrace{(\cos^2 v + \sin^2 v)}_{1} + \sin^2 u \cos^2 u}$$

$$= 4\sqrt{\cos^2 u \underbrace{(\cos^2 u + \sin^2 u)}_{1}} = 4|\cos u|$$

so $$\iint_D \|\vec{r}_u \times \vec{r}_v\|\, dA = \int_0^{\pi}\int_0^{2\pi} 4|\cos u|\, du\, dv$$

$$= \int_0^{\pi}\int_0^{\frac{\pi}{2}} 4\cos u\, du\, dv + \int_0^{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -4\cos u\, du\, dv + \int_0^{\pi}\int_{\frac{3\pi}{2}}^{2\pi} 4\cos u\, du\, dv$$

$$= \int_0^{\pi} 4\sin u \Big|_0^{\pi/2} dv + \int_0^{\pi} -4\sin u \Big|_{\frac{\pi}{2}}^{3\pi/2} dv + \int_0^{\pi} 4\sin u \Big|_{\frac{3\pi}{2}}^{2\pi} dv$$

$$= \int_0^{\pi} 4(1)\, dv + \int_0^{\pi} \underbrace{-4(-1-1)}_{8}\, dv + \int_0^{\pi} 4(0+1)\, dv$$

$$= 4v\Big|_0^{\pi} + 8v\Big|_0^{\pi} + 4v\Big|_0^{\pi}$$

$$= 4\pi + 8\pi + 4\pi = \boxed{16\pi}$$

= surface area of sphere

For surfaces $z = f(x,y)$, can use the same method by using parameters x & y, then

$$x = x \quad y = y \quad z = f(x,y)$$

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$$

$$\vec{r}_x(x,y) = \vec{i} + f_x(x,y)\vec{k}$$

$$\vec{r}_y(x,y) = \vec{j} + f_y(x,y)\vec{k}$$ and

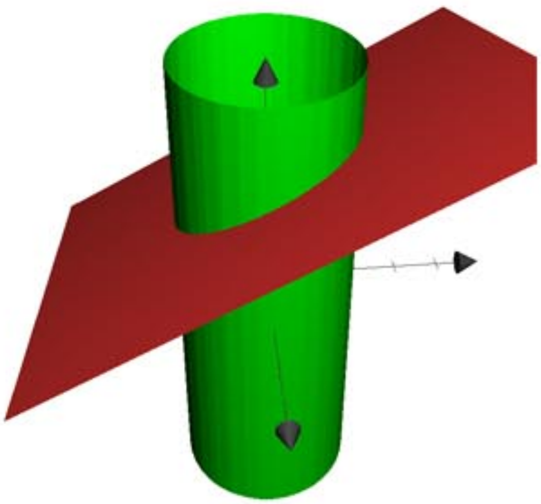
$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(x,y) \\ 0 & 1 & f_y(x,y) \end{vmatrix}$$

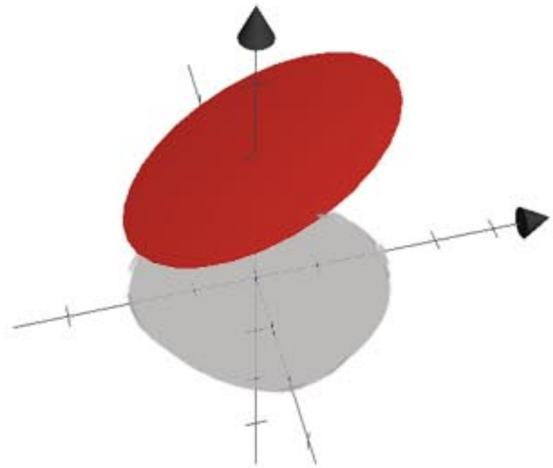
$$= -f_x(x,y)\vec{i} - f_y(x,y) + \vec{k}$$

so $\|\vec{r}_x \times \vec{r}_y\| = \sqrt{1 + (f_x(x,y))^2 + (f_y(x,y))^2}$

and Surface Area $= \iint\limits_D \sqrt{1 + (f_x(x,y))^2 + (f_y(x,y))^2}\, dA$

Ex. Find the area of the part of the plane $z = 10 + x + 2y$ that is inside the cylinder $x^2 + y^2 = 4$.





$$\text{Surface Area} = \iint_D \sqrt{1 + (f_x(x,y))^2 + (f_y(x,y))^2}\ dA$$

$f(x,y) = 10 + x + 2y$

$f_x(x,y) = 1$

$f_y(x,y) = 2$

$D = \{(x,y) : x^2+y^2 \leq 4\}$

$$\text{Surface Area} = \iint_D \sqrt{1 + 1^2 + 2^2}\ dA$$

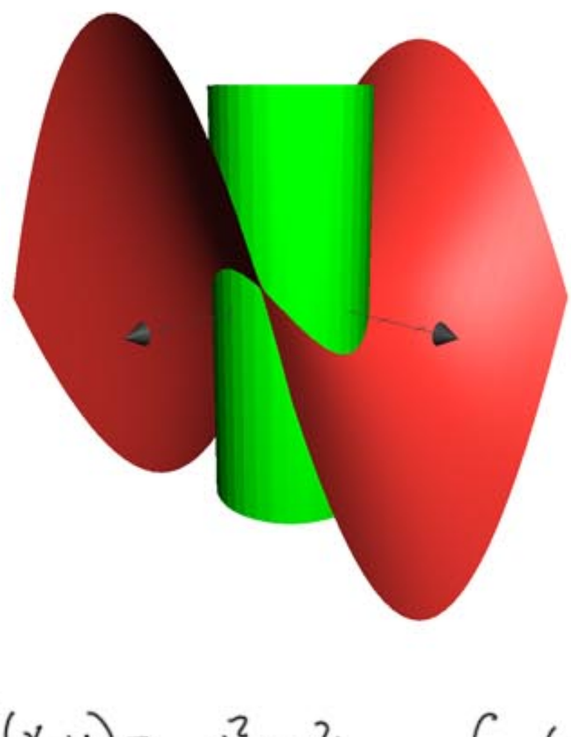
$$= \int_0^{2\pi}\int_0^2 \sqrt{6}\ r\,dr\,d\theta$$

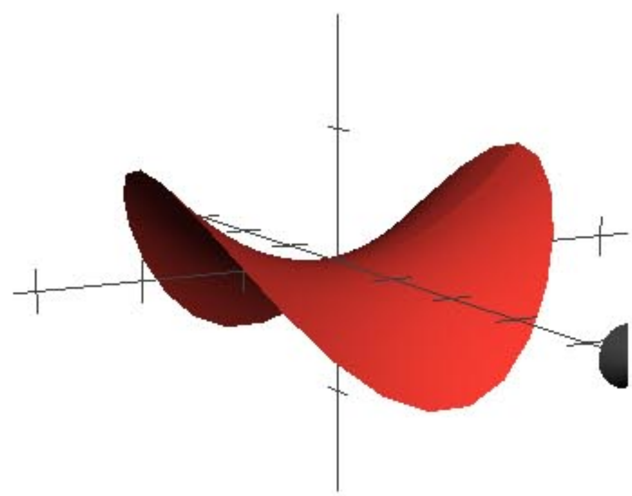
$$= \int_0^{2\pi} \frac{\sqrt{6}\,r^2}{2} \Big|_0^2 \, d\theta = \int_0^{2\pi} \frac{\sqrt{6}(4)}{2} \, d\theta = \int_0^{2\pi} 2\sqrt{6}\, d\theta$$

$$= 2\sqrt{6}\,\theta \Big|_0^{2\pi} = 4\sqrt{6}\,\pi .$$

Ex. Find the surface area of the hyperbolic paraboloid $z = x^2 - y^2$

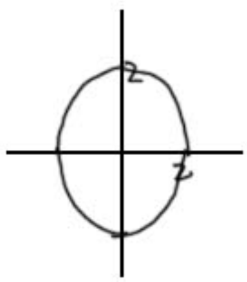
inside the cylinder $x^2 + y^2 = 4$.





$f(x,y) = x^2 - y^2 \qquad f_x(x,y) = 2x \qquad f_y(x,y) = -2y$

$$\text{Surface Area} = \iint_D \sqrt{1+(2x)^2+(-2y)^2}\; dA$$



$$= \iint_D \sqrt{1+4x^2+4y^2}\; dA$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1+4x^2+4y^2}\; dy\, dx$$

$$\text{OR} \quad = \int_0^{2\pi}\int_0^{2} \sqrt{1+4r^2\cos^2\theta+4r^2\sin^2\theta}\quad r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \sqrt{1+4r^2}\; r\,dr\,d\theta$$

$u = 1+4r^2$

$du = 8r\,dr$

$$= \frac{1}{8}\int_0^{2\pi}\int_0^{2} 8r\sqrt{1+4r^2}\; dr\,d\theta \quad =$$

$$= \frac{1}{8} \int_0^{2\pi} \int_1^{17} u^{1/2}\, du\, d\theta = \frac{1}{8} \int_0^{2\pi} \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{17}\, d\theta$$

$$= \frac{1}{8} \int_0^{2\pi} \frac{2}{3} \left(17^{3/2} - 1 \right) d\theta$$

$$= \frac{1}{12} \left(17^{3/2} - 1 \right) \theta \Big|_0^{2\pi} = \frac{\pi}{6} \left(17^{3/2} - 1 \right).$$