Math 20300
Calculus III

Lesson 22
Double Integrals Over General Regions

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Double Integrals Over General Regions

In lesson 21, we defined the double integral

$$\iint_R f(x,y)\,dA = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A_{ij}$$

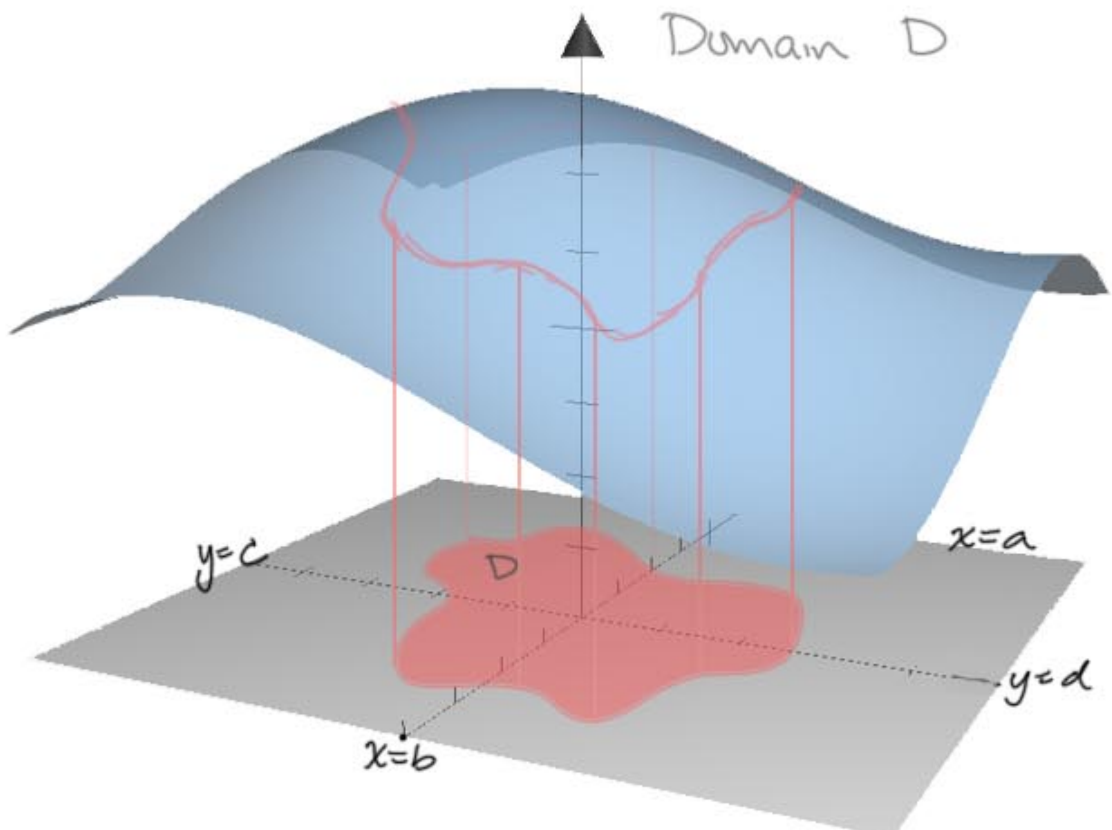
over rectangle $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

and recognized that this double integral gives the volume of the surface under the graph of $f(x,y)$ if $f(x,y) \ge 0$ on R (otherwise volumes under the xy plane get negative signs).

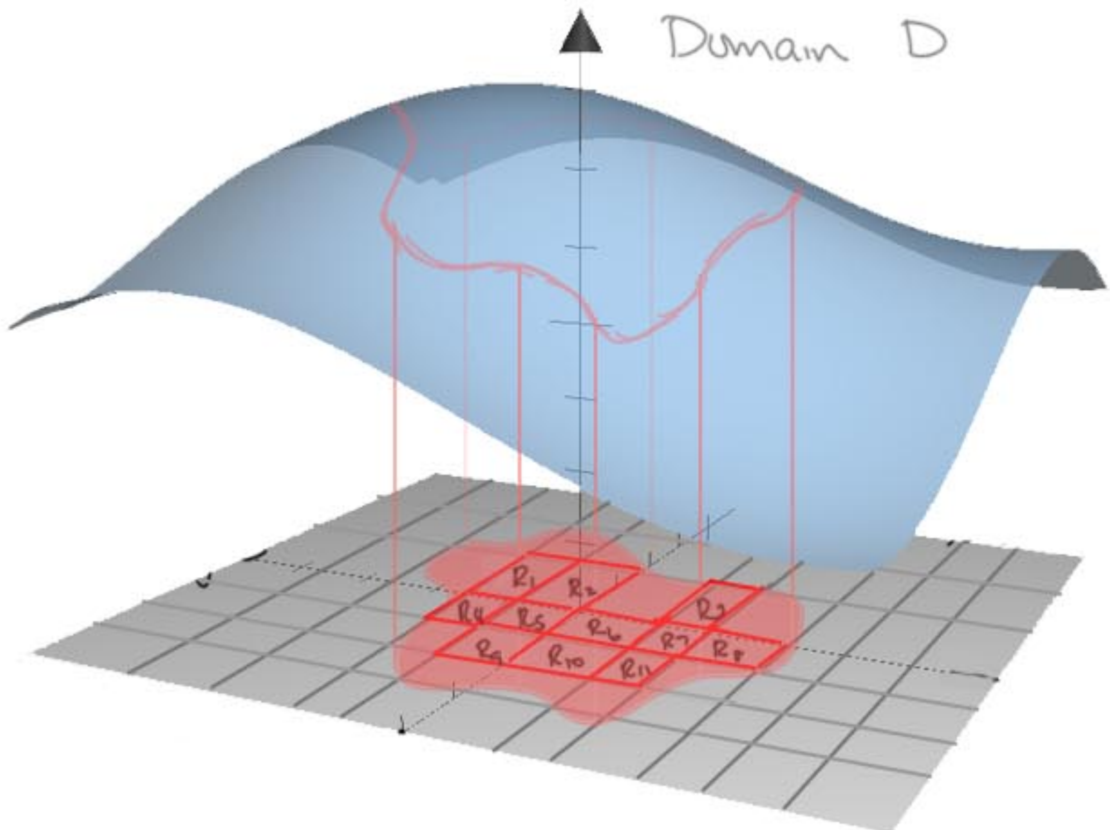
And we compute the double integral using Fubini's Theorem:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Now we are interested in volumes over general regions in $\mathbb{R}^2$.



So we need to define double integrals over a general region, and learn how to compute them.



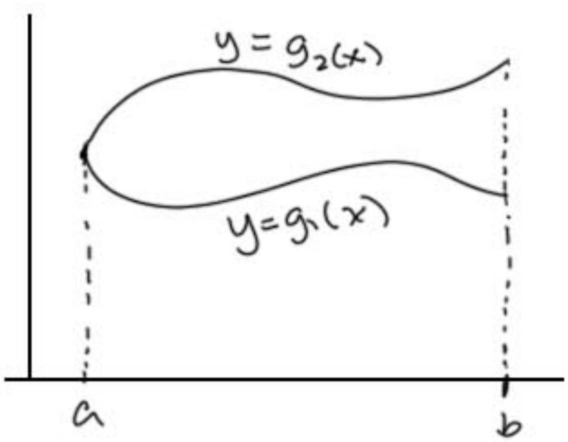
To define the double integral, we only compute over subrectangles that are completely in the domain D. So we'll number the rectangles $R_1, \dots R_n$, and let partition $P = \{R_1, R_2, \dots R_n\}$.

Let $\|P\|$ = "norm of P" = largest length or width of any $R_i \in P$. The evaluation point we'll call $(x_i^*, y_i^*) \in R_i$, and $\Delta A_i = \Delta x_i \Delta y_i$ on R_i.

$$\iint_D f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta A_i$$

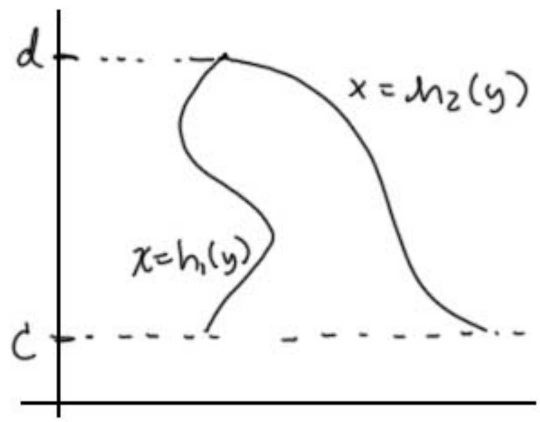
And to compute, we recognize the domain as one of two types:

Type I domain,
"top to bottom" region



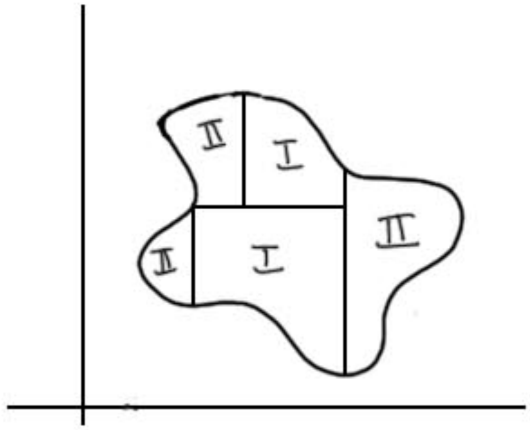
$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Type II domain
"right to left" region



$D = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}.$

Note: If the domain is neither Type I nor Type II, break the region up into Type I + Type II Regions:



To compute a Type I double integral:

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

To compute a Type II double integral:

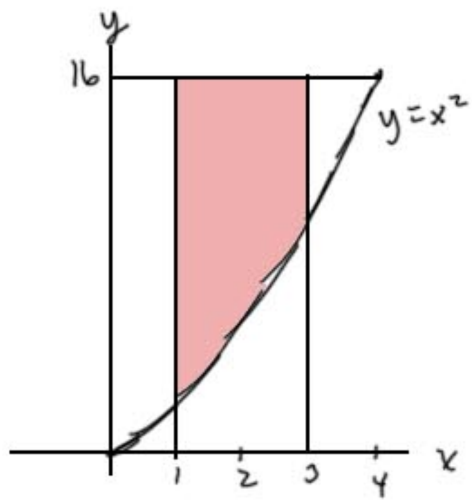
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Ex. Find the double integral of $f(x,y) = 24xy^2$

over the region bounded by $y = x^2$, $y = 16$,

$x = 1$, $x = 3$

first sketch the region in $\mathbb{R}^2$:



notice This is a top - bottom region, Type I

So $D = \{(x,y) : 1 \le x \le 3 ,\ x^2 \le y \le 16\}$

So $$\iint_D 24xy^2\,dA = \int_1^3 \int_{x^2}^{16} 24xy^2\,dy\,dx$$

$$= \int_1^3 24x\frac{y^3}{3}\Big|_{x^2}^{16}\,dx$$

$$= \int_1^3 \left[24x\frac{(16)^3}{3} - 24x\frac{(x^2)^3}{3} \right] dx$$

$$= \int_1^3 \left(8x(4096) - 8x \cdot x^6 \right) dx$$

$$= \int_1^3 (32768x - 8x^7)\,dx = \frac{32768x^2}{2} - x^8 \Big|_1^3$$

$$= (16384 \cdot 3^2 - 3^8) - (16384 - 1)$$

$$= 147456 - 6561 - 16383$$

$$= 124512.$$

Ex. Find the double integral of $f(x,y) = 2xy^2 + 2y\cos x$ over the region bounded by $y = \sqrt{x}$, $x = 0$, $y = 3$.

Either right-left or top-bottom!

Type 1 top-bottom: $\sqrt{x} \le y \le 3$, $0 \le x \le 9$

or

Type 2 right-left: $0 \le x \le y^2$, $0 \le y \le 3$

$$\int_0^9 \int_{\sqrt{x}}^3 (2xy^2 + 2y\cos x)\, dy\, dx \qquad \int_0^3 \int_0^{y^2} (2xy^2 + 2y\cos x)\, dx\, dy$$

$$\int_0^3 \int_0^{y^2} (2xy^2 + 2y\cos x)\, dx\, dy = \int_0^3 \left[x^2y^2 + 2y\sin x\right]_0^{y^2} dy$$

$$= \int_0^3 \left(\left[(y^2)^2 y^2 + 2y\sin(y^2)\right] - \left[0y^2 + 2y\sin(0)\right] \right) dy$$

$$= \int_0^3 \left(y^6 + 2y\sin(y^2)\right) dy = \frac{y^7}{7} - \cos(y^2) \Big|_0^3$$

$$\int 2y\sin(y^2)\, dy = \int \sin u\, du = -\cos u + C$$

$u = y^2$

$du = 2y\, dy$

$$= \frac{3^7}{7} - \cos(9) - \left(0 - \cos(0^2)\right)$$

$$\frac{2187}{7} - \cos 9 + 1 = \frac{2194}{7} - \cos 9.$$

Ex. Find the volume of the region in $\mathbb{R}^3$

bounded by:

$$\left\{ y = x^2 - x,\ y = x + 8,\ y = 2 \right\} \text{ in } \mathbb{R}^2, \text{ domain } D$$

$$= \left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4}$$

$$z = 1, \quad z = 5 + x.$$

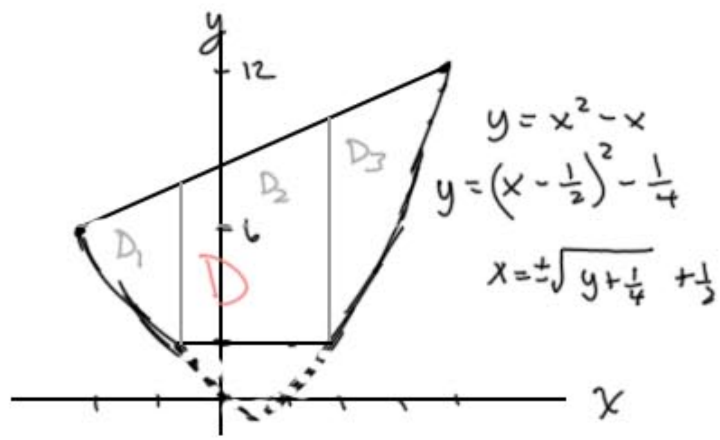
$$x^2 - x = x + 8$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = 4, \quad x = -2$$

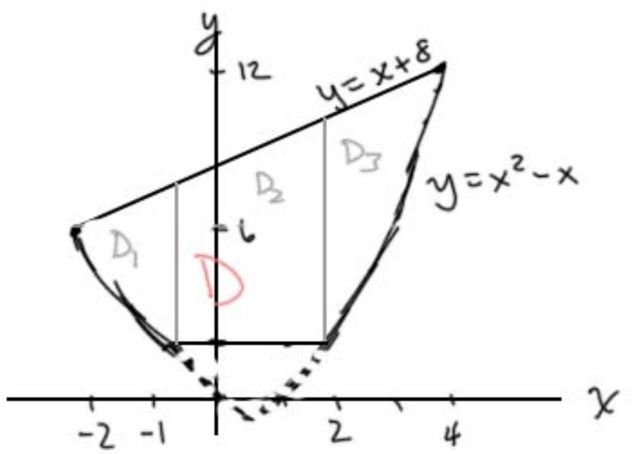
$$y = 12 \qquad y = 6$$



We need $\underbrace{\iint_D \underset{\text{top surface}}{(5+x)}\, dA}_{\text{volume from } z = 5+x \text{ down to xy plane}} - \underbrace{\iint_D \underset{\text{bottom surface}}{1}\, dA}_{\text{volume from } z = 1 \text{ to xy plane}}$

$$= \iint_D (5+x-1)\,dA = \iint_D (4+x)\,dA.$$

$$= \iint_{D_1} (4+x)\,dA + \iint_{D_2} (4+x)\,dA + \iint_{D_3} (4+x)\,dA$$



$y = x^2 - x = 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2,\ x = -1$

$D_1 : \{(x,y) : x^2 - x \le y \le x+8,\ -2 \le x \le -1\}$

$D_2 : \{(x,y) : 2 \le y \le x+8,\ -1 \le x \le 2\}.$

$D_3 : \{(x,y) : x^2 - x \le y \le x+8,\ 2 \le x \le 4\}$

$$\text{Volume} = \int_{-2}^{-1}\int_{x^2-x}^{x+8} (4+x)\,dy\,dx + \int_{-1}^{2}\int_{2}^{x+8} (4+x)\,dy\,dx + \int_{2}^{4}\int_{x^2-x}^{x+8} (4+x)\,dy\,dx$$

$= \ldots$ (compute)

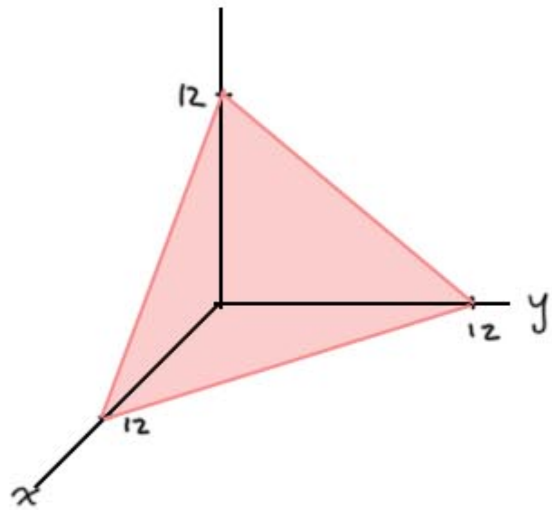
Ex. Set up two double integrals (one $dy\,dx$, other $dx\,dy$) to find the volume of the solid in the 1^{st} octant bounded by $x + y + z = 12$.

plane, intersects the axes where?

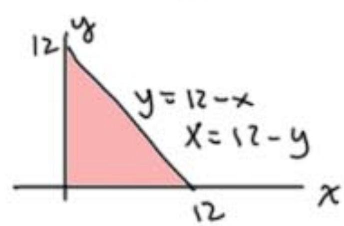
when $x = 0, y = 0, z = 12$

$x = 0, z = 0, y = 12$

$y = 0, z = 0, x = 12$



We need to describe the domain in $\mathbb{R}^2$ over which to integrate.



can slice either way:

Type I: $0 \le y \le 12 - x$, $0 \le x \le 12$ or Type II $0 \le x \le 12 - y$, $0 \le y \le 12$

$$\int_0^{12}\int_0^{12-x} (12 - x - y)\, dy\, dx \quad \text{or} \quad \int_0^{12}\int_0^{12-y} (12 - x - y)\, dx\, dy$$

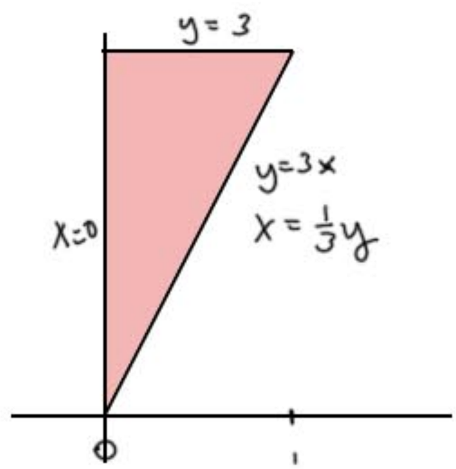
Changing the order of integration:

given integral → sketch of domain → new integral

Ex. $\displaystyle\int_0^1\int_{3x}^3 e^{y^2}\,dy\,dx$ can not integrate $\int e^{y^2}\,dy$

$3x \le y \le 3$ $y=3x$, $y=3$

$0 \le x \le 1$



given as top-bottom

now describe as right-left:

$0 \le x \le \frac{1}{3}y$

$0 \le y \le 3$

$$\int_0^3\int_0^{\frac{1}{3}y} e^{y^2}\,dx\,dy = \int_0^3 x e^{y^2}\Big|_0^{\frac{1}{3}y}\,dy$$

$$= \int_0^3 \frac{1}{3}y e^{y^2}\,dy$$

u-sub $u=y^2$

$du = 2y\,dy$

$$= \frac{1}{3} \cdot \frac{1}{2} \int_0^3 2y e^{y^2}\, dy = \frac{1}{6} \int_0^9 e^u \, du = \frac{1}{6} e^u \Big|_0^9$$

$$= \frac{1}{6} e^9 - \frac{1}{6} e^0 = \frac{1}{6}\left(e^9 - 1\right).$$

Ex. $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy$

$\sqrt{y} \le x \le 1$ $\quad x = \sqrt{y},\ x = 1$

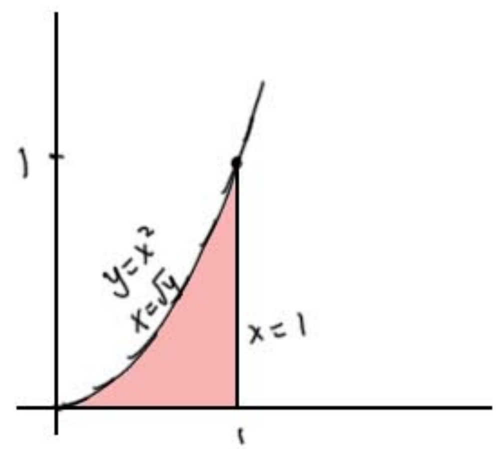
$0 \le y \le 1$

given as right-left

now describe as top-bottom

$0 \le y \le x^2$

$0 \le x \le 1$



$$\int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx = \int_0^1 y\sqrt{x^3+1}\Big|_0^{x^2} dx$$

$$= \int_0^1 x^2 \sqrt{x^3+1}\, dx$$

u-sub $\quad u = x^3 + 1$

$du = 3x^2\, dx$

$$= \frac{1}{3}\int_0^1 3x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int_1^2 u^{1/2}\,du =$$

$$\frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big|_1^2 = \frac{2}{9}\left(2^{3/2}-1^{3/2}\right)$$

$$= \frac{2}{9}\left(2^{3/2}-1\right).$$