Math 20300
Calculus III

Lesson 19
Absolute Extrema

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Absolute Extrema

Recall: for $y = f(x)$, f has a <u>global/absolute maximum</u> at $x = c$ if $f(c) \geq f(x)\ \forall x \in D$, where D is the domain of f. $f(c)$ is the maximum value of f on D.

f has a <u>global/absolute minimum</u> at $x = c$ if $f(c) \leq f(x)\ \forall x \in D$, where D is the domain of f. $f(c)$ is the minimum value of f on D.

<u>The Extreme Value Theorem</u>: if f is continuous on a closed interval $[a,b]$ then f attains an absolute maximum and an absolute minimum value on $[a,b]$

The absolute extrema can occur at critical points of f in $[a,b]$ or at endpoints of the interval $[a,b]$.

For $z = f(x,y)$,

f has a <u>global / absolute maximum</u> at (a,b) if $f(a,b) \geq f(x,y)$ $\forall (x,y)$ in the domain D of f.

$f(a,b)$ is the maximum value of f on D.

f has a <u>global / absolute minimum</u> at (a,b) if $f(a,b) \leq f(x,y)$ $\forall (x,y)$ in the domain D of f.

$f(a,b)$ is the minimum value of f on D.

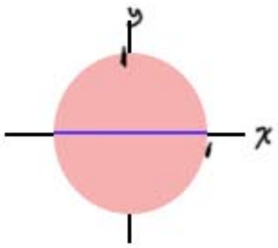
<u>The Extreme Value Theorem</u>: if f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains absolute minimum and absolute maximum values on D.

To find absolute extrema of a continuous function f on a closed, bounded set D:

1. find the values of f at the critical points of f that are in D
2. find the extreme values of f on the boundary ∂D

3. The largest value from steps 1 + 2 is the absolute maximum value of f on D, and the smallest value is the absolute minimum value of f on D.

Ex. Find the absolute extrema of $f(x,y) = xy^2$ on the $\underbrace{\text{unit disk}}$.



closed, bounded region in $\mathbb{R}^2$ whose boundary is the unit circle $x^2+y^2=1$

i.e., $x^2+y^2 \le 1$.

1. find critical points of f in D

$f_x(x,y) = y^2 = 0 \qquad f_y(x,y) = 2xy = 0$

$\Rightarrow y = 0 \qquad\qquad x$ can be anything

so the critical points in D are the points on the line segment from $(-1,0)$ to $(1,0)$ including both points.

The value of f at any of those points is

$$f(x,y) = xy^2 = x(0)^2 = 0.$$

2. The boundary of D is $x^2+y^2=1$

The value of f at any boundary point:

$f(x,y) = xy^2 = \underbrace{x(1-x^2)}$ we need the extreme values of this over $[-1,1]$

This part is a Calc I problem:

$$g(x) = x(1-x^2) = x - x^3 \qquad x \in [-1,1]$$

$$g'(x) = 1-3x^2 = 0 \quad \text{to find critical pts}$$

$$1 = 3x^2$$

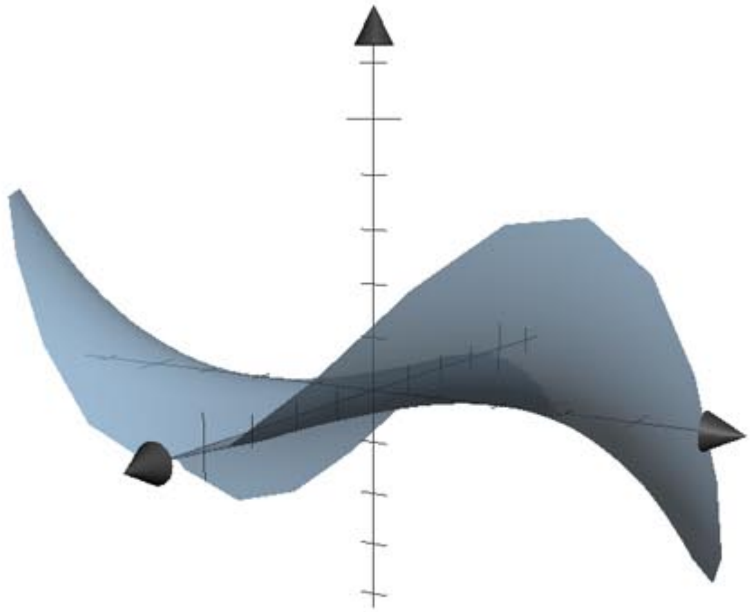
$$x = \pm\sqrt{\tfrac{1}{3}}$$

	x	$g(x) = x - x^3 = x(1-x^2)$	
crit pts	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}(1-\frac{1}{3}) = \sqrt{\frac{1}{3}}\cdot\frac{2}{3} = \frac{2\sqrt{3}}{9}$	← max on boundary
	$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}(1-\frac{1}{3}) = -\frac{2\sqrt{3}}{9}$	min on boundary
end pts	-1	$-1(1-1) = 0$	
	1	$1(1-1) = 0$	

3. max on boundary or critical pts is $\frac{2\sqrt{3}}{9}$
occurs at $\left(\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{2}{3}}\right)$

min on boundary or critical pts is $-\frac{2\sqrt{3}}{9}$

occurs at $\left(-\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{2}{3}}\right)$



Ex. Find the absolute extrema of
$f(x,y) = x^2+2y^2-x^2y$ over the triangular
region bounded by $(0,0)$ $(10,0)$ and $(0,10)$.

1. $f_x(x,y) = 2x - 2xy = 0$ $\qquad f_y(x,y) = 4y - x^2 = 0$

$$y = \frac{x^2}{4}$$

$$2x - 2x\left(\frac{x^2}{4}\right) = 0$$

$$2x - \frac{x^3}{2} = 0$$

$$4x - x^3 = 0$$

$$x(4-x^2) = 0$$

$$x = 0,\ x = \pm 2$$

$x = 0$, $y = \frac{0^2}{4} = 0$
$(0,0)$

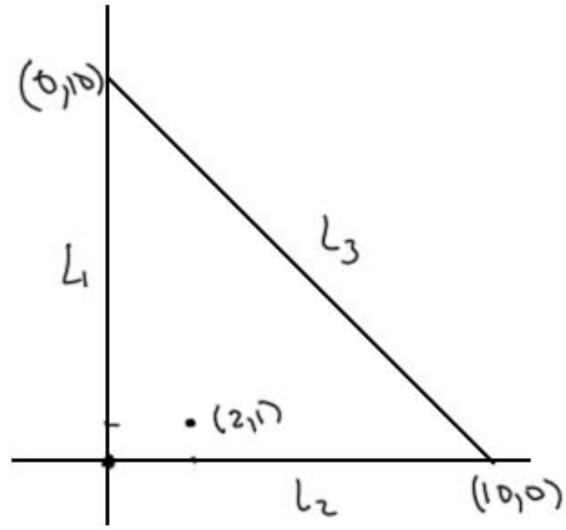
$x = 2$, $y = \frac{2^2}{4} = 1$
$(2,1)$

$\cancel{x=-2 \quad (-2,1)}$ not in domain

$f(x,y) = x^2+2y^2-x^2y$

$f(0,0) = 0$

$f(2,1) = 4+2-4 = 2$



2. find extrema on the boundary triangle

L_1: $x=0 \quad 0 \le y \le 10$

$f(x,y) = 2y^2$ min at $(0,0)$ $f(0,0)=0$

max at $(0,10)$ $f(0,10)=200$

l_2: $y=0 \quad 0 \le x \le 10$

$f(x,y) = x^2$ min at $(0,0)$ $f(0,0)=0$

max at $(10,0)$ $f(10,0)=100$

L_3: $y = -x+10$ for $0 \le x \le 10$

$$f(x,y) = x^2+2y^2-x^2y$$
$$= x^2+2(-x+10)^2-x^2(-x+10)$$
$$= x^2+2(x^2-20x+100)+x^3-10x^2$$

$$= x^3 - 7x^2 - 40x + 200 = g(x)$$

Calc I
abs. extrema

$$g'(x) = 3x^2 - 14x - 40 = 0$$

$$x = \frac{14 \pm \sqrt{196 - 4(3)(-40)}}{6}$$

$$\begin{array}{r} 196 \\ \underline{480} \\ 676 \end{array}$$

$$= \frac{14 \pm \sqrt{676}}{6}$$

$$x = \frac{14 \pm 26}{6} \qquad x = \frac{40}{6} = \frac{20}{3} \qquad \cancel{x = \frac{-12}{6} = -2}$$

not in our domain

$$y = 10 - \frac{20}{3} = \frac{10}{3}$$

$$f\left(\frac{20}{3}, \frac{10}{3}\right) = \left(\frac{20}{3}\right)^2 + 2\left(\frac{10}{3}\right)^2 - \left(\frac{20}{3}\right)^2\left(\frac{10}{3}\right)$$

$$= \frac{400}{9} + \frac{200}{9} - \frac{4000}{27} = -\frac{2200}{27} \approx -81.5$$

still on L_3: when $x = 0$ or $x = 10$, we've already considered these function values.

3. absolute maximum of f on D is 200 and occurs at $(0, 10)$

absolute minimum of f on D is $\frac{-2200}{27}$ and occurs at $\left(\frac{20}{3}, \frac{10}{3}\right)$.