Math 20300
Calculus III

Lesson 16
The Chain Rule

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The Chain Rule

Recall: for $y = f(x)$ and $x = g(t)$,

$$y = f(g(t))$$

and $$\frac{dy}{dt} = f'(g(t)) \cdot g'(t)$$

or $$\frac{dy}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}$$

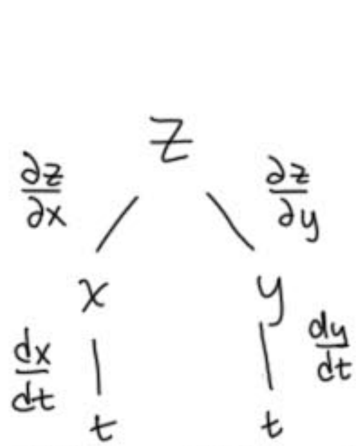
When composing functions of more than one variable, the chain rule will have multiple terms.

The easiest way to keep track of them is to use tree diagrams:

Ex. $z = f(x, y)$

$x = x(t)$

$y = y(t)$



z is a function of t, can talk about

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Ex. $z = 2\cos(xy)$

$x = e^{-t}$

$y = t^3$

$$\begin{tikzcd}[column sep=small] & z \arrow[dl, dash, "\frac{\partial z}{\partial x}"'] \arrow[dr, dash, "\frac{\partial z}{\partial y}"] & \\ x \arrow[d, dash, "\frac{dx}{dt}"'] & & y \arrow[d, dash, "\frac{dy}{dt}"] \\ t & & t \end{tikzcd}$$

find $\displaystyle \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$$\frac{\partial z}{\partial x} = -2\sin(xy) \cdot y \qquad \frac{dx}{dt} = -e^{-t} \qquad \frac{\partial z}{\partial y} = -2x\sin(xy)$$

$$= -2y\sin(xy) \qquad \frac{dy}{dt} = 3t^2$$

$$\frac{dz}{dt} = 2ye^{-t}\sin(xy) + -6t^2 x\sin(xy) \qquad \begin{array}{l} x = e^{-t} \\ y = t^3 \end{array}$$

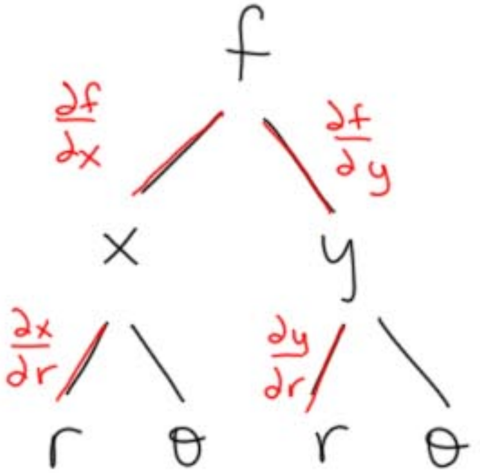
$$= 2t^3e^{-t}\sin(t^3e^{-t}) - 6t^2e^{-t}\sin(t^3e^{-t})$$

$$= 2t^2e^{-t}\sin(t^3e^{-t})\left[t - 3\right]$$

Ex. find $\frac{\partial f}{\partial r}$ for $f(x,y) = 3x^2y^3\cos(x^2) + e^{3xy}$

where $x = r\cos\theta \quad y = r\sin\theta$

(polar coordinates)



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(3x^2y^3\right)\cdot\cos(x^2) + \left(3x^2y^3\right)\cdot\frac{\partial}{\partial x}\left(\cos(x^2)\right) + \frac{\partial}{\partial x}\left(e^{3xy}\right)$$

$$= 6xy^3\cos(x^2) + 3x^2y^3(-\sin(x^2))\cdot 2x + 3ye^{3xy}$$

$$\frac{\partial f}{\partial x} = 6xy^3\cos(x^2) - 6x^3y^3\sin(x^2) + 3ye^{3xy}$$

$x = r\cos\theta \qquad y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$

$f(x,y) = 3x^2y^3\cos(x^2) + e^{3xy}$

$$\frac{\partial f}{\partial y} = 9x^2y^2\cos(x^2) + 3xe^{3xy}$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial r} = \left[6xy^3\cos(x^2) - 6x^3y^3\sin(x^2) + 3ye^{3xy}\right]\cos\theta \; +$$

$$+ \left[9x^2y^2\cos(x^2) + 3xe^{3xy}\right]\sin\theta$$

$$= \Big[6r\cos\theta\,(r\sin\theta)^3\cos\left((r\cos\theta)^2\right) - 6(r\cos\theta)^3(r\sin\theta)^3\sin\left((r\cos\theta)^2\right)$$

$$+ 3r\sin\theta\, e^{3r^2\cos\theta\sin\theta}\Big]\cos\theta \; +$$

$$+ \left[9(r\cos\theta)^2(r\sin\theta)^2\cos\left((r\cos\theta)^2\right) + 3r\cos\theta\, e^{3r^2\cos\theta\sin\theta}\right]\sin\theta$$

$$= \underline{6r^4\cos^2\theta\sin^3\theta\cos\left((r\cos\theta)^2\right)} - 6r^6\cos^4\theta\sin^3\theta\sin\left((r\cos\theta)^2\right)$$

$$+ \underline{3r\cos\theta\sin\theta\, e^{3r^2\cos\theta\sin\theta}} \; +$$

$$+ \underline{9r^4\cos^2\theta\sin^3\theta\cos\left((r\cos\theta)^2\right)} + \underline{3r\cos\theta\sin\theta\, e^{3r^2\cos\theta\sin\theta}}$$

$$= 15r^4\cos^2\theta\sin^3\theta\cos\left((r\cos\theta)^2\right) - 6r^6\cos^4\theta\sin^3\theta\sin\left((r\cos\theta)^2\right)$$

$$+ 6r\cos\theta\sin\theta\, e^{3r^2\cos\theta\sin\theta}$$

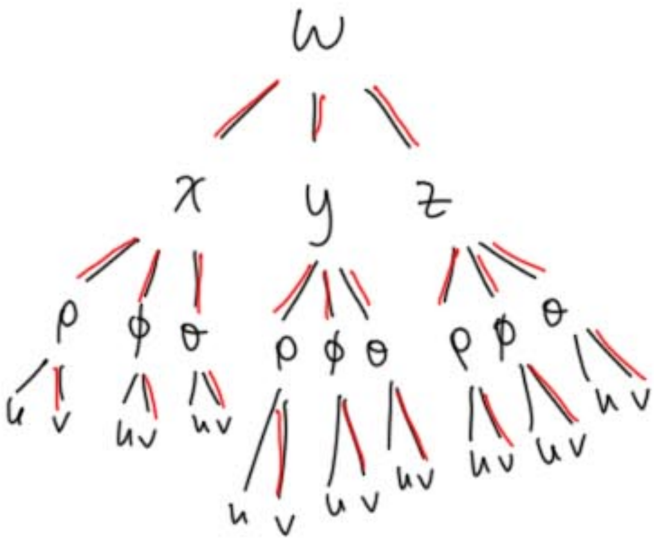
Ex. For $w = f(x, y, z)$, $x = x(\rho, \phi, \theta)$ ρ rho

$y = y(\rho, \phi, \theta)$ ϕ phi

$z = z(\rho, \phi, \theta)$ θ theta

and
$$\rho = \rho(u, v)$$
$$\phi = \phi(u, v)$$
$$\theta = \theta(u, v)$$

find the chain rule formula for $\frac{\partial w}{\partial v}$.



$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial \rho} \cdot \frac{\partial \rho}{\partial v} + \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial \phi} \cdot \frac{\partial \phi}{\partial v} + \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial \theta} \cdot \frac{\partial \theta}{\partial v} +$$

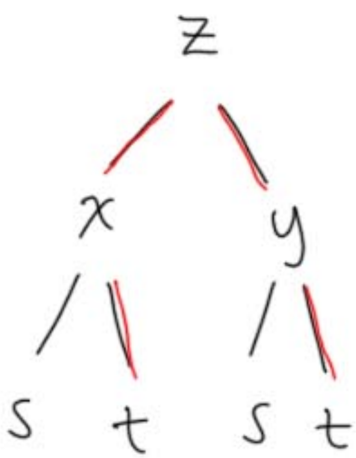
$$+ \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial \rho} \cdot \frac{\partial \rho}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial \phi} \cdot \frac{\partial \phi}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial \theta} \cdot \frac{\partial \theta}{\partial v} +$$

$$+ \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial \rho} \cdot \frac{\partial \rho}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial \phi} \cdot \frac{\partial \phi}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial v}$$

Second Partials with the Chain Rule.

$z = f(x,y) \quad x = x(s,t) \quad y = y(s,t)$

find formulas for $\frac{\partial z}{\partial t}$, $\frac{\partial^2 z}{\partial t^2}$



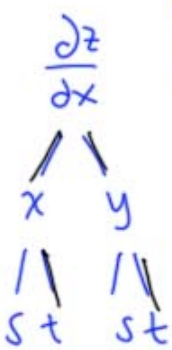
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

for $\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right)$

$$= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}\right)$$

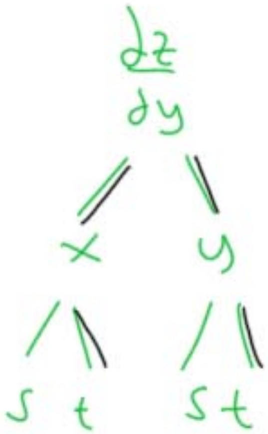
$$= \underbrace{\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial x} \cdot \underbrace{\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t}\right)} + \underbrace{\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\right)} \cdot \frac{\partial y}{\partial t} + \frac{\partial z}{\partial y} \cdot \underbrace{\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)}$$

$$\left[\qquad \right] \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial x} \cdot \frac{\partial^2 x}{\partial t^2} + \left[\qquad \right] \frac{\partial y}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial^2 y}{\partial t^2}$$



$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \cdot \frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y \partial x} \cdot \frac{\partial y}{\partial t}$$



$$\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)}\cdot\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial t}$$

$$\left[\frac{\partial^2 z}{\partial x\,\partial y}\cdot\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y^2}\cdot\frac{\partial y}{\partial t}\right]$$

$$\frac{\partial^2 z}{\partial t^2} = \left(\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y\,\partial x}\cdot\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial t^2} \quad +$$

$$+ \left(\frac{\partial^2 z}{\partial x\,\partial y}\cdot\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y^2}\cdot\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial t^2}$$

Ex. $z = 2x^2y^3 \qquad x = s+t \qquad y = s-t$

Work on this problem
on your own

$$z = 2x^2y^3 \qquad x = s+t \qquad y = s-t$$

$$\frac{\partial z}{\partial x} = 4xy^3 \qquad \frac{\partial z}{\partial y} = 6x^2y^2 \qquad \frac{\partial x}{\partial t} = 1 \qquad \frac{\partial^2 x}{\partial t^2} = 0$$

$$\frac{\partial^2 z}{\partial x^2} = 4y^3 \qquad \frac{\partial^2 z}{\partial y^2} = 12x^2y \qquad \frac{\partial y}{\partial t} = -1 \qquad \frac{\partial^2 y}{\partial t^2} = 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 12xy^2$$

$$\frac{\partial^2 z}{\partial t^2} = \left(\frac{\partial^2 z}{\partial x^2} \cdot \underbrace{\frac{\partial x}{\partial t}}_{1} + \frac{\partial^2 z}{\partial y \partial x} \cdot \underbrace{\frac{\partial y}{\partial t}}_{-1}\right)\underbrace{\frac{\partial x}{\partial t}}_{1} + \cancel{\frac{\partial z}{\partial x} \cdot \overset{0}{\frac{\partial^2 x}{\partial t^2}}} \quad +$$

$$+ \left(\frac{\partial^2 z}{\partial x \partial y} \cdot \underbrace{\frac{\partial x}{\partial t}}_{1} + \frac{\partial^2 z}{\partial y^2} \cdot \underbrace{\frac{\partial y}{\partial t}}_{-1}\right)\underbrace{\frac{\partial y}{\partial t}}_{-1} + \cancel{\frac{\partial z}{\partial y} \cdot \overset{0}{\frac{\partial^2 y}{\partial t^2}}}$$

$$= \frac{\partial^2 z}{\partial x^2} \underbrace{- \frac{\partial^2 z}{\partial y \partial z} - \frac{\partial^2 z}{\partial x \partial y}} + \frac{\partial^2 z}{\partial y^2}$$

$$= 4y^3 - 24xy^2 + 12x^2y$$ answer in s + t

$$= 4(s-t)^3 - 24(s+t)(s-t)^2 + 12(s+t)^2(s-t)$$

$$= 4(s-t)^3 - 24(s^2-t^2)(s-t) + 12(s^2-t^2)(s+t)$$

$$= 4(s-t)^3 + 12(s^2-t^2)\left[-2(s-t) + s + t\right]$$

$$-2s + 2t + s + t$$

$$3t - s$$

$$= 4(s-t)^3 + 12(s^2-t^2)(3t-s).$$

Implicit Differentiation

Recall: $\quad x^2 + y^2 = 4 \quad$ find $\frac{dy}{dx}$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(4)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

Consider $F(x,y) = 0 \qquad F(x,y) = x^2 + y^2 - 4 = 0$

for y as an implicit function of x

$$\begin{tikzcd}[column sep=small] & F \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[d, dash] & & y \arrow[d, dash] \\ x & & x \end{tikzcd}$$

$$F(x,y) = 0$$

$$\underbrace{\frac{d}{dx}\left(F(x,y)\right)}_{\text{chain rule}} = \frac{d}{dx}(0)$$

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$F_x + F_y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

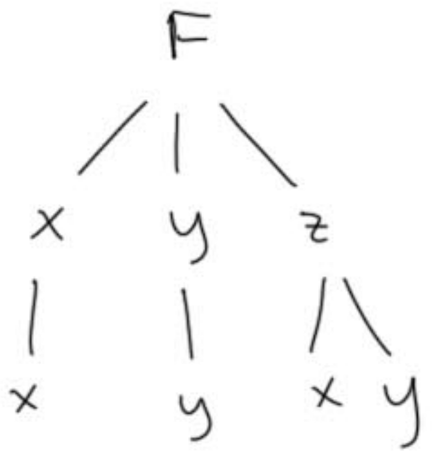
for $F(x,y) = x^2 + y^2 - 4 = 0$

$$F_x = 2x \qquad \therefore \frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-2x}{2y} = \frac{-x}{y} \quad \checkmark \text{ same}$$

$$F_y = 2y$$

The Implicit Function Theorem says that if F_x and F_y are continuous on an open disk containing the point (x_0, y_0) where $F(x_0,y_0) = 0$ and $F_y(x_0,y_0) \neq 0$, then $F(x,y) = 0$ implicitly defines y as a function of x near (x_0, y_0).

Similarly, for $F(x,y,z)=0$ implicitly defining $z=f(x,y)$ with f differentiable.



$$F(x,y,z)=0$$

$$\frac{\partial}{\partial x}\left(F(x,y,z)\right) = \frac{\partial}{\partial x}(0)$$

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$F_x + F_z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

And $\frac{\partial}{\partial y}\left(F(x,y,z)\right) = \frac{\partial}{\partial y}(0)$

$$\frac{\partial F}{\partial y}\cdot\frac{dy}{dy} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y} = 0$$

$$F_y + F_z \frac{\partial z}{\partial y} = 0 \qquad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

And the Implicit Function Theorem says that if F_x, F_y, F_z are continuous in a sphere around (x_0, y_0, z_0) where $F(x_0, y_0, z_0) = 0$ and $F_z(x_0, y_0, z_0) \neq 0$, then $F(x,y,z) = 0$ implicitly defines z as a function of x and y near (x_0, y_0, z_0).

Ex. find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $xy^2z^3 = \sin(xyz)$

need a $F(x,y,z) = 0 \qquad F(x,y,z) = xy^2z^3 - \sin(xyz)$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(y^2z^3 - yz\cos(xyz))}{3xy^2z^2 - xy\cos(xyz)}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(2xyz^3 - xz\cos(xyz))}{3xy^2z^2 - xy\cos(xyz)}$$

Ex. $xyz - xz^4 + 2y^2 z = 2$ find $\left.\frac{\partial z}{\partial y}\right|_{(1,1,1)}$.

Work on this problem on your own

$$F(x,y,z) = xyz - xz^4 + 2y^2 z - 2$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(xz + 4yz)}{xy - 4xz^3 + 2y^2}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1,1)} = \frac{-(1+4)}{1-4+2} = \frac{-5}{-1} = 5.$$