Math 20300
Calculus III

Lesson 15
Tangent Planes and Linear Approximations

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Tangent Planes + Linear Approximations

Recall: for $y=f(x)$, the tangent line to

$f(x)$ at $x = x_0$ is $y - \underbrace{f(x_0)}_{y_0} = \underbrace{f'(x_0)}_{\text{slope } m} (x-x_0)$

at (x_0, y_0)

$(x_0, f(x_0))$

$$y = f'(x_0)(x-x_0) + f(x_0)$$

Slope of tangent line = slope of f at x_0.

And we can use the tangent line to approximate function values for x-values near x_0, ie $f(x) \approx f'(x_0)(x-x_0) + f(x_0)$

Can say $L(x) = f'(x)(x-x_0) + f(x_0)$ is the local linearization of f at x_0.

Ex. $f(x) = 8 - x^2$ tangent line at $x = 1$

$f'(x) = -2x \quad f'(1) = -2(1) = -2$

$f(1) = 8 - (1)^2 = 7$

tangent line $\quad y - 7 = -2(x-1)$

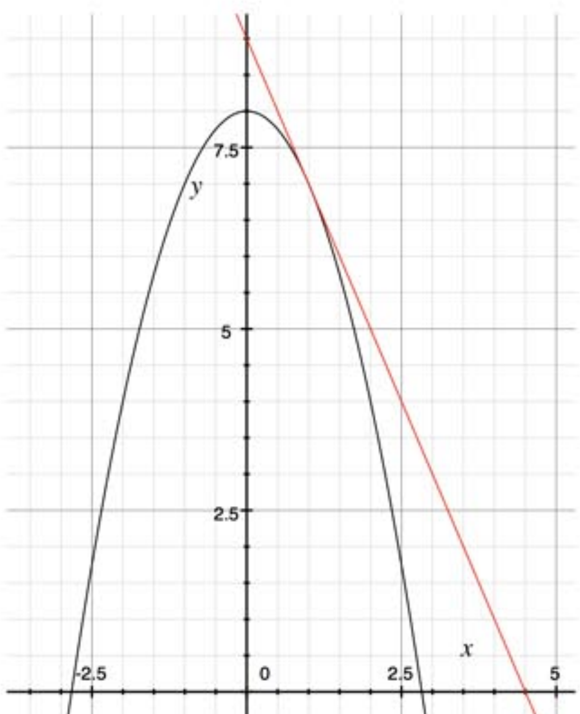
$y = -2(x-1) + 7$.

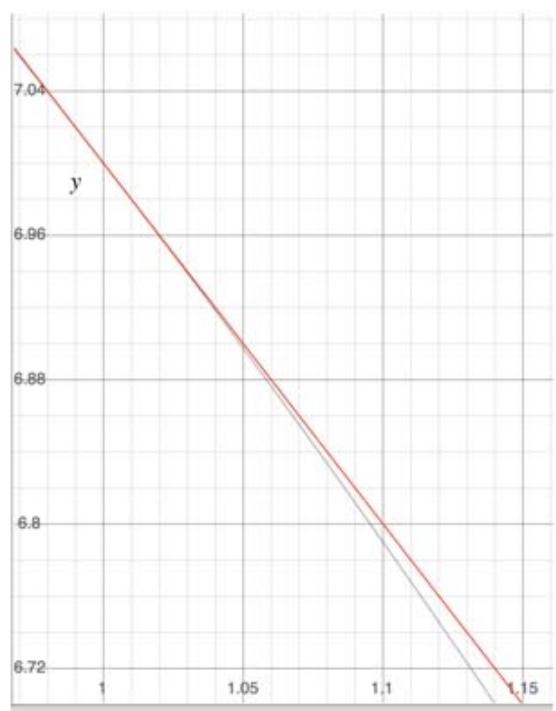
then to approximate $f(1.1)$,

$f(1.1) \approx -2(1.1-1) + 7 = -2(.1) + 7$

$= -.2 + 7 = 6.8$

actual value $f(1.1) = 8 - (1.1)^2 = 8 - 1.21 = 6.79$





For $z = f(x,y)$, we have a tangent plane

at $(x_0, y_0, \underbrace{z_0}_{f(x_0,y_0)})$ and can use the equation of

the tangent plane to approximate function

values for (x,y) near (x_0,y_0).

For The tangent plane:

equation of plane $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

where (x_0,y_0,z_0) is a point on The plane and

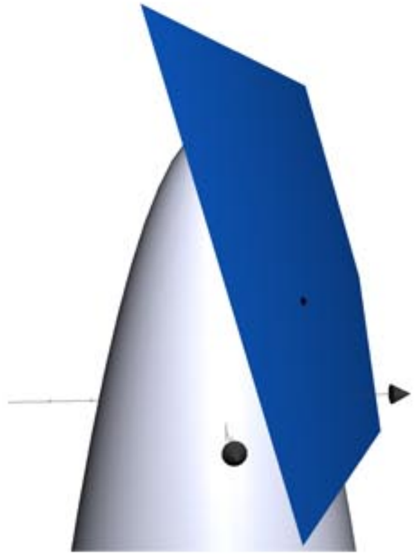
$\langle a,b,c \rangle$ is a normal vector to The plane.

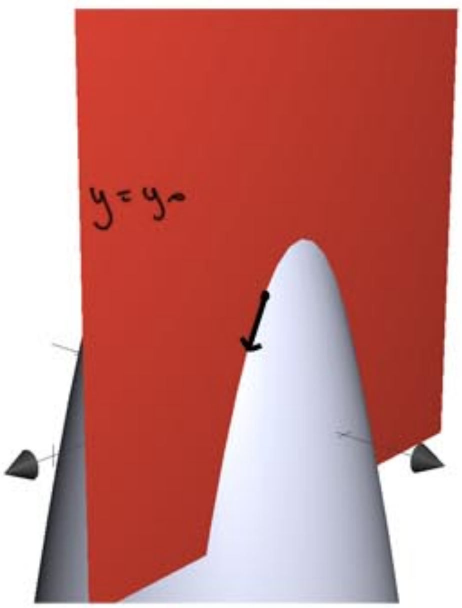
We have a point $(x_0, y_0, f(x_0,y_0))$

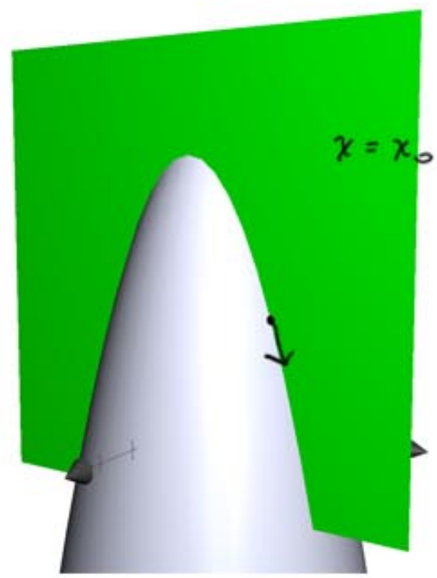
how do we find a normal vector?

Based on The partial derivatives $f_x(x_0,y_0)$

and $f_y(x_0,y_0)$, we'll find two vectors in The plane. Their cross product will be a normal vector to the plane.







slope in x direction $= f_x(x_0, y_0)$

$$\frac{\partial z}{\partial x} = f_x(x_0, y_0)$$

tan vector $\langle 1, 0, f_x(x_0, y_0) \rangle$
to curve must be in tangent plane.

slope in y direction $= f_y(x_0, y_0)$

$$\frac{\partial z}{\partial y} = f_y(x_0, y_0)$$

$\langle 0, 1, f_y(x_0, y_0) \rangle$
tan vector in tangent plane

$$\langle 0, 1, f_y(x_0,y_0)\rangle \times \langle 1, 0, f_x(x_0,y_0)\rangle =$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & f_y(x_0,y_0) \\ 1 & 0 & f_x(x_0,y_0) \end{vmatrix} = f_x(x_0,y_0)\,\vec{i} + f_y(x_0,y_0)\,\vec{j} - \vec{k}$$

$$= \langle f_x(x_0,y_0), f_y(x_0,y_0), -1\rangle$$

$= \langle a, b, c\rangle$ normal vector to plane.

Equation of tangent plane to $z = f(x,y)$ at $(x_0, y_0, \underbrace{f(x_0,y_0)}_{z_0})$ is

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

solved for z:

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$

Then

$$f(x,y) \approx f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$

for (x,y) near (x_0,y_0) (linear approximation)

Can say $L(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$

is the local linearization of f at (x_0,y_0).

Can also talk about the normal line to

$z = f(x,y)$ at $(x_0, y_0, \underbrace{f(x_0,y_0)}_{z_0})$, which is given by

$$\left.\begin{aligned} x &= x_0 + f_x(x_0,y_0)t \\ y &= y_0 + f_y(x_0,y_0)t \\ z &= z_0 - t \end{aligned}\right\}$$ given by the same normal vector

$\langle f_x(x_0,y_0), f_y(x_0,y_0), -1 \rangle$

Ex. $f(x,y) = 8 - x^2 - y^2$

find equations for the tangent plane and normal line through $(1,2,3)$

$(x_0, y_0, \underbrace{f(x_0,y_0)}_{z_0})$

$f_x(x,y) = -2x \qquad f_x(x_0,y_0) = f_x(1,2) = -2(1) = -2$

$f_y(x,y) = -2y \qquad f_y(x_0,y_0) = f_y(1,2) = -2(2) = -4$

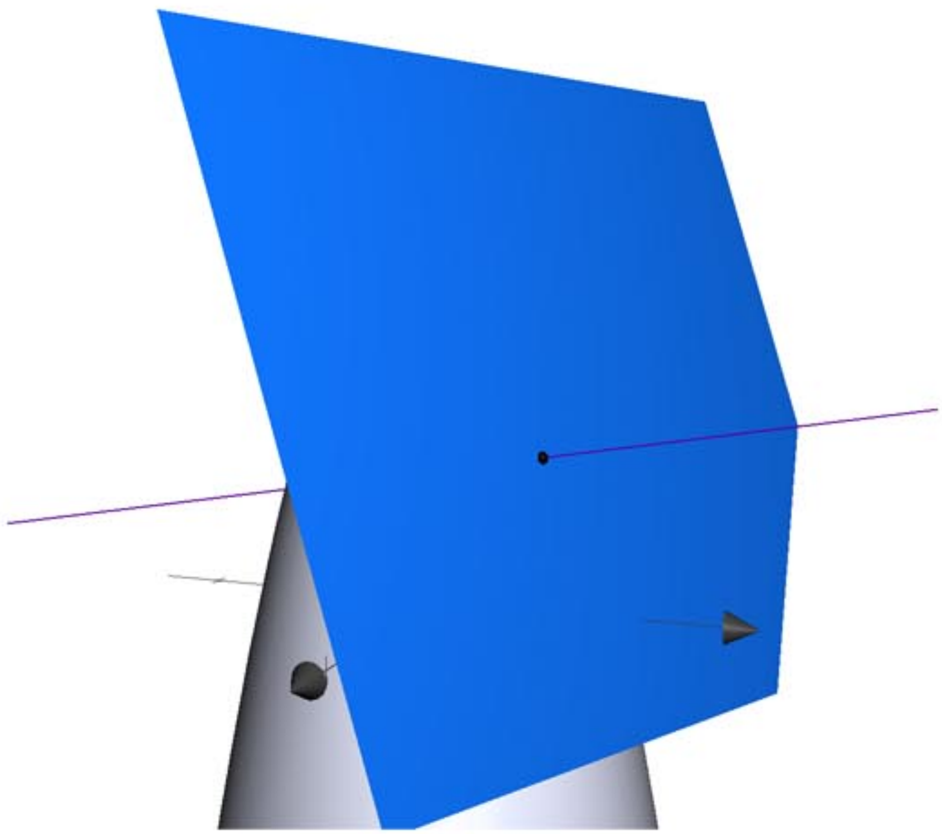
Equation of the tangent plane:

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0$$

$$-2(x-1)-4(y-2)-(z-3)=0$$

Equations of the normal line

$$x = 1 - 2t$$
$$y = 2 - 4t$$
$$z = 3 - t$$



We can use the equation of the tangent plane to approximate $f(1.2, 2.1)$:

tangent plane:

$$-2(x-1) - 4(y-2) - (z-3) = 0$$

solving for z:

$$z = -2(x-1) - 4(y-2) + 3$$

So $L(x,y) = -2(x-1) - 4(y-2) + 3$

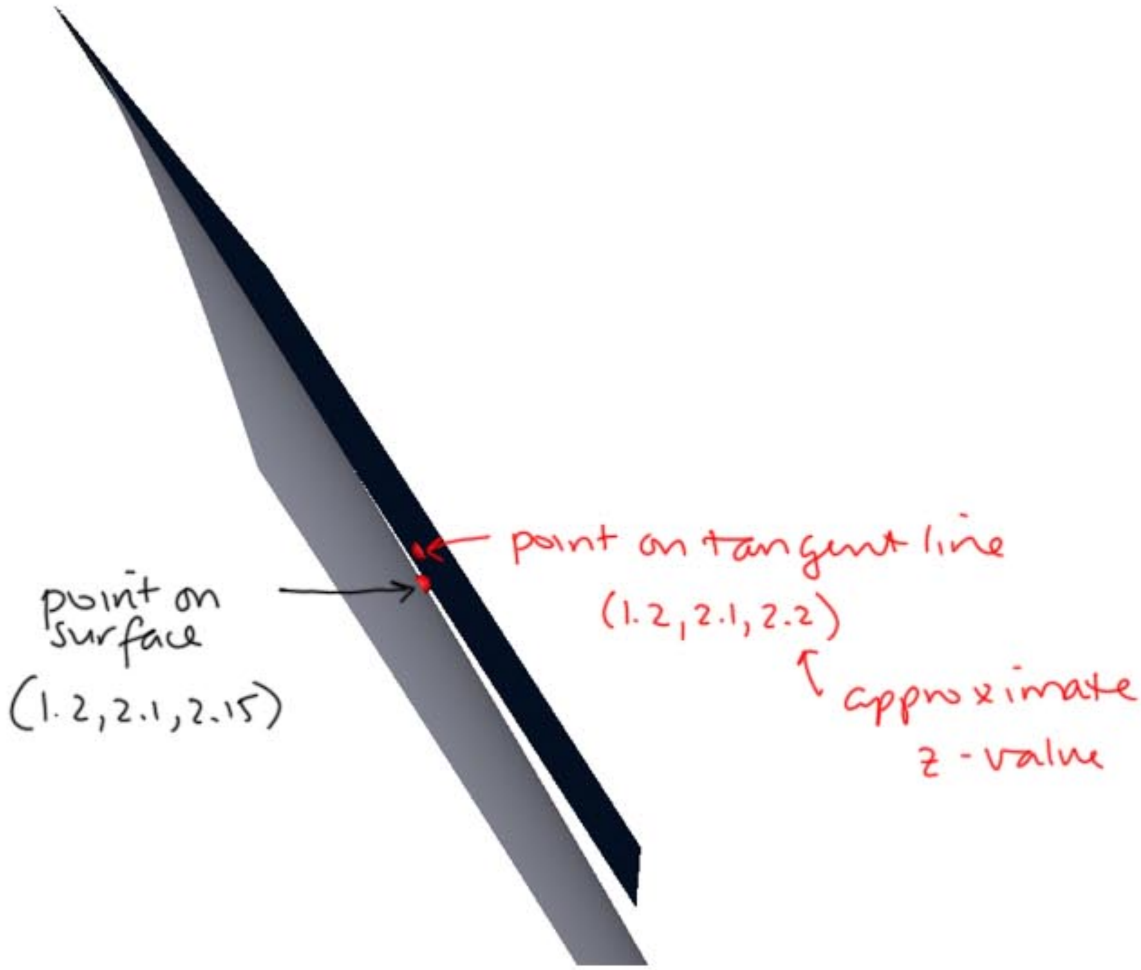
and $f(1.2, 2.1) \approx L(1.2, 2.1) = -2(1.2-1) - 4(2.1-2) + 3$

$$= -2(.2) - 4(.1) + 3$$

$$= -.4 - .4 + 3 = 2.2$$

actual value: $f(1.2, 2.1) = 8 - (1.2)^2 - (2.1)^2 =$

$$= 8 - 1.44 - 4.41 = 2.15$$


point on tangent line
(1.2, 2.1, 2.2)
approximate
z - value
point on
surface
(1.2, 2.1, 2.15)

Tangent Planes to Parametric Surfaces

Recall from lesson 10:

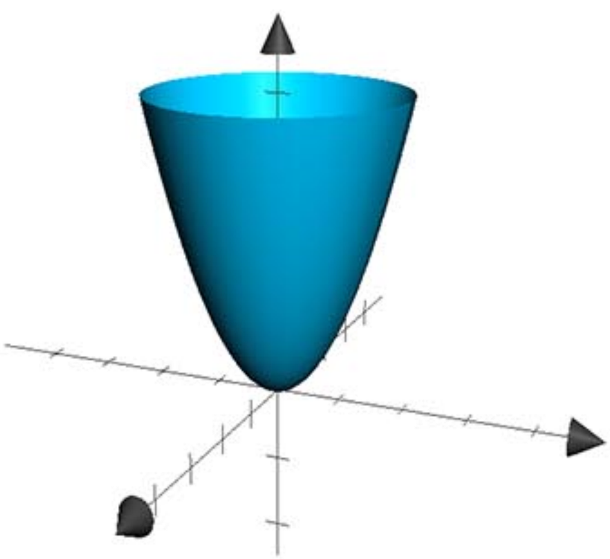
parametric surface defined by $x(u,v)$
$y(u,v)$
$z(u,v)$.

For example, we saw the paraboloid $z = x^2 + y^2$

described as: $x = v\sin u$

$y = v\cos u$

$z = v^2$



Consider the position vector function

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

our ex: $\vec{r}(u,v) = (v\sin u)\vec{i} + (v\cos u)\vec{j} + v^2\vec{k}$

To find a tangent plane at (u_0, v_0),

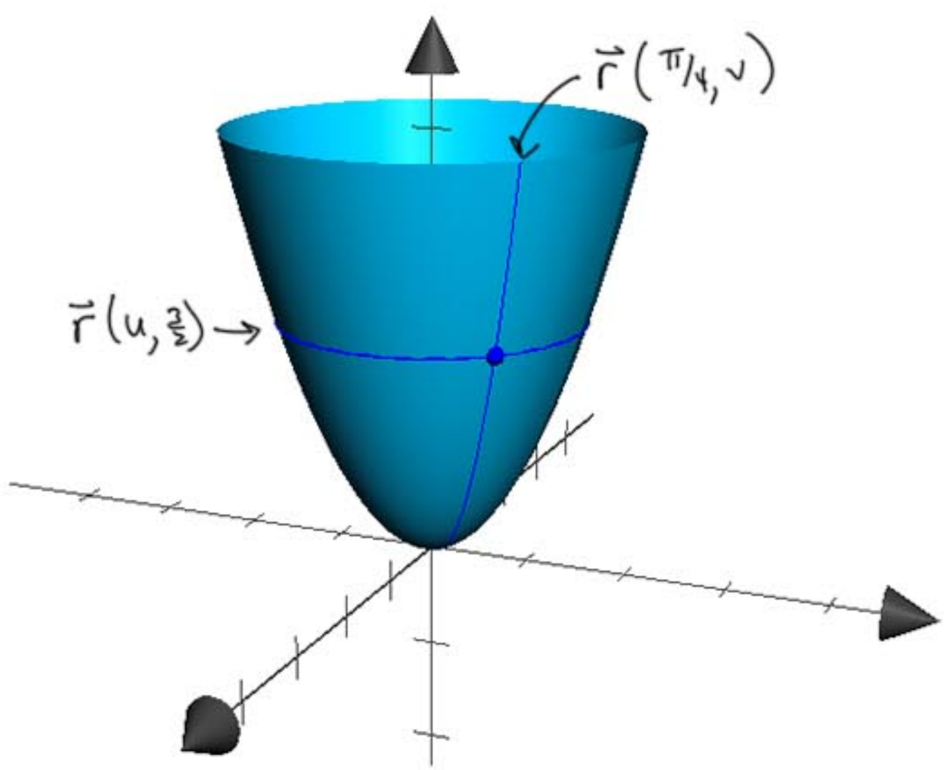
say $\left(\pi/4, 3/2\right)$ in our example,

we consider the curves obtained by holding

each variable constant:

$$\vec{r}\left(\pi/4, v\right) = \left(v \cdot \sin \pi/4\right)\vec{i} + \left(v \cdot \cos \pi/4\right)\vec{j} + v^2\vec{k}$$

and $\vec{r}\left(u, 3/2\right) = \left(\frac{3}{2}\sin u\right)\vec{i} + \left(\frac{3}{2}\cos u\right)\vec{j} + \frac{9}{4}\vec{k}$



then $\vec{r}_u(u_0, v_0) = \frac{\partial x}{\partial u}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial u}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial u}(u_0,v_0)\vec{k}$

and $\vec{r}_v(u_0, v_0) = \frac{\partial x}{\partial v}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial v}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial v}(u_0,v_0)\vec{k}$

are tangent vectors to the surface.

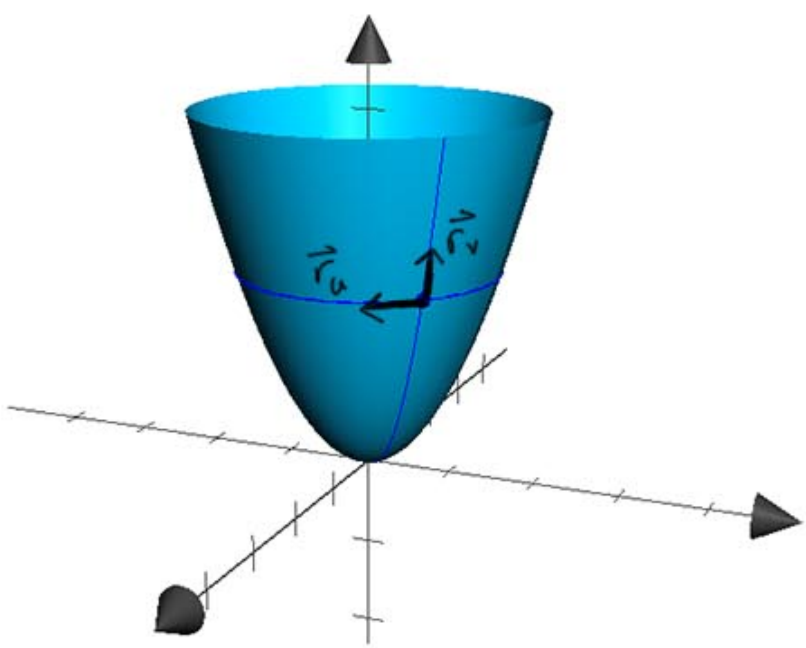
our example:

$$\vec{r}_u = (v\cos u)\vec{i} - (v\sin u)\vec{j} + 0\vec{k}$$

$$\vec{r}_v = (\sin u)\vec{i} + (\cos u)\vec{j} + 2v\vec{k}$$

$$\vec{r}_u(\pi/4, 3/2) = \frac{3}{2}\left(\frac{\sqrt{2}}{2}\right)\vec{i} - \frac{3}{2}\cdot\frac{\sqrt{2}}{2}\vec{j} = \left\langle \frac{3\sqrt{2}}{4}, -\frac{3\sqrt{2}}{4}, 0\right\rangle$$

$$\vec{r}_v(\pi/4, 3/2) = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j} + 3\vec{k} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 3\right\rangle$$



So the cross product $\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$ is a normal vector to the surface:

$$\left\langle \frac{3\sqrt{2}}{4}, -\frac{3\sqrt{2}}{4}, 0 \right\rangle \times \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 3 \right\rangle =$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{3\sqrt{2}}{4} & -\frac{3\sqrt{2}}{4} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 3 \end{vmatrix} = \begin{vmatrix} -\frac{3\sqrt{2}}{4} & 0 \\ \frac{\sqrt{2}}{2} & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{3\sqrt{2}}{4} & 0 \\ \frac{\sqrt{2}}{2} & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{3\sqrt{2}}{4} & -\frac{3\sqrt{2}}{4} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix} \vec{k}$$

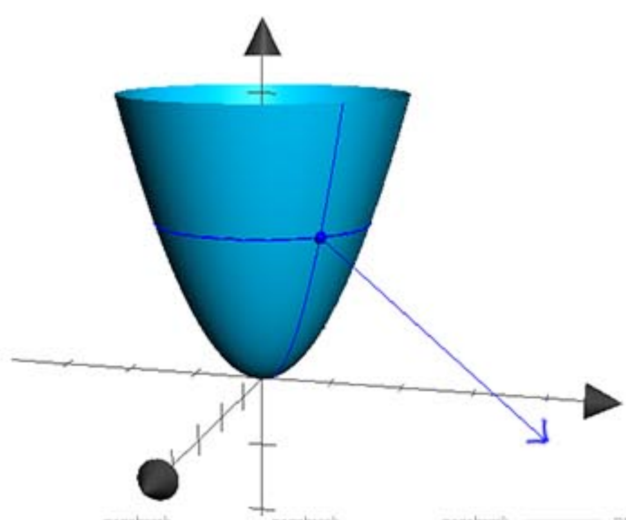
$$= -\frac{9\sqrt{2}}{4}\vec{i} - \frac{9\sqrt{2}}{4}\vec{j} + \frac{3}{2}\vec{k}$$

points inward, so we can choose

$\frac{9\sqrt{2}}{4}\vec{i} + \frac{9\sqrt{2}}{4}\vec{j} - \frac{3}{2}\vec{k}$ to view:

and can multiply by 4 for ease of computation:

$\langle 9\sqrt{2}, 9\sqrt{2}, -6 \rangle$



So $x_0 = x(u_0, v_0) = x(\pi/4, 3/2) = \frac{3}{2}\sin\frac{\pi}{4} = \frac{3\sqrt{2}}{4}$

$y_0 = y(u_0, v_0) = y(\pi/4, 3/2) = \frac{3}{2}\cos\frac{\pi}{4} = \frac{3\sqrt{2}}{4}$

$z_0 = z(u_0, v_0) = z(\pi/4, 3/2) = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

with normal vector $\langle 9\sqrt{2}, 9\sqrt{2}, -6\rangle = \langle a, b, c\rangle$

tangent plane: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$$9\sqrt{2}\left(x - \frac{3\sqrt{2}}{4}\right) + 9\sqrt{2}\left(y - \frac{3\sqrt{2}}{4}\right) - 6\left(z - \frac{9}{4}\right) = 0.$$

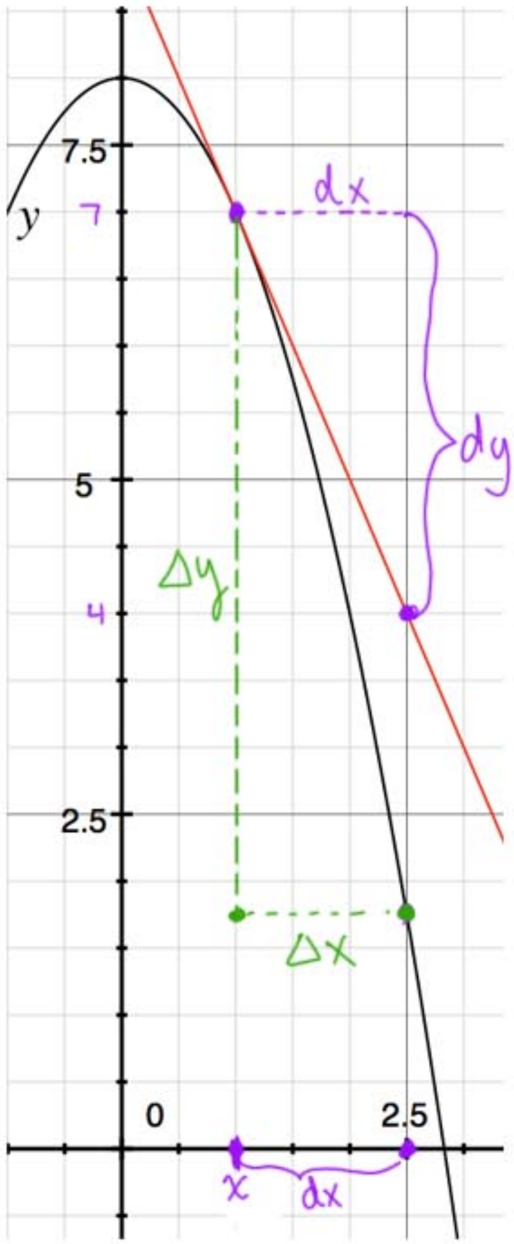
Increments + Differentials

Recall: if $y = f(x)$ with f differentiable,

the differential dx is an independent variable (can take any value) and

the differential $dy = f'(x)\,dx$

So for example, with $f(x) = 8 - x^2$ as above,



here $x = 1$

$dx = 1.5$

$f'(x) = -2x$

so $f'(1) = -2(1) = -2$

$$dy = f'(x)\,dx$$
$$= -2(1.5) = -3$$

dy is the change in y from moving along the tangent line.

Compare dy to Δy which is the change in y from moving along the actual function

$$\Delta y = f(x + \Delta x) - f(x)$$

same as dx to compare

increment

In the notation of differentials, the linearization (tangent line approximation) of f at $x = x_0$ looks like:

$$L(x) = f'(x)\underbrace{(x - x_0)}_{dx} + f(x_0)$$

$$\underbrace{L(x)}_{\substack{\text{approximation} \\ \text{of } f \text{ at new} \\ x \text{ value}}} = \underbrace{f(x_0)}_{\substack{\text{old} \\ y\text{-value}}} + \underbrace{dy}_{\substack{\text{change in } y \text{ by} \\ \text{moving along the} \\ \text{tangent line.}}}$$

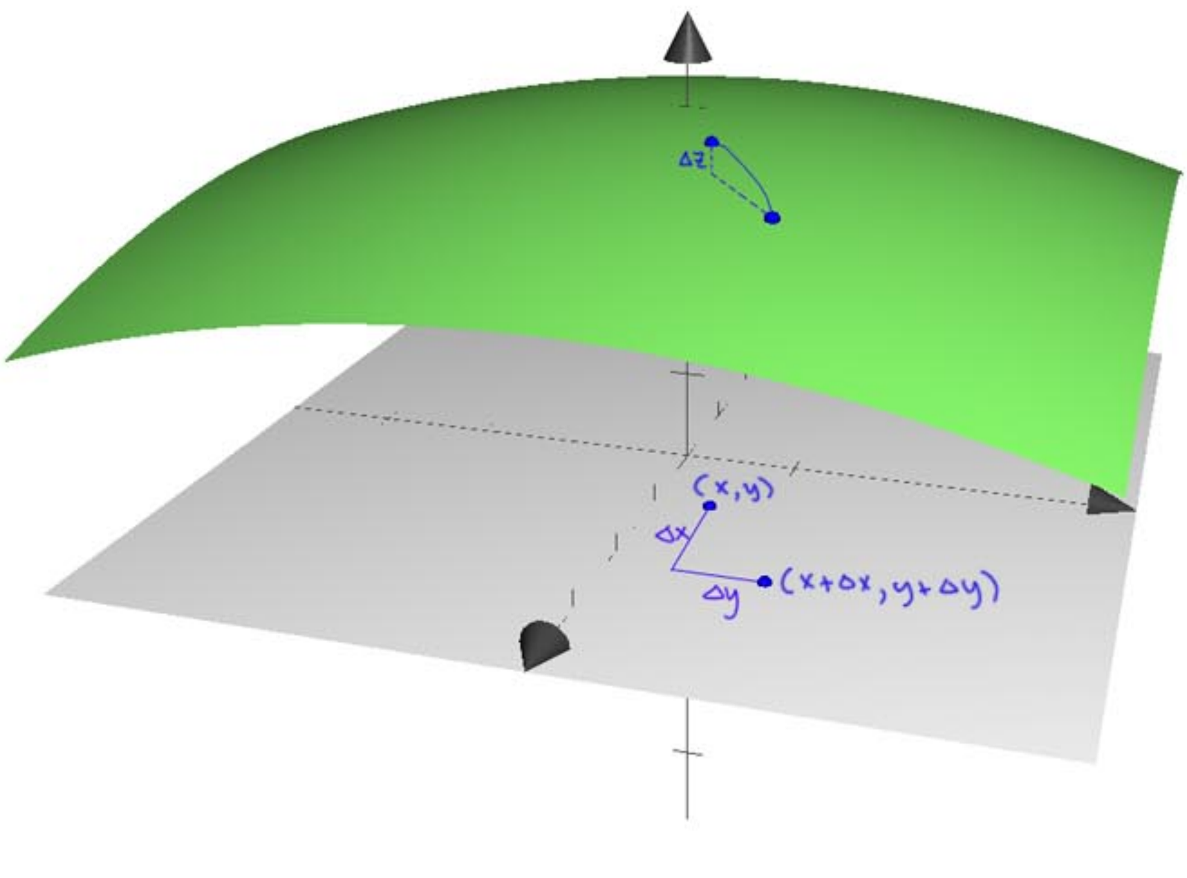
Note that the idea of using a linear approx. and differentials relies on f being differentiable, ie $f'(x)$ existing at and near x_0.

To be able to use a linear approximation by tangent plane to $z = f(x,y)$, we need more than f_x and f_y existing at and near (x_0, y_0). So we define differentiability of f as follows.

Notation:

Δx and Δy are independent changes in x + y

then $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$



For $z=f(x,y)$, f is differentiable at (x_0,y_0) if Δz can be expressed in the form:

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Theorem: if the partial derivatives f_x and f_y exist near (x_0,y_0) and are continuous at (x_0,y_0), then f is differentiable at (x_0,y_0).

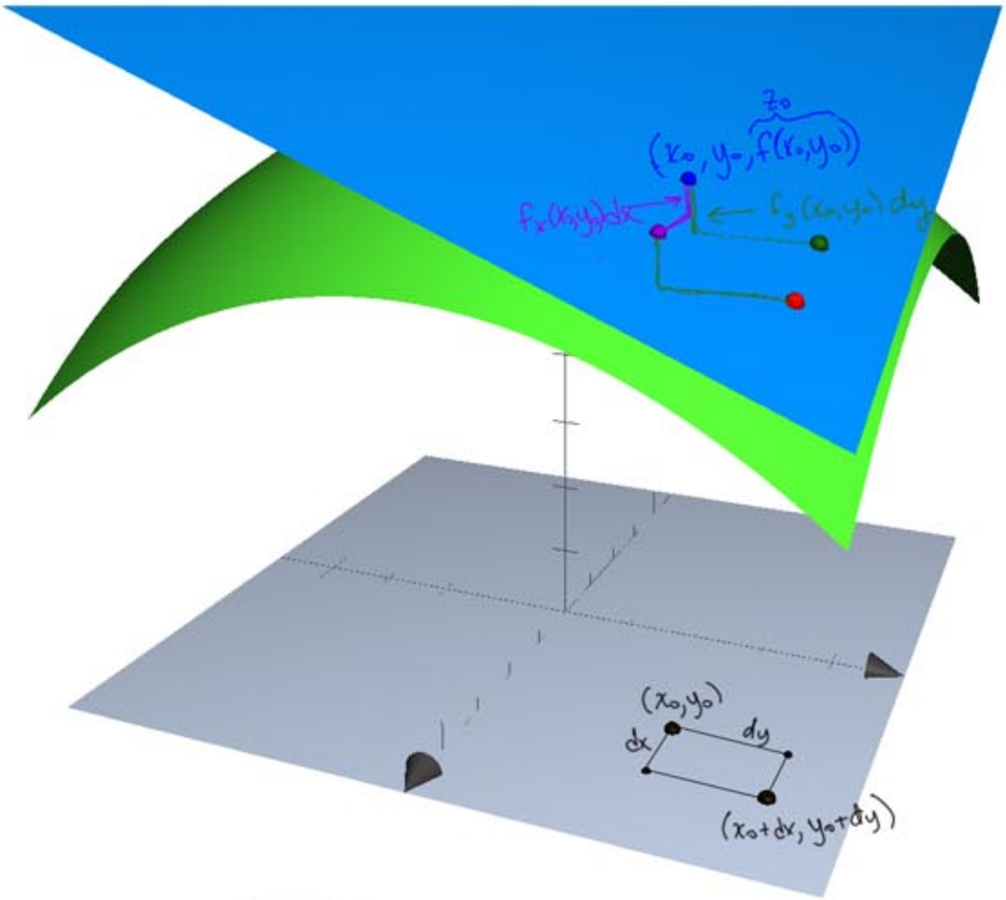
Differentials for $z=f(x,y)$

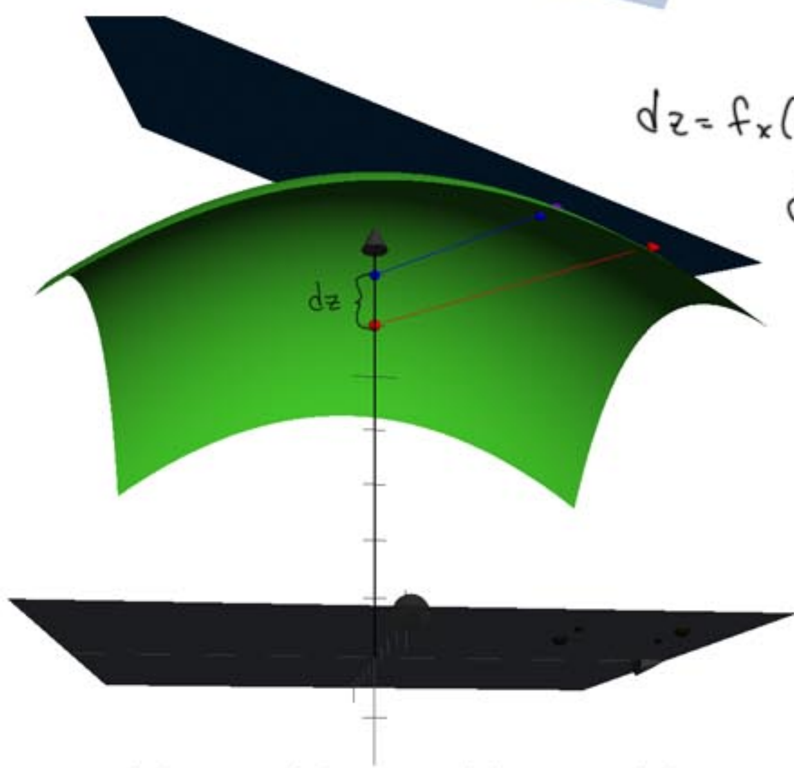
now dx + dy are independent variables,

total differential $dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

Then $L(x,y) = f_x(x_0,y_0)\underbrace{(x-x_0)}_{dx} + f_y(x_0,y_0)\underbrace{(y-y_0)}_{dy} + z_0$

$$L(x,y) = z_0 + dz$$
$$= f(x_0, y_0) + dz.$$





$$dz = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy$$

dz = change in z by moving along the tangent plane

$$L(x,y) = z_0 + dz$$
$$= f(x_0, y_0) + dz.$$

Ex. Use differentials to find an approximation to $\sqrt[3]{(2.02)^2 + 3.97}$

What is $f(x,y)$? $\quad f(x,y) = \sqrt[3]{x^2 + y}$

What is (x_0, y_0)? $\quad (x_0, y_0) = (2, 4)$ convenient values near 2.02 & 3.97

Then $f(2.02, 3.97) \approx f(2,4) + dz$
$$= \sqrt[3]{2^2+4} + f_x(2,4)\,dx + f_y(2,4)\,dy$$

$$f_x(x,y) = \frac{1}{3}(x^2+y)^{-2/3}(2x) = \frac{2x}{3(x^2+y)^{2/3}} \qquad f_x(2,4) = \frac{4}{3(8)^{2/3}} = \frac{1}{3}$$

$$f_y(x,y) = \frac{1}{3}(x^2+y)^{-2/3}(1) = \frac{1}{3(x^2+y)^{2/3}} \qquad f_y(2,4) = \frac{1}{3(8)^{2/3}} = \frac{1}{12}$$

so $f(2.02, 3.97) \approx 2 + \underbrace{\frac{1}{3}(.02)}_{2 + .00\overline{6}} + \underbrace{\frac{1}{12}(-.03)}_{-.0025} \approx 2.00417$

actual function value $\sqrt[3]{(2.02)^2 + 3.97} \approx 2.00419$

For $w = f(x,y,z)$, the local linearization at (x_0, y_0, z_0) is:

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0)(y - y_0) + f_z(x_0,y_0,z_0)(z - z_0)$$

Ex. Find the linear approximation to

$$f(x,y,z) = \ln(x + 2y - z) \quad \text{at } (0,1,1)$$

and use it to approximate $f(.1, .9, 1.2)$

$$f_x(x,y,z) = \frac{1}{x+2y-z} \cdot 1 = \frac{1}{x+2y-z} \qquad f_x(0,1,1) = 1$$

$$f_y(x,y,z) = \frac{1}{x+2y-z}\cdot 2 = \frac{2}{x+2y-z} \qquad f_y(0,1,1) = 2$$

$$f_z(x,y,z) = \frac{1}{x+2y-z}\cdot(-1) = \frac{-1}{x+2y-z} \qquad f_z(0,1,1) = -1$$

$$L(x,y,z) = \underbrace{f(0,1,1)}_{0} + 1(x-0) + 2(y-1) - (z-1)$$

$$= x + 2y - 2 - z + 1 = x + 2y - z - 1.$$

$$f(.1,.9,1.2) \approx L(.1,.9,1.2) = .1 + 2(.9) - 1.2 - 1$$

$$= .1 + 1.8 - 1.2 - 1 = -.3$$

$$\left(\text{actual value } \ln(.1 + 2(.9) - 1.2) = -.357\right)$$